

ROMANIAN MATHEMATICAL MAGAZINE

Let $p, q, r > 0$ such that $pqr \geq 1$. Prove that:

$$\frac{p^n \sqrt{p^n q^n + r^n p^n + 2q^n r^n}}{q^n + r^n} + \frac{q^n \sqrt{q^n r^n + p^n q^n + 2r^n p^n}}{r^n + p^n} + \frac{r^n \sqrt{r^n p^n + q^n r^n + 2p^n q^n}}{p^n + q^n} \geq 3 \left(\frac{1}{p^5 + q^2 + r^2} + \frac{1}{q^5 + r^2 + p^2} + \frac{1}{r^5 + p^2 + q^2} \right); \text{ where } n \text{ is a real number } \geq 1.$$

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Solution by Jenish Rijal-Bardiya-Nepal

Let $x = p^n, y = q^n$, and $z = r^n$. Then the given inequality becomes,

$$\sum_{cyc} \frac{x \sqrt{xy + zx + 2yz}}{y + z} \geq 3 \sum_{cyc} \frac{1}{p^5 + q^2 + r^2}$$

We have, $\frac{x \sqrt{xy + zx + 2yz}}{y + z} + \frac{x \sqrt{xy + zx + 2yz}}{y + z} + \frac{x(y + z)^2}{xy + zx + 2yz} \stackrel{AM-GM}{\geq} 3x$

$$\therefore 2 \sum_{cyc} \frac{x \sqrt{xy + zx + 2yz}}{y + z} + \sum_{cyc} \frac{x(y + z)^2}{xy + zx + 2yz} \geq 3(x + y + z)$$

Also, $\sum_{cyc} \frac{x(y + z)^2}{xy + zx + 2yz} = \sum_{cyc} \frac{(\sqrt{xy} + \sqrt{zx})^2}{(xy + yz) + (yz + zx)} \stackrel{Bergstrom}{\geq} \leq \sum_{cyc} \frac{xy^2}{(xy + yz)} + \sum_{cyc} \frac{z^2 x}{(yz + zx)} = \sum_{cyc} \frac{xy}{x + z} + \sum_{cyc} \frac{zx}{y + x} = x + y + z$

So, $\sum_{cyc} \frac{x \sqrt{xy + zx + 2yz}}{y + z} \geq \frac{1}{2} \left[3(x + y + z) - \underbrace{\sum_{cyc} \frac{x(y + z)^2}{xy + zx + 2yz}}_{\leq x + y + z} \right] \geq x + y + z$

$$\therefore \sum_{cyc} \frac{p^n \sqrt{p^n q^n + r^n p^n + 2q^n r^n}}{q^n + r^n} \geq p^n + q^n + r^n \stackrel{AM-GM}{\geq} 3 \sqrt[3]{(pqr)^n} \stackrel{pqr \geq 1}{\geq} 3 \sqrt[3]{1} = 3$$

Thus, $\sum_{cyc} \frac{p^n \sqrt{p^n q^n + r^n p^n + 2q^n r^n}}{q^n + r^n} \geq 3 \cdot 1 \stackrel{\sum p^2 \geq 3}{\geq} 3 \cdot \frac{3}{p^2 + q^2 + r^2} = \frac{9}{p^2 + q^2 + r^2}$

Hence, it suffices to prove that: $\sum_{cyc} \frac{1}{p^5 + q^2 + r^2} \leq \frac{3}{p^2 + q^2 + r^2}$

Here, $(p^5 + q^2 + r^2) \left(\frac{1}{p} + q^2 + r^2 \right) \stackrel{Cauchy}{\geq} (p^2 + q^2 + r^2)^2 \Leftrightarrow \frac{1}{p^5 + q^2 + r^2} \leq \frac{\left(\frac{1}{p} + q^2 + r^2 \right)}{(p^2 + q^2 + r^2)^2}$

$$\begin{aligned} \therefore \sum_{cyc} \frac{1}{p^5 + q^2 + r^2} &\leq \sum_{cyc} \frac{\left(\frac{1}{p} + q^2 + r^2 \right)}{(p^2 + q^2 + r^2)^2} = \frac{2 \sum p^2 + \sum \frac{1}{p}}{(\sum p^2)^2} \stackrel{\sum \frac{1}{p} \leq \sum pq}{\leq} \\ &\leq \frac{2 \sum p^2 + \sum pq}{(\sum p^2)^2} \stackrel{\sum pq \leq \sum p^2}{\leq} \frac{2 \sum p^2 + \sum p^2}{(\sum p^2)^2} = \frac{3}{\sum p^2} \end{aligned}$$

Equality holds if and only if $p = q = r = 1$.