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If $a,b,c>0, a+b+c=1$ and $\lambda\geq 0$ then :

$$\left(\sum_{cyc} ab\right)\left(\sum_{cyc}\frac{a}{b^2+\lambda b}\right)\geq\frac{3}{3\lambda+1}$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{\sqrt{b+\lambda}}=\sum_{cyc}\frac{a^2}{a.\sqrt{b+\lambda}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}\left(a.\sqrt{b+\lambda}\right)}=\frac{\left(\sum_{cyc}a\right)^2}{\sum_{cyc}\left(\sqrt{a}.\sqrt{ab+\lambda a}\right)}$$

$$\overset{\text{CBS}}{\geq}\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\sum_{cyc}ab+\lambda\sum_{cyc}a}}\geq\frac{\left(\sum_{cyc}a\right)^2}{\sqrt{\sum_{cyc}a}.\sqrt{\frac{\left(\sum_{cyc}a\right)^2}{3}+\lambda\sum_{cyc}a}}=\frac{1}{\sqrt{\frac{1}{3}+\lambda}}$$

$$\left(\because\sum_{cyc}a=1\right)\therefore\sum_{cyc}\frac{a}{\sqrt{b+\lambda}}\geq\sqrt{\frac{3}{3\lambda+1}}\rightarrow\textcircled{1}\ \text{and now,}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{a}{b^2+\lambda b}\right)$$

$$\overset{\text{Reverse CBS}}{\geq}\left(\sum_{cyc}\frac{a}{\sqrt{b+\lambda}}\right)^2\overset{\text{via }\textcircled{1}}{\geq}\frac{3}{3\lambda+1}\ \text{and so,}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}\frac{a}{b^2+\lambda b}\right)\geq\frac{3}{3\lambda+1}$$

$$\forall\, a,b,c>0\,\big|\,a+b+c=1\ \text{and}\ \lambda\geq 0,''=''\ \text{iff}\ a=b=c=\frac{1}{3}\ (\text{QED})$$