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If $a, b, c > 0$, $\sum_{cyc} a^2 = 3$ and $n \in \mathbb{N}$ then :

$$\sum_{cyc} \left(\frac{a^2 + b^2}{bc(bc + a^2)} \right)^n \geq 3$$

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$$\begin{aligned} \sum_{cyc} \frac{a^2 + b^2}{bc(bc + a^2)} &\stackrel{\text{AM-GM}}{\geq} \sum_{cyc} \frac{2ab}{bc(bc + a^2)} = 2 \cdot \sum_{cyc} \frac{\left(\sqrt{\frac{a}{c}}\right)^2}{bc + a^2} \stackrel{\text{Bergstrom}}{\geq} \\ &2 \cdot \frac{\left(\sum_{cyc} \sqrt{\frac{b}{a}}\right)^2}{\sum_{cyc} ab + \sum_{cyc} a^2} \stackrel{\text{AM-GM}}{\geq} 2 \cdot \frac{\left(3 \cdot \sqrt[3]{\sqrt{\frac{b}{a} \cdot \frac{a}{c} \cdot \frac{c}{b}}}\right)^2}{\sum_{cyc} ab + \sum_{cyc} a^2} \geq \frac{18}{2 \sum_{cyc} a^2} = 3 \left(\because \sum_{cyc} a^2 = 3 \right) \end{aligned}$$

$$\text{and so, via Holder, } \sum_{cyc} \left(\frac{a^2 + b^2}{bc(bc + a^2)} \right)^n \geq \frac{\left(\sum_{cyc} \frac{a^2 + b^2}{bc(bc + a^2)} \right)^n}{3^{n-1}} \geq \frac{3^n}{3^{n-1}}$$

$$(\because n \geq 0 \text{ as } n \in \mathbb{N}) = 3 \text{ and so, } \sum_{cyc} \left(\frac{a^2 + b^2}{bc(bc + a^2)} \right)^n \geq 3$$

$$\forall a, b, c > 0 \mid \sum_{cyc} a^2 = 3 \text{ and } n \in \mathbb{N}, " = " \text{ iff } a = b = c = 1 \text{ (QED)}$$