ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ then prove that :

$$\sum_{cyc} \frac{a(b+c)}{a^2+bc} \ge 1 + \frac{16abc}{(a+b)(b+c)(c+a)}$$

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Assigning $b+c = x, c+a = y, a+b = z \Rightarrow x+y-z = 2c > 0$,
$y+z-x = 2a > 0$ and $z+x-y = 2b > 0 \Rightarrow x+y > z, y+z > x, z+x > y$
$\Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius $= s, R, r$ (say); then : $abc = r^2 s, \sum_{cyc} a = s, abc = r^2 s, \sum_{cyc} ab = 4Rr + r^2$,

$$\sum_{cyc} a^2 = s^2 - 8Rr - 2r^2, \sum_{cyc} a^3 = s^3 - 12Rrs, \sum_{cyc} a^2b^2 = r^2((4R+r)^2 - 2s^2),$$

$$\sum_{cyc} a^3b^3 = (4Rr+r^2)^3 - 12Rr^3s^2 \text{ and then : } \sum_{cyc} \frac{a(b+c)}{a^2+bc} - 1 =$$

$$\frac{\sum_{cyc} a^4b^2 + \sum_{cyc} a^2b^4 - abc\sum_{cyc} a^3 - \sum_{cyc} a^3b^3 + 3abc\left(\sum_{cyc} a^2b + \sum_{cyc} ab^2\right) - 2a^2b^2c^2}{\sum_{cyc} a^3b^3 + abc\sum_{cyc} a^3 + 2a^2b^2c^2}$$

$$= \frac{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^2b^2\right) - 14a^2b^2c^2 - abc\sum_{cyc} a^3 - \sum_{cyc} a^3b^3 + 3abc\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right)}{\sum_{cyc} a^3b^3 + abc\sum_{cyc} a^3 + 2a^2b^2c^2}$$

$$\overset{?}{\ge} \frac{16abc}{(a+b)(b+c)(c+a)}$$

$$\Leftrightarrow \frac{(s^2-8Rr-2r^2)(r^2)((4R+r)^2-2s^2) - 14r^4s^2 - r^2s(s^3-12Rrs) - (4Rr+r^2)^3 + 12Rr^3s^2 + 3r^2s^2(4Rr+r^2)}{(4Rr+r^2)^3 - 12Rr^3s^2 + r^2s(s^3-12Rrs) + r^4s^2} \overset{?}{\ge} \frac{16r^2s}{4Rrs}$$

$$\Leftrightarrow -(3R+4r)s^4 + (16R^3+60R^2r+90Rr^2-8r^3)s^2 -$$
$$r(192R^4+400R^3r+228R^2r^2+51Rr^3+4r^4) \boxed{\overset{?}{\ge} \ (1)} \ 0$$

Now, Rouche $\Rightarrow s^2 - (m-n) \ge 0$ and $s^2 - (m+n) \le 0$, where $m =$
$2R^2 + 10Rr - r^2$ & $n = 2(R-2r)\sqrt{R^2-2Rr} \therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right) \le 0$
$\Rightarrow s^4 - s^2(2m) + m^2 - n^2 \le 0 \Rightarrow s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3 \le 0$
$\Rightarrow P = -(3R+4r)(s^4 - s^2(4R^2+20Rr-2r^2) + r(4R+r)^3) \ge 0$

$\therefore$ in order to prove (1), it suffices to prove : LHS of (1) $\overset{?}{\ge}$ P

$\Leftrightarrow 4Rr^2s^2(R-2r)^2 \overset{?}{\ge} 0 \rightarrow$ true $\Rightarrow$ (1) is true

$$\therefore \sum_{cyc} \frac{a(b+c)}{a^2+bc} \ge 1 + \frac{16abc}{(a+b)(b+c)(c+a)} \ \forall\, a, b, c > 0,$$

$'' = ''$ iff $a = b = c$ (QED)