ROMANIAN MATHEMATICAL MAGAZINE

Let a, b, c be positive real numbers such that $ab + bc + ca = 1$ prove that

$$\sqrt{\frac{1}{a^2}+1}+\sqrt{\frac{1}{b^2}+1}+\sqrt{\frac{1}{c^2}+1}\geq 6$$

Proposed by Jenish Rijal-Bardiya-Nepal

Solution by Sahel Koley-Tarakeswar-India

For any $a, b, c > 0$ with $ab + bc + ca = 1$, we can find a triangle ABC such that

$$a=\tan\frac{A}{2},\quad b=\tan\frac{B}{2},\quad c=\tan\frac{C}{2}$$

and also $A + B + C = \pi$. Now, we can write that:

$$\sqrt{\frac{1}{a^2}+1}=\sqrt{\cot^2\frac{A}{2}+1}=\sqrt{\csc^2\frac{A}{2}}=\csc\frac{A}{2}$$

as $0<\frac{A}{2}<\frac{\pi}{2}$, so, we have to prove for $A + B + C = \pi$ that

$$\csc\left(\frac{A}{2}\right)+\csc\left(\frac{B}{2}\right)+\csc\left(\frac{C}{2}\right)\geq 6$$

Let, $f(x) = \csc x$, $\quad 0 < x < \frac{\pi}{2}$, $\quad f'(x) = -\csc x \cot x$, f''(x)= cscx (cot²x + csc²x)>0

so f is strictly convex on $\left(0,\frac{\pi}{2}\right)$. Therefore, by Jensen inequality, we get

$$f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)\geq 3f\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)$$

$$\Rightarrow \sum\csc\left(\frac{A}{2}\right)\geq 3\csc\left(\frac{A+B+C}{6}\right)\Rightarrow \sum\csc\left(\frac{A}{2}\right)\geq 3\csc\left(\frac{\pi}{6}\right)=3.2=6$$

$$\sqrt{\frac{1}{a^2}+1}+\sqrt{\frac{1}{b^2}+1}+\sqrt{\frac{1}{c^2}+1}\geq 6$$

Equality occurs for $A=B=C=\frac{\pi}{6}$, i.e. $a=b=c=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$