

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$, $\sum_{\text{cyc}} x = 1$ and $-3 \leq \lambda \leq \frac{15}{4}$ then :

$$\sum_{\text{cyc}} x^3 + \lambda xyz \geq \frac{\lambda + 3}{27}$$

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An observation : since $\lambda \geq -3 \therefore \sum_{\text{cyc}} x^3 + \lambda xyz \geq \sum_{\text{cyc}} x^3 - 3xyz$

$\stackrel{\text{AM-GM}}{\geq} 0$ and also, $\frac{\lambda + 3}{27} \geq 0$; now $\sum_{\text{cyc}} x^3 + \lambda xyz \stackrel{?}{\geq} \frac{\lambda + 3}{27}$

$$\Leftrightarrow \sum_{\text{cyc}} x^3 - \frac{1}{9} \left(\sum_{\text{cyc}} x \right)^3 \stackrel{?}{\geq} \frac{\lambda}{27} \left(\left(\sum_{\text{cyc}} x \right)^3 - 27xyz \right) \text{ and } \therefore$$

$\left(\sum_{\text{cyc}} x \right)^3 - 27xyz \stackrel{\text{AM-GM}}{\geq} 0$ and $\frac{\lambda}{27} \leq \frac{5}{36} \therefore$ it suffices to prove :

$$4 \left(9 \sum_{\text{cyc}} x^3 - \left(\sum_{\text{cyc}} x \right)^3 \right) \stackrel{?}{\geq} 5 \left(\left(\sum_{\text{cyc}} x \right)^3 - 27xyz \right)$$

$$\Leftrightarrow \sum_{\text{cyc}} x^3 + 3xyz \stackrel{?}{\geq} \sum_{\text{cyc}} x^2y + \sum_{\text{cyc}} xy^2 \rightarrow \text{true via Schur}$$

$$\therefore \sum_{\text{cyc}} x^3 + \lambda xyz \geq \frac{\lambda + 3}{27} \forall x, y, z > 0 \mid \sum_{\text{cyc}} x = 1 \text{ and } -3 \leq \lambda \leq \frac{15}{4},$$

$$'' = '' \text{ iff } x = y = z = \frac{1}{3} \text{ (QED)}$$