ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0; a + b = 2$ then:

$$5(a^2 + b^2) + \frac{4(ab + a + b)}{a^2 + b^2 + 1} \geq 14$$

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Denote: $S = a + b; P = ab \Rightarrow S = 2$

$$2 = S = a + b \overset{AM-GM}{\geq} 2\sqrt{ab} = 2\sqrt{P} \Rightarrow \sqrt{P} \leq 1$$

$$\Rightarrow P \leq 1 \Rightarrow P - 1 \leq 0 \quad (1)$$

$$P \leq 1 \Rightarrow 10P \leq 10 \Rightarrow 10P - 19 \leq 10 - 19 = -9 < 0$$

$$\Rightarrow 10P - 19 < 0 \quad (2)$$

By (1); (2) $\Rightarrow (P - 1)(10P - 19) \geq 0 \quad (3)$

$$5(a^2 + b^2) + \frac{4(ab + a + b)}{a^2 + b^2 + 1} \geq 14 \Rightarrow 5(S^2 - 2P) + \frac{4(P + S)}{S^2 - 2P + 1} - 14 \geq 0$$

$$\Rightarrow 5(4 - 2P) + \frac{4(P + 2)}{5 - 2P} - 14 \geq 0$$

$$20 - 10P + \frac{4P + 8}{5 - 2P} - 14 \geq 0$$

$$6 - 10P \geq \frac{4P + 8}{2P - 5} \Leftrightarrow 3 - 5P \geq \frac{2P + 4}{2P - 5} \Leftrightarrow$$

$$(3 - 5P)(2P - 5) \leq 2P + 4 \Leftrightarrow 6P - 15 - 10P^2 + 25P \leq 2P + 4$$

$$\Leftrightarrow -10P^2 + 31P - 15 - 2P - 4 \leq 0$$

$$-10P^2 + 29P - 19 \leq 0$$

$$10P^2 - 29P + 19 \geq 0$$

$$(P - 1)(10P - 19) \geq 0 \quad \text{which is true: (3)}$$

Equality holds for $a = b = 1$.