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If $a,b,c>0$ are such that $abc=1$ then prove that:

$$\frac{a^{15}}{(b^2+c^2)\sqrt{c^3+c+1}}+\frac{b^{15}}{(c^2+a^2)\sqrt{a^3+a+1}}+\frac{c^{15}}{(a^2+b^2)\sqrt{b^3+b+1}}\ge\frac{\sqrt{3}}{2}$$

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Solution by Jenish Rijal – Bardiya – Nepal:

Here, $\sqrt{3(c^3+c+1)}\overset{AM-GM}{\le}\frac{3+c^3+c+1}{2}\quad\therefore\sqrt{c^3+c+1}\le\frac{c^3+c+4}{2\sqrt{3}}$

So, $\sum_{cyc}\frac{a^{15}}{(b^2+c^2)\sqrt{c^3+c+1}}\ge\sum_{cyc}\frac{a^{15}}{(b^2+c^2)(c^3+c+4)}\cdot 2\sqrt{3}$

Hence, it suffices to prove that: $\sum_{cyc}\frac{a^{15}}{(b^2+c^2)(c^3+c+4)}\ge\frac{1}{4}$

Lemma: If $a,b,c>0$ such that $abc=1$ then $\sum a^p\ge\sum a^q$ *if* $p\ge q\ge 0$.

Proof: W.L.O.G. assume that

$a\ge b\ge c$. The sequences (a^q,b^q,c^q) and $(a^{p-q},b^{p-q},c^{p-q})$ are similarly sorted.

Hence, by Chebyshev's inequality,

$$\sum a^p=\sum a^q\cdot a^{p-q}\ge\frac{1}{3}\left(\sum a^q\right)\left(\sum a^{p-q}\right)\overset{AM-GM}{\ge}\frac{1}{3}\left(\sum a^q\right)\cdot 3\cdot\left(\sqrt[3]{(abc)^{p-q}}\right)=$$

$$=\left(\sum a^q\right)\cdot\left(\sqrt[3]{(1)^{p-q}}\right)=\sum a^q$$

Therefore, by the lemma, we have:

$$\sum a^5\ge\sum a^3\ge\sum a^2\ge\sum a\overset{AM-GM}{\ge}3\sqrt[3]{abc}=3$$

Now, $\frac{a^{15}}{(b^2+c^2)(c^3+c+4)}+\frac{b^2+c^2}{24}+\frac{c^3+c+4}{72}\overset{AM-GM}{\ge}3\sqrt[3]{\frac{a^{15}}{1728}}=3\cdot\frac{a^5}{12}=\frac{a^5}{4}$

$$\therefore\sum_{cyc}\frac{a^{15}}{(b^2+c^2)(c^3+c+4)}+\sum_{cyc}\frac{b^2+c^2}{24}+\sum_{cyc}\frac{c^3+c+4}{72}\ge\sum_{cyc}\frac{a^5}{4}$$

$$\Leftrightarrow\sum_{cyc}\frac{a^{15}}{(b^2+c^2)(c^3+c+4)}+\frac{\sum a^2}{12}+\frac{\sum a^3+\sum a+12}{72}\ge\sum_{cyc}\frac{a^5}{4}$$

$$\Leftrightarrow\sum_{cyc}\frac{a^{15}}{(b^2+c^2)(c^3+c+4)}\ge\frac{\sum a^5}{4}-\frac{\sum a^3}{72}-\frac{\sum a^2}{12}-\frac{\sum a}{72}-\frac{12}{72}\overset{Lemma}{\ge}$$

$$\ge\frac{\sum a^5}{4}-\frac{\sum a^5}{72}-\frac{\sum a^5}{12}-\frac{\sum a^5}{72}-\frac{1}{6}\ge\frac{5}{36}\cdot\left(\sum a^5\right)-\frac{1}{6}\ge\frac{5}{36}\cdot(3)-\frac{1}{6}=\frac{1}{4}$$

Equality holds if and only if $a=b=c=1$.