

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z, t > 0$, $\sum_{cyc} x = 4$ and $\lambda \geq 2 \in \mathbb{N}$ then :

$$\sum_{cyc} \frac{x^2}{x^2 + \lambda(y + z + t)} \geq \frac{4}{3\lambda + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{x^2}{x^2 + \lambda(y + z + t)} &= \frac{x^2}{x^2 + \lambda(4 - x)} \stackrel{?}{\geq} \frac{7\lambda x - 4\lambda + 1}{(3\lambda + 1)^2} \\ \Leftrightarrow x^2(3\lambda + 1)^2 - (x^2 + \lambda(4 - x))(7\lambda x - 4\lambda + 1) &\stackrel{?}{\geq} 0 \\ \Leftrightarrow \lambda(16\lambda(x - 1)^2 - (7x^3 - 10x^2 - x + 4)) &\stackrel{?}{\geq} 0 \\ \Leftrightarrow \lambda(16\lambda(x - 1)^2 - (x - 1)^2(7x + 4)) &\stackrel{?}{\geq} 0 \\ \Leftrightarrow \lambda(x - 1)^2(16(\lambda - 2) + 7(4 - x)) &\stackrel{?}{\geq} 0 \\ \Leftrightarrow \lambda(x - 1)^2(16(\lambda - 2) + 7(y + z + t)) &\stackrel{?}{\geq} 0 \rightarrow \text{true} \because \lambda \geq 2 \text{ and } y, z, t > 0 \\ \therefore \frac{x^2}{x^2 + \lambda(y + z + t)} &\geq \frac{7\lambda x - 4\lambda + 1}{(3\lambda + 1)^2} \text{ and analogs} \Rightarrow \sum_{cyc} \frac{x^2}{x^2 + \lambda(y + z + t)} \geq \\ \frac{1}{(3\lambda + 1)^2} \cdot \left(7\lambda \left(\sum_{cyc} x \right) - 16\lambda + 4 \right) &= \frac{1}{(3\lambda + 1)^2} \cdot (7\lambda(4) - 16\lambda + 4) \\ \left(\because \sum_{cyc} x = 4 \right) &= \frac{4(3\lambda + 1)}{(3\lambda + 1)^2} = \frac{4}{3\lambda + 1} \text{ and so, } \sum_{cyc} \frac{x^2}{x^2 + \lambda(y + z + t)} \geq \frac{4}{3\lambda + 1} \\ \forall x, y, z, t > 0 \mid \sum_{cyc} x = 4 \text{ and } \lambda \geq 2 \in \mathbb{N}, " &= " \text{ iff } x = y = z = t = 1 \text{ (QED)} \end{aligned}$$