ROMANIAN MATHEMATICAL MAGAZINE

If $a,b,c>0, ab+bc+ca=1$ then:

$$\frac{(ab+1)^2}{a^2+4ab+b^2}+\frac{(bc+1)^2}{b^2+4bc+c^2}+\frac{(ca+1)^2}{c^2+4ca+a^2}\ge\frac{8}{3}$$

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$$a^2+b^2+c^2\ge ab+bc+ca=1$$

$$a^2+b^2+c^2\ge 1 \qquad (1)$$

$$\frac{(ab+1)^2}{a^2+4ab+b^2}+\frac{(bc+1)^2}{b^2+4bc+c^2}+\frac{(ca+1)^2}{c^2+4ca+a^2}=\sum_{cyc}\frac{(ab+1)^2}{a^2+4ab+b^2}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(ab+1+bc+1+ca+1)^2}{2(a^2+b^2+c^2)+4(ab+bc+ca)}=\frac{(1+3)^2}{2(a^2+b^2+c^2)+4}=$$

$$=\frac{16}{2(a^2+b^2+c^2)+4}\ge\frac{8}{a^2+b^2+c^2+2}\overset{(1)}{\ge}\frac{8}{1+2}=\frac{8}{3}$$

Equality holds for $a=b=c=\frac{\sqrt{3}}{8}$.