

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$, $\sum_{cyc} ab = 3$ and $n \in \mathbb{N}$ then :

$$\sum_{cyc} \left(\frac{a^2 + b^2}{a^2 + b^2 + 2} \right)^n \geq \frac{3}{2^n}$$

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$$\begin{aligned} & \sum_{cyc} \frac{a^2 + b^2}{a^2 + b^2 + 2} \stackrel{?}{\geq} \frac{3}{2} \\ \Leftrightarrow & 3 \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} a^2 b^2 \right) - 3a^2 b^2 c^2 + 2 \left(\sum_{cyc} a^2 \right)^2 + 2 \sum_{cyc} a^2 b^2 \stackrel{?}{\geq} 8 \sum_{cyc} a^2 + 24 \\ \Leftrightarrow & 3 \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} a^2 b^2 \right) - 3a^2 b^2 c^2 + \frac{2}{3} \left(\sum_{cyc} ab \right) \left(\sum_{cyc} a^2 \right)^2 + \\ & \frac{2}{3} \left(\sum_{cyc} ab \right) \left(\sum_{cyc} a^2 b^2 \right) \stackrel{?}{\underset{(*)}{\geq}} \frac{8}{9} \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} ab \right)^2 + \frac{24}{27} \left(\sum_{cyc} ab \right)^3 \left(\because \sum_{cyc} ab = 3 \right) \end{aligned}$$

Assigning $b + c = x, c + a = y, a + b = z \Rightarrow x + y - z = 2c > 0, y + z - x = 2a > 0$ and $z + x - y = 2b > 0 \Rightarrow x + y > z, y + z > x, z + x > y \Rightarrow x, y, z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius = s, R, r (say);

$$\text{then : } abc = r^2 s, \sum_{cyc} a = s, abc = r^2 s, \sum_{cyc} ab = 4Rr + r^2, \sum_{cyc} a^2 = s^2 - 8Rr - 2r^2,$$

$$\sum_{cyc} a^2 b^2 = r^2 ((4R + r)^2 - 2s^2), \text{ and then : } (*) \Leftrightarrow$$

$$\begin{aligned} & 27r^2(s^2 - 8Rr - 2r^2)((4R + r)^2 - 2s^2) - 27r^4 s^2 + 6(4Rr + r^2)(s^2 - 8Rr - 2r^2)^2 \\ & + 6r^2(4Rr + r^2) \left(\frac{(4R + r)^2}{2s^2} - 1 \right) \stackrel{?}{\geq} 8(s^2 - 8Rr - 2r^2)(4Rr + r^2)^2 + 8(4Rr + r^2)^3 \end{aligned}$$

$$\Leftrightarrow (3R - 6r)s^4 - r(10R^2 - 43Rr - 8r^2)s^2 - 2r^2(4R + r)^3 \stackrel{?}{\geq} 0 \text{ and since :}$$

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Gerretsen
and
Euler

$(3R - 6r)(s^2 - 16Rr + 5r^2)^2 \stackrel{?}{\geq} 0 \therefore$ in order to prove ①, suffices to prove :

$$\text{LHS of ①} \stackrel{?}{\geq} (3R - 6r)(s^2 - 16Rr + 5r^2)^2 \Leftrightarrow$$

$$(86R^2 - 179Rr + 68r^2)s^2 \stackrel{?}{\geq} r(896R^3 - 1920R^2r + 1059Rr^2 - 148r^3)$$

& finally, $(86R^2 - 179Rr + 68r^2)s^2 \stackrel{\text{Gerretsen}}{\geq} (86R^2 - 179Rr + 68r^2)(16Rr - 5r^2)$

$$\stackrel{?}{\geq} \text{RHS of ②} \Leftrightarrow 80t^3 - 229t^2 + 154t - 32 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(80t^2 - 69t + 16) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \text{②} \Rightarrow \text{① is true}$$

$$\therefore \sum_{\text{cyc}} \frac{a^2 + b^2}{a^2 + b^2 + 2} \geq \frac{3}{2} \therefore \sum_{\text{cyc}} \left(\frac{a^2 + b^2}{a^2 + b^2 + 2} \right)^n \stackrel{\text{Holder}}{\geq} \frac{1}{3^{n-1}} \cdot \left(\sum_{\text{cyc}} \frac{a^2 + b^2}{a^2 + b^2 + 2} \right)^n$$

$$\geq \frac{1}{3^{n-1}} \cdot \left(\frac{3}{2} \right)^n \quad (\because n \in \mathbb{N}) = \frac{3}{2^n} \text{ and so, } \sum_{\text{cyc}} \left(\frac{a^2 + b^2}{a^2 + b^2 + 2} \right)^n \geq \frac{3}{2^n}$$

$$\forall a, b, c > 0 \mid \sum_{\text{cyc}} ab = 3 \text{ and } n \in \mathbb{N}, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$