ROMANIAN MATHEMATICAL MAGAZINE

If $x, y, z > 0, \sum_{cyc} xy = 3$ and $\frac{9}{4} \leq \lambda \leq 6$ then :

$$\sum_{cyc} x^3 + \lambda xyz \geq \lambda + 3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc} x\right)^2 \geq 3\sum_{cyc} xy = 9 \Rightarrow \sum_{cyc} x \geq 3 \Rightarrow \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \geq 9$$

$$\Rightarrow \sum_{cyc} x^3 + \lambda xyz - (\lambda + 3) \geq \sum_{cyc} x^3 + \lambda xyz - \left(\frac{\lambda + 3}{9}\right)\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)$$

$$= \sum_{cyc} x^3 - \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - \frac{\lambda}{9}\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right)$$

$$\overset{\lambda \leq 6}{\geq} \sum_{cyc} x^3 - \frac{1}{3}\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - \frac{6}{9}\left(\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) - 9xyz\right)$$

$$\left(\because \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) \overset{AM-GM}{\geq} 9xyz\right)$$

$$= \sum_{cyc} x^3 - \left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right) + 6xyz = \sum_{cyc} x^3 + 3xyz - \sum_{cyc} x^2y - \sum_{cyc} xy^2 \overset{Schur}{\geq} 0$$

$$\therefore \sum_{cyc} x^3 + \lambda xyz \geq \lambda + 3 \ \forall \ x, y, z > 0 \ \middle| \ \sum_{cyc} xy = 3 \text{ and } \frac{9}{4} \leq \lambda \leq 6,$$

$$'' ='' \text{ iff } x = y = z = 1 \ (\text{QED})$$