ROMANIAN MATHEMATICAL MAGAZINE

Let $a, \mathrm{b}, \mathrm{c} > 0$ *such that* $\mathrm{abc} = 1$ then prove that :

$$\frac{a^{15}}{(\mathrm{b}^2+\mathrm{c}^2)\sqrt{\mathrm{c}^3+\mathrm{c}+1}}+\frac{\mathrm{b}^{15}}{(\mathrm{c}^2+a^2)\sqrt{a^3+a+1}}+\frac{\mathrm{c}^{15}}{(a^2+\mathrm{b}^2)\sqrt{\mathrm{b}^3+\mathrm{b}+1}}\ge\frac{3}{2\sqrt{3}}$$

Proposed by Hikmat Mammadov-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\mathrm{LHS}=\sum_{\mathrm{cyc}}\frac{a^{16}}{a(\mathrm{b}^2+\mathrm{c}^2).\sqrt{\mathrm{c}^3+\mathrm{c}+1}}\overset{\mathrm{Bergstrom}}{\ge}\frac{\left(\sum_{\mathrm{cyc}}a^8\right)^2}{\sum_{\mathrm{cyc}}\left(a(\mathrm{b}^2+\mathrm{c}^2).\sqrt{\mathrm{c}^3+\mathrm{c}+1}\right)}$$

$$\overset{\mathrm{CBS}}{\ge}\frac{\left(\sum_{\mathrm{cyc}}a^8\right)^2}{\sqrt{\sum_{\mathrm{cyc}}\left(a(\mathrm{b}^2+\mathrm{c}^2)(\mathrm{c}^3+\mathrm{c}+1)\right)}.\sqrt{\sum_{\mathrm{cyc}}\left(a(\mathrm{b}^2+\mathrm{c}^2)\right)}}\therefore \mathrm{LHS}\ge$$

$$\frac{\left(\sum_{\mathrm{cyc}}a^8\right)^2}{\sqrt{\mathrm{P}}.\sqrt{\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}ab\right)-3abc}}\to(*);\ \mathrm{P}=\sum_{\mathrm{cyc}}\left(a\left(\sum_{\mathrm{cyc}}a^2-a^2\right)(\mathrm{c}^3+\mathrm{c}+1)\right)$$

$$\overset{\mathrm{abc}=1}{=}\left(\sum_{\mathrm{cyc}}a^2\right)\left(\sum_{\mathrm{cyc}}a^3\mathrm{b}+\sum_{\mathrm{cyc}}ab\right)+\sum_{\mathrm{cyc}}a^3+\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}ab\right)-$$

$$\left(3a^2\mathrm{b}^2\mathrm{c}^2+\sum_{\mathrm{cyc}}a^3\mathrm{b}^3\right)-\sum_{\mathrm{cyc}}ab^3-\sum_{\mathrm{cyc}}a^3\overset{\mathrm{Schur}}{\le}\left(\sum_{\mathrm{cyc}}a^2\right)\left(\sum_{\mathrm{cyc}}a^3\mathrm{b}+\sum_{\mathrm{cyc}}ab\right)+$$

$$\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}ab\right)-\left(abc\left(\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}ab\right)-3abc\right)\right)-\sum_{\mathrm{cyc}}ab^3\overset{\mathrm{AM-GM}}{\le}$$

$$\left(\sum_{\mathrm{cyc}}a^2\right)\left(\sum_{\mathrm{cyc}}a^3\mathrm{b}+\sum_{\mathrm{cyc}}ab\right)+\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}ab\right)-\left(\sum_{\mathrm{cyc}}a\right)\left(\sum_{\mathrm{cyc}}ab\right)+3-$$

$$3.\sqrt[3]{a^4\mathrm{b}^4\mathrm{c}^4}\ (\because abc\ =\ 1)\overset{\mathrm{Vasc}}{\le}\frac{1}{3}\left(\sum_{\mathrm{cyc}}a^2\right)^3+\left(\sum_{\mathrm{cyc}}a^2\right)^2+3-3\ (\because abc\ =\ 1)$$

$$\le\frac{1}{3}\left(\sum_{\mathrm{cyc}}a^2\right).3\sum_{\mathrm{cyc}}a^4+3\sum_{\mathrm{cyc}}a^4\therefore \mathrm{P}\le\left(\sum_{\mathrm{cyc}}a^2\right)\left(\sum_{\mathrm{cyc}}a^4\right)+3\sum_{\mathrm{cyc}}a^4\to(**)$$

