ROMANIAN MATHEMATICAL MAGAZINE

$$\text{If } x,y,z>0, \sum_{cyc} xy = 1 \text{ and } \lambda \ge 6 \text{ then :}$$

$$\sum_{cyc} \frac{1}{\lambda x^2+1} \ge \frac{9}{\lambda+3}$$

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$$\sum_{cyc} \frac{1}{\lambda x^2+1} = \sum_{cyc} \frac{\frac{1}{x^2}}{\lambda+\frac{1}{x^2}} \overset{\text{Bergstrom}}{\ge} \frac{\left(\sum_{cyc}\frac{1}{x}\right)^2}{3\lambda+\sum_{cyc}\frac{1}{x^2}} \overset{?}{\ge} \frac{9}{\lambda+3}$$

$$\Leftrightarrow (\lambda+3).\sum_{cyc}\frac{1}{x^2} + 2(\lambda+3).\sum_{cyc}\frac{1}{xy} \overset{?}{\ge} 27\lambda + 9\sum_{cyc}\frac{1}{x^2}$$

$$\Leftrightarrow (\lambda-6).\sum_{cyc}\frac{1}{x^2} + 2(\lambda+3).\sum_{cyc}\frac{1}{xy} \underbrace{\overset{?}{\ge}}_{(*)} 27\lambda$$

$$\text{Now, LHS of } (*) \overset{\text{Bergstrom}}{\ge} (\lambda-6).\sum_{cyc}\frac{1}{x^2} + 2(\lambda+3).\frac{9}{\sum_{cyc} xy}$$

$$= (\lambda-6).\sum_{cyc}\frac{1}{x^2} + 18(\lambda+3) \left(\because \sum_{cyc} xy = 1\right)$$

$$\ge (\lambda-6).\sum_{cyc}\frac{1}{xy} + 18(\lambda+3)\ (\because \lambda \ge 6) \overset{\text{Bergstrom}}{\ge} (\lambda-6).\frac{9}{\sum_{cyc} xy} + 18(\lambda+3)$$

$$= 9(\lambda-6) + 18(\lambda+3) \left(\because \sum_{cyc} xy = 1\right) = 27\lambda \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{1}{\lambda x^2+1} \ge \frac{9}{\lambda+3}\ \forall\, x,y,z>0 \,\Big|\, \sum_{cyc} xy = 1 \text{ and } \lambda \ge 6,$$

$$'' = ''\text{ iff } x=y=z=\frac{1}{\sqrt{3}}\ (\text{QED})$$