

ROMANIAN MATHEMATICAL MAGAZINE

If $a, b, c > 0$ & $abc = 1$ then prove that :

$$\sum_{cyc} \sqrt{\frac{a^4 + b^4 + a^2b^2 + 1}{2(a^2 + b^2)}} + \sqrt[4]{\frac{(a^3 + b^3 + c^3)^5}{9(a+b+c)(a^2 + b^2 + c^2)^2}} + \prod_{cyc} \frac{a^2 + b^2 + 2}{a + b + 2} \geq 5$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Tapas Das-India

$$2a^4 + a^2b^2 + 1 \stackrel{AM-GM}{\geq} 4a^{\frac{5}{2}}b^{\frac{1}{2}} = 4a^2\sqrt{ab} \text{ \& } 2b^4 + a^2b^2 + 1 \stackrel{AM-GM}{\geq} 4b^{\frac{5}{2}}a^{\frac{1}{2}} = 4b^2\sqrt{ab}$$

$$(2a^4 + a^2b^2 + 1) + (2b^4 + a^2b^2 + 1) \geq 4\sqrt{ab}(a^2 + b^2)$$

$$\frac{a^4 + b^4 + a^2b^2 + 1}{2(a^2 + b^2)} \geq \sqrt{ab}$$

$$\begin{aligned} (a^3 + b^3 + c^3) &\stackrel{Chebyshev}{\geq} \frac{1}{3}(a^2 + b^2 + c^2)(a + b + c) \\ \sqrt[4]{\frac{(a^3 + b^3 + c^3)^5}{9(a+b+c)(a^2 + b^2 + c^2)^2}} &\geq \sqrt[4]{\frac{(a^2 + b^2 + c^2)^3(a + b + c)^4}{9 \times 3^5}} \stackrel{AM-GM}{\geq} \\ &\geq \sqrt[4]{\frac{\left(3(abc)^{\frac{2}{3}}\right)^3 \cdot \left(3(abc)^{\frac{1}{3}}\right)^4}{9 \times 3^5}} \stackrel{abc=1}{=} 1 \\ a^2 + b^2 + 2 &= \frac{a^2}{1} + \frac{b^2}{1} + \frac{1^2}{1} + \frac{1^2}{1} \stackrel{Radon}{\geq} \frac{1}{4}(a + b + 2)^2 \end{aligned}$$

$$\begin{aligned} \prod_{cyc} \frac{a^2 + b^2 + 2}{a + b + 2} &\geq \frac{1}{64} \prod_{cyc} (a + b + 2) = \frac{1}{64} \prod_{cyc} (a + b + 1 + 1) \stackrel{AM-GM}{\geq} \\ &\geq \frac{1}{64} \prod_{cyc} 4(a \cdot b \cdot 1 \cdot 1)^{\frac{1}{4}} = \frac{1}{64} (a^2b^2c^2)^{\frac{1}{4}} \times 64 \stackrel{abc=1}{=} 1 \end{aligned}$$

$$L.H.S \geq 3 + 1 + 1 = 5$$

Equality holds for $a=b=c=1$.