$$\sum_{\mathrm{cyc}}a^4\ge\frac{\left(\sum_{\mathrm{cyc}}a^2\right)^2}{3}\overset{\mathrm{AM-GM}}{\ge}\frac{1}{3}.3\sqrt[3]{a^2\mathrm{b}^2\mathrm{c}^2}.\sum_{\mathrm{cyc}}a^2\overset{abc=1}{=}\sum_{\mathrm{cyc}}a^2\therefore\sum_{\mathrm{cyc}}a^4\ge\sum_{\mathrm{cyc}}a^2$$

$\rightarrow (***)$; Again, $\sum_{cyc} a^4 \geq \sum_{cyc} a^2 \geq \sum_{cyc} ab \therefore \sum_{cyc} a^4 \geq \sum_{cyc} ab \rightarrow (****)$ and $\sum_{cyc} a^4 \geq$

$\sum_{cyc} a^2 \geq \frac{1}{3}\left(\sum_{cyc} a\right)^2 \overset{AM-GM}{\geq} \frac{1}{3}.3\sqrt[3]{abc}.\sum_{cyc} a \overset{abc=1}{=} \sum_{cyc} a \therefore \sum_{cyc} a^4 \geq \sum_{cyc} a \rightarrow (*****)$

$$\sum_{cyc} a^8 \geq \frac{1}{3}\left(\sum_{cyc} a^4\right)^2 \overset{?}{\geq} \frac{3}{\sqrt{6}}.\sqrt{\left(\sum_{cyc} a^4\right)^2 - 3} \Leftrightarrow 2t^4 \overset{?}{\geq} 27(t^2-3)\ \left(t = \sum_{cyc} a^4\right)$$

$$\Leftrightarrow (t^2-9)(2t^2-9) \overset{?}{\geq} 0 \rightarrow \text{true} \because t \overset{AM-GM}{\geq} 3.\sqrt[3]{a^4b^4c^4} \overset{abc=1}{=} 3 \Rightarrow t^2 \geq 9$$

$$\therefore \sum_{cyc} a^8 \geq \frac{3}{\sqrt{6}}.\sqrt{\left(\sum_{cyc} a^4\right)^2 - 3} \overset{\text{via } (****) \text{ and } (*****)}{\geq} \frac{3}{\sqrt{6}}.\sqrt{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3}$$

and $\because abc = 1 \therefore \sum_{cyc} a^8 \overset{(\blacksquare)}{\geq} \frac{3}{\sqrt{6}}.\sqrt{\left(\sum_{cyc} a\right)\left(\sum_{cyc} ab\right) - 3abc}$ and finally,

$$\sum_{cyc} a^8 \geq \frac{1}{3}\left(\sum_{cyc} a^4\right)^2 \overset{?}{\geq} \frac{1}{\sqrt{2}}.\sqrt{\left(\sum_{cyc} a^4\right)^2 + 3\sum_{cyc} a^4} \Leftrightarrow 2t^4 \overset{?}{\geq} 9(t^2+3t)$$

$$\Leftrightarrow t(t-3)(2t^2+6t+9) \overset{?}{\geq} 0 \rightarrow \text{true} \therefore \sum_{cyc} a^8 \geq \frac{1}{\sqrt{2}}.\sqrt{\left(\sum_{cyc} a^4\right)^2 + 3\sum_{cyc} a^4}$$

$$\overset{\text{via } (***)}{\geq} \frac{1}{\sqrt{2}}.\sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^4\right) + 3\sum_{cyc} a^4} \overset{\text{via } (**)}{\geq} \frac{1}{\sqrt{2}}.\sqrt{P} \therefore \sum_{cyc} a^8 \overset{(\blacksquare\blacksquare)}{\geq} \frac{1}{\sqrt{2}}.\sqrt{P}$$

$$\therefore (\blacksquare), (\blacksquare\blacksquare), (*) \Rightarrow \sum_{cyc} \frac{a^{15}}{(b^2+c^2).\sqrt{c^3+c+1}} \geq \frac{3}{2\sqrt{3}}\ \forall\, a, b, c > 0 \,|\, abc = 1,$$

$'' = ''$ iff $a = b = c = 1$ (QED)