

ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0$ and $\lambda \leq 3$ then :

$$\sum_{cyc} \frac{1}{xy} + \frac{\lambda}{\sum_{cyc} x^2} \geq \frac{\lambda + 9}{\sum_{cyc} xy}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \sum_{cyc} \frac{1}{xy} + \frac{\lambda}{\sum_{cyc} x^2} \stackrel{?}{\geq} \frac{\lambda + 9}{\sum_{cyc} xy} \Leftrightarrow \frac{\sum_{cyc} x}{xyz} + \frac{\lambda}{\sum_{cyc} x^2} \stackrel{?}{\geq} \frac{\lambda + 9}{\sum_{cyc} xy} \\ \Leftrightarrow & \left(\left(\sum_{cyc} x \right) \left(\sum_{cyc} xy \right) - 9xyz \right) \left(\sum_{cyc} x^2 \right) \stackrel{?}{\geq} \lambda \cdot xyz \left(\sum_{cyc} x^2 - \sum_{cyc} xy \right) \text{ and } \because \\ & \sum_{cyc} x^2 - \sum_{cyc} xy \geq 0 \text{ and } \lambda \leq 3 \therefore \text{it suffices to prove :} \\ & \left(\left(\sum_{cyc} x \right) \left(\sum_{cyc} xy \right) - 9xyz \right) \left(\sum_{cyc} x^2 \right) \stackrel{?}{\geq} \lambda \cdot xyz \left(\sum_{cyc} x^2 - \sum_{cyc} xy \right) \end{aligned}$$

Assigning $y + z = a, z + x = b, x + y = c \Rightarrow a + b - c = 2z > 0, b + c - a = 2x > 0$ and $c + a - b = 2y > 0 \Rightarrow a + b > c, b + c > a, c + a > b$

$\Rightarrow a, b, c$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$$= s, R, r \text{ (say) yielding } 2 \sum_{cyc} x = \sum_{cyc} a = 2s \Rightarrow \sum_{cyc} x = s$$

$$\Rightarrow x = s - a, y = s - b, z = s - c \Rightarrow xyz = r^2 s, \sum_{cyc} xy = 4Rr + r^2,$$

$$\sum_{cyc} x^2 = s^2 - 8Rr - 2r^2 \text{ and so, } (*) \Leftrightarrow$$

$$(s(4Rr + r^2) - 9r^2 s)(s^2 - 8Rr - 2r^2) \stackrel{?}{\geq} 3r^2 s(s^2 - 8Rr - 2r^2 - 4Rr - r^2)$$

$$\Leftrightarrow (4R - 11r)s^2 \stackrel{?}{\geq} r(32R^2 - 92Rr - 25r^2)$$

Case 1 $4R - 11r \geq 0$ and then : LHS of $(**)$ $\stackrel{\text{Gerretsen}}{\geq} (4R - 11r)(16Rr - 5r^2)$
 $\stackrel{?}{\geq}$ RHS of $(**)$ $\Leftrightarrow 8r(R - 2r)(4R - 5r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (**)$ is true

Case 2 $4R - 11r < 0$; then : LHS of $(**)$ $\stackrel{\text{Gerretsen}}{\geq} (4R - 11r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\geq}$
 RHS of $(**)$ $\Leftrightarrow 4t^3 - 15t^2 + 15t - 2 \stackrel{?}{\geq} 0 \left(t = \frac{R}{r} \right) \Leftrightarrow (t - 2)(4t^2 - 7t + 1) \stackrel{?}{\geq} 0$

$\rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (**)$ is true $\therefore$ combining both cases, $(**) \Rightarrow (*)$ is true

$$\forall \Delta XYZ_{s,R,r} \text{ and so, } \sum_{cyc} \frac{1}{xy} + \frac{\lambda}{\sum_{cyc} x^2} \geq \frac{\lambda + 9}{\sum_{cyc} xy} \quad \forall x, y, z > 0 \text{ and } \lambda \leq 3,$$

" = " iff $x = y = z$ (QED)