ROMANIAN MATHEMATICAL MAGAZINE

If $a, \mathrm{b} > 0, a^6 + \mathrm{b}^6 \geq 2$ and $\mathrm{n} \in \mathbb{N}$ then :

$$\left(\frac{a^3}{\mathrm{b}} + \frac{\mathrm{b}^3}{a}\right)^{\mathrm{n}} \geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^3}{\mathrm{b}} + \frac{\mathrm{b}^3}{a} \overset{?}{\geq} 2 \Leftrightarrow a^4 + \mathrm{b}^4 \overset{?}{\geq} 2a\mathrm{b} \Leftrightarrow (a^4 + \mathrm{b}^4)^3 \overset{?}{\geq} 8a^3\mathrm{b}^3 \text{ and } \because$$

$$8a^3\mathrm{b}^3 = 2(4a^3\mathrm{b}^3) \leq (4a^3\mathrm{b}^3)(a^6 + \mathrm{b}^6) \therefore \text{it suffices to prove :}$$

$$(a^4 + \mathrm{b}^4)^3 \overset{?}{\geq} (4a^3\mathrm{b}^3)(a^6 + \mathrm{b}^6) \Leftrightarrow (\mathrm{t}^4 + 1)^3 \overset{?}{\geq} 4\mathrm{t}^3(\mathrm{t}^6 + 1) \ \left(\mathrm{t} = \frac{a}{\mathrm{b}}\right)$$

$$\Leftrightarrow \mathrm{t}^{12} - 4\mathrm{t}^9 + 3\mathrm{t}^8 + 3\mathrm{t}^4 - 4\mathrm{t}^3 + 1 \overset{?}{\geq} 0$$

$$\Leftrightarrow (\mathrm{t} - 1)^2(\mathrm{t}^{10} + 2\mathrm{t}^9 + 3\mathrm{t}^8 + 3\mathrm{t}^2 + 2\mathrm{t} + 1) \overset{?}{\geq} 0 \to \text{true} \because \mathrm{t} > 0 \therefore \boxed{\frac{a^3}{\mathrm{b}} + \frac{\mathrm{b}^3}{a} \geq 2}$$

$$\Rightarrow \left(\frac{a^3}{\mathrm{b}} + \frac{\mathrm{b}^3}{a}\right)^{\mathrm{n}} \geq 2^{\mathrm{n}} \ (\because \mathrm{n} \geq 1 > 0) \overset{?}{\geq} 2 \Leftrightarrow (\mathrm{n} - 1)(\ln 2) \overset{?}{\geq} 0$$

$$\to \text{true} \because \mathrm{n} \geq 1 \text{ and } \ln 2 > 0 \text{ and so,} \left(\frac{a^3}{\mathrm{b}} + \frac{\mathrm{b}^3}{a}\right)^{\mathrm{n}} \geq 2$$

$$\forall\, a, \mathrm{b} > 0 \,\middle|\, a^6 + \mathrm{b}^6 \geq 2 \text{ and } \mathrm{n} \in \mathbb{N}, '' = '' \text{ iff } a = \mathrm{b} = 1 \text{ and } \mathrm{n} = 1 \text{ (QED)}$$