ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } x,y,z>0,\sum_{cyc}x=3 \text{ and } 1\le\lambda\le 2 \text{ then :}$$

$$\sum_{cyc}\frac{x}{y}+\lambda\sum_{cyc}\left(xy(x+y)\right)\ge(2\lambda+1)\sum_{cyc}xy$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{x}{y}+\lambda\sum_{cyc}\left(xy(x+y)\right)\ge(2\lambda+1)\sum_{cyc}xy\ge$$

$$\frac{9\sum_{cyc}x^2}{\left(\sum_{cyc}x\right)^2}+\lambda\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-3xyz\right)-(2\lambda+1)\sum_{cyc}xy$$

$$\overset{\sum_{cyc}x=3}{=}\frac{1}{3}\cdot\sum_{cyc}x\cdot\frac{9\sum_{cyc}x^2}{9}+\lambda\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-3xyz\right)-$$

$$\left(\frac{2\lambda+1}{3}\right)\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)$$

$$=\frac{1}{3}\cdot\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)+\lambda\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\right)\right)$$

$$\overset{\lambda\ge 1}{\ge}\frac{1}{3}\cdot\left(\left(\sum_{cyc}x\right)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)+\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\right)$$

$$\left(\because\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)-9xyz\overset{AM-GM}{\ge}0\right)=\frac{1}{3}\cdot\left(\sum_{cyc}x^3+\sum_{cyc}x^2y+\sum_{cyc}xy^2-9xyz\right)$$

$$\overset{AM-GM}{\ge}\frac{1}{3}\cdot(3xyz+3xyz+3xyz-9xyz)=0 \text{ and so,}$$

$$\sum_{cyc}\frac{x}{y}+\lambda\sum_{cyc}\left(xy(x+y)\right)\ge(2\lambda+1)\sum_{cyc}xy\ \forall\ x,y,z>0\ \Big|\ \sum_{cyc}x=3 \text{ and } 1\le\lambda\le 2,$$

$$''='' \text{ iff } x=y=z=1 \text{ (QED)}$$