

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0$; $a + b = 2$ then:

$$a^4 + b^4 + 6ab \geq 8$$

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Denote: $S = a + b$; $P = ab$

$$\begin{aligned} S &= a + b = 2; 2 = a + b \stackrel{AM-GM}{\geq} 2\sqrt{ab} = 2\sqrt{P} \\ 2 &\geq 2\sqrt{P} \Rightarrow \sqrt{P} \leq 1 \Rightarrow P \leq 1 \Rightarrow P - 1 \leq 0 \Rightarrow P - 4 \leq -3 < 0 \\ P - 1 &\leq 0; P - 4 < 0 \Rightarrow (P - 1)(P - 4) \geq 0 \\ P^2 - 5P + 4 &\geq 0 \Rightarrow 2P^2 - 10P + 8 \geq 0 \\ 16 + 4P^2 - 16P - 2P^2 + 6P &\geq 8 \\ (4 - 2P)^2 - 2P^2 + 6P &\geq 8 \\ (S^2 - 2P)^2 - 2P^2 + 6P &\geq 8 \\ a^4 + b^4 + 6ab &\geq 8 \end{aligned}$$

Equality holds for $a = b = 1$.