ROMANIAN MATHEMATICAL MAGAZINE

If $x,y,z>0,\sum_{cyc}x=xyz$ then :

$$\sum_{cyc}\frac{1}{3+2x^2}\geq\frac{1}{3}$$

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$$\sum_{cyc}\frac{1}{3+2x^2}\overset{?}{\geq}\frac{1}{3}\Leftrightarrow\frac{27+12\sum_{cyc}x^2+4\sum_{cyc}x^2y^2}{27+8x^2y^2z^2+12\sum_{cyc}x^2y^2+18\sum_{cyc}x^2}\overset{?}{\geq}\frac{1}{3}$$

$$\Leftrightarrow 27+9\sum_{cyc}x^2\overset{?}{\geq}4x^2y^2z^2\Leftrightarrow\frac{27xyz}{\sum_{cyc}x}+9\sum_{cyc}x^2\overset{?}{\geq}4x^2y^2z^2.\frac{\left(\sum_{cyc}x\right)^2}{x^2y^2z^2}$$

$$\Leftrightarrow 27xyz+9\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\overset{?}{\geq}4\left(\sum_{cyc}x\right)^3$$

$$\Leftrightarrow 5\sum_{cyc}x^3+3xyz\overset{?}{\geq}\underset{①}{3}\sum_{cyc}x^2y+3\sum_{cyc}xy^2$$

$$\text{Now, } 5\sum_{cyc}x^3+15xyz\overset{\text{Schur}}{\geq}5\sum_{cyc}x^2y+5\sum_{cyc}xy^2\rightarrow②$$

$$2\sum_{cyc}x^2y+2\sum_{cyc}xy^2\overset{AM-GM}{\geq}12xyz\rightarrow③\ \therefore ②+③\Rightarrow①\text{ is true}$$

$$\therefore\sum_{cyc}\frac{1}{3+2x^2}\geq\frac{1}{3}\ \forall\ x,y,z>0\ \Big|\ \sum_{cyc}x=xyz,''='' \text{ iff } x=y=z=\sqrt{3}\ (\text{QED})$$