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If $a, b, c > 0$, $abc = 8$ and $0 \leq \lambda \leq 1$ then :

$$\sum_{\text{cyc}} \frac{1}{a + \lambda} \geq \frac{3}{\lambda + 2}$$

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$$\begin{aligned} \sum_{\text{cyc}} \frac{1}{a + \lambda} &\stackrel{?}{\geq} \frac{3}{\lambda + 2} \Leftrightarrow \frac{3\lambda^2 + 2\lambda \sum_{\text{cyc}} a + \sum_{\text{cyc}} ab}{abc + \lambda \sum_{\text{cyc}} ab + \lambda^2 \sum_{\text{cyc}} a + \lambda^3} \stackrel{?}{\geq} \frac{3}{\lambda + 2} \\ &\Leftrightarrow \left(6 - \sum_{\text{cyc}} a\right) \lambda^2 + \left(4 \sum_{\text{cyc}} a - 2 \sum_{\text{cyc}} ab\right) \lambda + 2 \sum_{\text{cyc}} ab - 3abc \stackrel{?}{\geq} 0 \quad (*) \\ \text{Firstly, } \sum_{\text{cyc}} a &\stackrel{\text{AM-GM}}{\geq} 3 \cdot \sqrt[3]{abc} = 3 \cdot \sqrt[3]{8} = 6 \Rightarrow \sum_{\text{cyc}} a \geq 6 \rightarrow \textcircled{1} \text{ and secondly,} \end{aligned}$$

$$\begin{aligned} 2 \sum_{\text{cyc}} ab &\stackrel{\text{AM-GM}}{\geq} 6 \cdot \sqrt[3]{a^2 b^2 c^2} = 6 \cdot \sqrt[3]{\frac{a^3 b^3 c^3}{abc}} \stackrel{abc=8}{=} 6 \cdot \sqrt[3]{\frac{a^3 b^3 c^3}{8}} = 3abc \\ &\Rightarrow 2 \sum_{\text{cyc}} ab \geq 3abc \rightarrow \textcircled{2} \end{aligned}$$

Case 1 $4 \sum_{\text{cyc}} a - 2 \sum_{\text{cyc}} ab \geq 0$ and $\because 0 \leq \lambda \leq 1 \Rightarrow \lambda \geq \lambda^2$ and $1 \geq \lambda^2$, then, via $\textcircled{2}$,

$$\begin{aligned} \text{LHS} &\geq \left(6 - \sum_{\text{cyc}} a\right) \lambda^2 + \left(4 \sum_{\text{cyc}} a - 2 \sum_{\text{cyc}} ab\right) \lambda^2 + \left(2 \sum_{\text{cyc}} ab - 3abc\right) \lambda^2 \\ &= \lambda^2 \left(3 \sum_{\text{cyc}} a + 6 - 3abc\right) \stackrel{\text{AM-GM}}{\geq} \lambda^2 (9 \cdot \sqrt[3]{abc} + 6 - 3abc) (\because \lambda^2 \geq 0) \stackrel{abc=8}{=} 0 \\ &\Rightarrow (*) \text{ is true} \end{aligned}$$

Case 2 $4 \sum_{\text{cyc}} a - 2 \sum_{\text{cyc}} ab < 0$ and then, since $\lambda, \lambda^2 \leq 1$, then via $\textcircled{1}$, LHS of $(*) \geq$

$$\begin{aligned} 6 - \sum_{\text{cyc}} a + 4 \sum_{\text{cyc}} a - 2 \sum_{\text{cyc}} ab + 2 \sum_{\text{cyc}} ab - 3abc &= 3 \sum_{\text{cyc}} a + 6 - 3abc \stackrel{\text{AM-GM}}{\geq} \\ 9 \cdot \sqrt[3]{abc} + 6 - 3abc &\stackrel{abc=8}{=} 0 \Rightarrow (*) \text{ is true and so, combining both cases,} \\ \sum_{\text{cyc}} \frac{1}{a + \lambda} &\geq \frac{3}{\lambda + 2} \quad \forall a, b, c > 0 \mid abc = 8 \text{ and } 0 \leq \lambda \leq 1, \end{aligned}$$

" = " iff $a = b = c = 2$ (QED)