ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0; abc = 1$ then:

$$\frac{(a+2)^2}{a^3} + \frac{(b+2)^2}{b^3} + \frac{(c+2)^2}{c^3} \ge 27$$

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$$\frac{(a+2)^2}{a^3} + \frac{(b+2)^2}{b^3} + \frac{(c+2)^2}{c^3} \overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\frac{\left((a+2)(b+2)(c+2)\right)^2}{(abc)^2}} = 3\sqrt[3]{\left((a+2)(b+2)(c+2)\right)^2}$$

Remains to prove:

$$3\sqrt[3]{\left((a+2)(b+2)(c+2)\right)^2} \ge 27 \Leftrightarrow \sqrt[3]{\left((a+2)(b+2)(c+2)\right)^2} \ge 9$$

$$\left((a+2)(b+2)(c+2)\right)^2 \ge 9^3 (a+2)(b+2)(c+2) \ge 3^3$$

$$abc + 2(ab+bc+ca) + 4(a+b+c) + 8 \ge 27$$

$$2(ab+bc+ca) + 4(a+b+c) \ge 18$$

$$ab+bc+ca+2(a+b+c) \ge 9 \qquad \text{(to prove)}$$

$$ab+bc+ca+2(a+b+c) \overset{AM-GM}{\ge} 9\sqrt[9]{ab\cdot bc\cdot ca\cdot a^2\cdot b^2\cdot c^2} = 9\sqrt[3]{(abc)^4} = 9\cdot 1 = 9$$

Equality holds for $a = b = c = 1$.