ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b > 0$ and $n \in \mathbb{N}$ such that $a^{3n} + b^{3n} = 2$. Prove that:

$$(a^{2n} + b^{2n} + 1)(a^{6n} + b^{6n} + 1) \geq \frac{27}{a^n + b^n + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Adgozel Adgozalov-Azerbaijan

$$(a^{2n} + b^{2n} + 1)(a^{6n} + b^{6n} + 1) \geq_{(CS)} (a^{4n} + b^{4n} + 1)^2 \geq_{(CS)} [(a^{3n} + b^{3n} + 1)^2/(a^{2n} + b^{2n} + 1)]^2$$

$$= 81/(a^{2n} + b^{2n} + 1)^2 \geq_{(AM\text{-}GM)} 81/9 = 9 \geq 27/(a^n + b^n + 1)$$

Note: By AM-GM inequality:

$$2a^{3n} + 1 \geq 3a^{2n}, \quad 2b^{3n} + 1 \geq 3b^{2n} \Rightarrow 2(a^{3n} + b^{3n}) + 2 \geq 3(a^{2n} + b^{2n}) \Rightarrow a^{2n} + b^{2n} + 1 \leq 3$$

$$a^{3n} + 2 \geq 3a^n, \quad b^{3n} + 2 \geq 3b^n \Rightarrow a^{3n} + b^{3n} + 4 \geq 3(a^n + b^n) \Rightarrow a^n + b^n + 1 \leq 3$$

$$\Rightarrow 81/(a^{2n} + b^{2n} + 1)^2 \geq 81/3^2 = 9 \quad \text{and} \quad 27/(a^n + b^n + 1) \leq 27/3 = 9$$