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If $x,y,z>0,\sum_{cyc}x=1$ then :

$$\sum_{cyc}\sqrt{\frac{yz}{1-x^2}}\le\frac{3}{2\sqrt{2}}$$

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Assigning $y+z=a,z+x=b,x+y=c\Rightarrow a+b-c=2z>0,b+c-a=2x>0$ and $c+a-b=2y>0\Rightarrow a+b>c,b+c>a,c+a>b\Rightarrow a,b,c$ form sides of a triangle ABC with semiperimeter, circumradius and inradius $=s,R,r$ (say) yielding $2\sum_{cyc}x=\sum_{cyc}a=2s\Rightarrow\sum_{cyc}x=s\Rightarrow\sum_{cyc}xy=4Rr+r^2$ and so,

$$\sum_{cyc}\sqrt{\frac{yz}{1-x^2}}\overset{CBS}{\le}\sqrt{\sum_{cyc}xy}.\sqrt{\sum_{cyc}\frac{1}{1-x^2}}\overset{\sum_{cyc}x=1}{=}\sqrt{\sum_{cyc}xy}.\sqrt{\sum_{cyc}\frac{1}{(y+z)(x+y+z+x)}}$$

$$=\sqrt{4Rr+r^2}.\sqrt{\sum_{cyc}\frac{1}{a(b+c)}}$$

$$=\sqrt{4Rr+r^2}.\sqrt{\frac{1}{4Rrs.2s(s^2+2Rr+r^2)}.\sum_{cyc}\left(bc\left(a^2+\sum_{cyc}ab\right)\right)}$$

$$=\sqrt{4Rr+r^2}.\sqrt{\frac{8Rrs^2+(s^2+4Rr+r^2)^2}{8Rrs^2(s^2+2Rr+r^2)}}\overset{?}{\le}\frac{3}{2\sqrt{2}}$$

$$\Leftrightarrow(5R-r)s^4-r(46R^2+15Rr+2r^2)s^2-r^2(4R+r)^3\boxed{\overset{?}{\ge}}\ 0\ (1)$$ and $\because$

$(5R-r)(s^2-16Rr+5r^2)^2\overset{Gerretsen}{\ge}0\ \therefore$ in order to prove (1), it suffices to prove :

$$\text{LHS of }(1)\overset{?}{\ge}(5R-r)(s^2-16Rr+5r^2)^2\Leftrightarrow$$

$$(114R^2-97Rr+8r^2)s^2\boxed{\overset{?}{\ge}}\ r(1344R^3-1008R^2r+297Rr^2-24r^3)\ (2)$$ and finally,

$$(114R^2-97Rr+8r^2)s^2\overset{Gerretsen}{\ge}(114R^2-97Rr+8r^2)(16Rr-5r^2)\overset{?}{\ge}$$

$$\text{RHS of }(2)\Leftrightarrow 240R^3-557R^2r+158Rr^2-8r^3\overset{?}{\ge}0$$

$$\Leftrightarrow(R-2r)(114R^2-77Rr+4r^2)\overset{?}{\ge}0\to\text{true}\because R\overset{Euler}{\ge}2r\Rightarrow(2)\Rightarrow(1)\text{ is true}$$

$$\therefore\sum_{cyc}\sqrt{\frac{yz}{1-x^2}}\le\frac{3}{2\sqrt{2}}\ \forall x,y,z>0,\sum_{cyc}x=1,''=''\text{ iff }x=y=z=\frac{1}{3}\ (QED)$$