ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b}, \mathrm{c} > 0$ and $a + \mathrm{b} + \mathrm{c} \geq \frac{1}{a} + \frac{1}{\mathrm{b}} + \frac{1}{\mathrm{c}}$ then prove that :

$$\sum_{\mathrm{cyc}} \sqrt{2a^2 + 4a + 3} - \sum_{\mathrm{cyc}} a \geq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \mathrm{f}(x) = \sqrt{2x^2 + 4x + 3}\ \forall\, x > 0 \text{ and then : } \mathrm{f}''(x) = \frac{2}{(2x^2 + 4x + 3)^{\frac{3}{2}}}$$

$$> 0 \therefore \mathrm{f}(x) \text{ is convex} \Rightarrow \sum_{\mathrm{cyc}} \sqrt{2a^2 + 4a + 3} \overset{\text{Jensen}}{\geq} 3.\sqrt{2\mathrm{t}^2 + 4\mathrm{t} + 3}\ \left(\mathrm{t} = \frac{\sum_{\mathrm{cyc}} a}{3}\right)$$

$$\overset{?}{\geq} \sum_{\mathrm{cyc}} a + 6 = 3\mathrm{t} + 6 \Leftrightarrow 2\mathrm{t}^2 + 4\mathrm{t} + 3 \overset{?}{\geq} \mathrm{t}^2 + 4\mathrm{t} + 4 \Leftrightarrow \mathrm{t}^2 \overset{?}{\geq} 1 \Leftrightarrow \frac{\sum_{\mathrm{cyc}} a}{3} \overset{?}{\geq} 1$$

$$\Leftrightarrow \sum_{\mathrm{cyc}} a \overset{?}{\geq} 3 \rightarrow \text{true} \because \sum_{\mathrm{cyc}} a \geq \sum_{\mathrm{cyc}} \frac{1}{a} \overset{\text{Bergstrom}}{\geq} \frac{9}{\sum_{\mathrm{cyc}} a} \Rightarrow \sum_{\mathrm{cyc}} a \geq 3$$

$$\therefore \sum_{\mathrm{cyc}} \sqrt{2a^2 + 4a + 3} - \sum_{\mathrm{cyc}} a \geq 6\ \forall\, a, \mathrm{b}, \mathrm{c} > 0 \,\Big|\, a + \mathrm{b} + \mathrm{c} \geq \frac{1}{a} + \frac{1}{\mathrm{b}} + \frac{1}{\mathrm{c}},$$

$$'' = ''\ \text{iff}\ \ a = \mathrm{b} = \mathrm{c} = 1\ (\text{QED})$$