

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $\sum_{cyc} x = 3$ and $3n \geq 4\lambda \geq 0, k = 3n - \lambda$ then :

$$\lambda xyz \geq n \sum_{cyc} xy - k$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \lambda xyz &\stackrel{?}{\geq} n \sum_{cyc} xy - k \Leftrightarrow \lambda xyz \stackrel{?}{\geq} n \sum_{cyc} xy - 3n + \lambda \\ &\Leftrightarrow \frac{n}{9} \left(\sum_{cyc} x \right) \left(\left(\sum_{cyc} x \right)^2 - 3 \sum_{cyc} xy \right) \stackrel{?}{\geq} \frac{\lambda}{27} \left(\left(\sum_{cyc} x \right)^3 - 27xyz \right) \\ &\left(\because \sum_{cyc} x = 3 \right) \text{ and } \because \frac{n}{9} \geq \frac{4\lambda}{27} \geq 0 \therefore \text{it suffices to prove :} \\ &4 \left(\sum_{cyc} x \right) \left(\left(\sum_{cyc} x \right)^2 - 3 \sum_{cyc} xy \right) \stackrel{?}{\geq} \left(\sum_{cyc} x \right)^3 - 27xyz \\ &\Leftrightarrow \sum_{cyc} x^3 + 3xyz \stackrel{?}{\geq} \sum_{cyc} x^2y + \sum_{cyc} xy^2 \rightarrow \text{true via Schur } \therefore \lambda xyz \geq n \sum_{cyc} xy - k \\ &\forall x, y, z > 0 \mid \sum_{cyc} x = 3 \text{ and } 3n \geq 4\lambda \geq 0, k = 3n - \lambda, \\ &\quad \quad \quad " = " \text{ iff } x = y = z = 1 \text{ (QED)} \end{aligned}$$