

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0, abc = 1$ then :

$$\sum_{\text{cyc}} \frac{1}{\sqrt[4]{(2+a^2)(1+a^2+b^2)}} \leq \sqrt{3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

Since $abc = 1$, we can assign : $a = \frac{x}{y}, b = \frac{y}{z}, c = \frac{z}{x}$ and so,

$$\begin{aligned} \sum_{\text{cyc}} \frac{1}{\sqrt[4]{(2+a^2)(1+a^2+b^2)}} &= \sum_{\text{cyc}} \frac{1}{\sqrt[4]{\left(2+\frac{x^2}{y^2}\right)\left(1+\frac{x^2}{y^2}+\frac{y^2}{z^2}\right)}} \\ &= \sqrt{3} \cdot \sum_{\text{cyc}} \frac{y \cdot \sqrt{z}}{\sqrt[4]{(y^2+y^2+x^2)(x^2z^2+y^2z^2+y^4)(1+1+1)(1+1+1)}} \\ &\stackrel{\text{Holder}}{\leq} \sqrt{3} \cdot \sum_{\text{cyc}} \frac{y \cdot \sqrt{z}}{\sqrt{xyz} + y \cdot \sqrt{z} + y \cdot \sqrt{x}} = \sqrt{3} \cdot \sum_{\text{cyc}} \frac{y \cdot \sqrt{z}}{\sqrt{y} \cdot (\sqrt{zx} + \sqrt{yz} + \sqrt{xy})} \\ &= \sqrt{3} \cdot \sum_{\text{cyc}} \frac{\sqrt{yz}}{\sqrt{zx} + \sqrt{yz} + \sqrt{xy}} = \sqrt{3} \cdot \frac{\sqrt{yz} + \sqrt{zx} + \sqrt{xy}}{\sqrt{zx} + \sqrt{yz} + \sqrt{xy}} = \sqrt{3} \text{ and so,} \\ \sum_{\text{cyc}} \frac{1}{\sqrt[4]{(2+a^2)(1+a^2+b^2)}} &\leq \sqrt{3} \forall a, b, c > 0 \mid abc = 1, \\ &\quad " = " \text{ iff } a = b = c = 1 \text{ (QED)} \end{aligned}$$