

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ then :

$$\left(\frac{\sum_{\text{cyc}} xy}{\sum_{\text{cyc}} x^2} \right)^2 + \frac{\sum_{\text{cyc}} x^3}{3xyz} \geq 2$$

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$$\begin{aligned} & \left(\frac{\sum_{\text{cyc}} xy}{\sum_{\text{cyc}} x^2} \right)^2 + \frac{\sum_{\text{cyc}} x^3}{3xyz} \stackrel{?}{\geq} 2 \\ \Leftrightarrow & \frac{\sum_{\text{cyc}} x^3 - 3xyz}{3xyz} \stackrel{?}{\geq} \frac{(\sum_{\text{cyc}} x^2 - \sum_{\text{cyc}} xy)(\sum_{\text{cyc}} x^2 + \sum_{\text{cyc}} xy)}{(\sum_{\text{cyc}} x^2)^2} \\ \Leftrightarrow & \frac{(\sum_{\text{cyc}} x)(\sum_{\text{cyc}} x^2 - \sum_{\text{cyc}} xy)}{3xyz} \stackrel{?}{\geq} \frac{(\sum_{\text{cyc}} x^2 - \sum_{\text{cyc}} xy)(\sum_{\text{cyc}} x^2 + \sum_{\text{cyc}} xy)}{(\sum_{\text{cyc}} x^2)^2} \\ \Leftrightarrow & \left(\sum_{\text{cyc}} x \right) \left(\sum_{\text{cyc}} x^2 \right)^2 \stackrel{?}{\geq} 3xyz \left(\sum_{\text{cyc}} x^2 + \sum_{\text{cyc}} xy \right) \left(\because \sum_{\text{cyc}} x^2 - \sum_{\text{cyc}} xy \geq 0 \right) \end{aligned}$$

$$\text{and since : } \left(\sum_{\text{cyc}} x \right) \left(\sum_{\text{cyc}} x^2 \right) \stackrel{\text{AM-GM}}{\geq} (3 \cdot \sqrt[3]{xyz}) (3 \cdot \sqrt[3]{x^2 y^2 z^2}) = 9xyz$$

$$\begin{aligned} \therefore & \text{ it suffices to prove : } 9xyz \left(\sum_{\text{cyc}} x^2 \right) \stackrel{?}{\geq} 3xyz \left(\sum_{\text{cyc}} x^2 + \sum_{\text{cyc}} xy \right) \\ \Leftrightarrow & 3 \sum_{\text{cyc}} x^2 \stackrel{?}{\geq} \sum_{\text{cyc}} x^2 + \sum_{\text{cyc}} xy \Leftrightarrow 2 \sum_{\text{cyc}} x^2 \stackrel{?}{\geq} \sum_{\text{cyc}} xy \rightarrow \text{true (strict inequality)} \\ \therefore & \sum_{\text{cyc}} x^2 \geq \sum_{\text{cyc}} xy \text{ and so, } \left(\frac{\sum_{\text{cyc}} xy}{\sum_{\text{cyc}} x^2} \right)^2 + \frac{\sum_{\text{cyc}} x^3}{3xyz} \geq 2 \forall x, y, z > 0, \\ & \text{" = " iff } x = y = z \text{ (QED)} \end{aligned}$$