ROMANIAN MATHEMATICAL MAGAZINE

If $a, b > 0, a + b = 2$ and $n \in \mathbb{N}$ then :

$$\sum_{cyc}\left(\frac{a^3}{b^4+1}\right)^n \geq \frac{1}{2^{n-1}}$$

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$$\text{Let } ab = t \text{ and } \because a + b = 2 \therefore a^2 + b^2 \overset{(*)}{=} 4 - 2t$$

$$\Rightarrow a^3 + b^3 = (a+b)(a^2+b^2-ab) = 2(4-2t-t) \therefore a^3 + b^3 \overset{(**)}{=} 2(4-3t);$$

$$a^4 + b^4 = (a^2+b^2)^2 - 2a^2b^2 \therefore a^4 + b^4 \overset{(***)}{=} (4-2t)^2 - 2t^2 \text{ and}$$

$$a^6 + b^6 = (a^2+b^2)(a^4+b^4-a^2b^2) = (4-2t)((4-2t)^2 - 2t^2 - t^2)$$

$$\therefore a^6 + b^6 \overset{(****)}{=} (4-2t)((4-2t)^2 - 3t^2) \text{ and now,}$$

$$\frac{a^3}{b^4+1} + \frac{b^3}{a^4+1} \overset{?}{\geq} 1 \Leftrightarrow a^7 + b^7 + a^3 + b^3 \overset{?}{\geq} a^4 + b^4 + a^4b^4 + 1$$

$$\Leftrightarrow (a+b)\left((a^6+b^6) - ab(a^4+b^4) + a^2b^2(a^2+b^2) - a^3b^3\right) + a^3 + b^3$$

$$\overset{?}{\geq} a^4 + b^4 + a^4b^4 + 1 \overset{\text{via}(*),(**),(***) \text{ and } (****)}{=}$$

$$2\left((4-2t)((4-2t)^2-3t^2) - t((4-2t)^2-2t^2) + t^2(4-2t) - t^3\right) + 2(4-3t)$$

$$\overset{?}{\geq} (4-2t)^2 - 2t^2 + t^4 + 1 \Leftrightarrow -t^4 - 14t^3 + 110t^2 - 214t + 119 \overset{?}{\geq} 0$$

$$\Leftrightarrow (1-t)(t^3 + 15t^2 + 95(1-t) + 24) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because 2 = a + b \overset{\text{AM-GM}}{\geq} 2\sqrt{t} \Rightarrow 0 < t \leq 1 \therefore \frac{a^3}{b^4+1} + \frac{b^3}{a^4+1} \overset{(1)}{\geq} 1$$

$$\text{Now, } \sum_{cyc}\left(\frac{a^3}{b^4+1}\right)^n \overset{\text{Holder}}{\geq} \frac{1}{2^{n-1}} \cdot \left(\frac{a^3}{b^4+1} + \frac{b^3}{a^4+1}\right)^n \overset{\text{via } (1)}{\geq} \frac{1}{2^{n-1}}$$

$$(\because n \in \mathbb{N} \Rightarrow n \geq 0) \text{ and so, } \sum_{cyc}\left(\frac{a^3}{b^4+1}\right)^n \geq \frac{1}{2^{n-1}} \forall\, a, b > 0 \,\middle|\, a + b = 2 \text{ and } n \in \mathbb{N},$$

$$'' = '' \text{ iff } a = b = 1 \text{ (QED)}$$