ROMANIAN MATHEMATICAL MAGAZINE

Let $a, \mathrm{b}, \mathrm{c} > 0$ *such that* $\mathrm{abc} = 1$; then prove that :

$$\frac{a^7}{(\mathrm{b}^2+\mathrm{c}^2)(\mathrm{c}^3+\mathrm{c}+1)}+\frac{\mathrm{b}^7}{(\mathrm{c}^2+a^2)(a^3+a+1)}+\frac{\mathrm{c}^7}{(a^2+\mathrm{b}^2)(\mathrm{b}^3+\mathrm{b}+1)}\geq\frac{1}{2}$$

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$$\sum_{\text{cyc}}\frac{a^7}{(\mathrm{b}^2+\mathrm{c}^2)(\mathrm{c}^3+\mathrm{c}+1)}=\sum_{\text{cyc}}\frac{a^8}{a(\mathrm{b}^2+\mathrm{c}^2)(\mathrm{c}^3+\mathrm{c}+1)}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{\text{cyc}}a^4\right)^2}{\mathrm{P}}$$

$$\rightarrow(*);\mathrm{P}=\sum_{\text{cyc}}\left(a(\mathrm{b}^2+\mathrm{c}^2)(\mathrm{c}^3+\mathrm{c}+1)\right)=\sum_{\text{cyc}}\left(a\left(\sum_{\text{cyc}}a^2-a^2\right)(\mathrm{c}^3+\mathrm{c}+1)\right)$$

$$=\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a^3\mathrm{b}+\sum_{\text{cyc}}a\mathrm{b}+\sum_{\text{cyc}}a\right)-\sum_{\text{cyc}}a^3\mathrm{b}^3-\sum_{\text{cyc}}a\mathrm{b}^3-\sum_{\text{cyc}}a^3$$

$$=\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a^3\mathrm{b}+\sum_{\text{cyc}}a\mathrm{b}\right)+\sum_{\text{cyc}}a^3+\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}a\mathrm{b}\right)-$$

$$\left(3a\mathrm{bc}+\sum_{\text{cyc}}a^3\mathrm{b}^3\right)-\sum_{\text{cyc}}a\mathrm{b}^3-\sum_{\text{cyc}}a^3\overset{a\mathrm{bc}=1}{=}$$

$$\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a^3\mathrm{b}+\sum_{\text{cyc}}a\mathrm{b}\right)+\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}a\mathrm{b}\right)-$$

$$\left(3a^2\mathrm{b}^2\mathrm{c}^2+\sum_{\text{cyc}}a^3\mathrm{b}^3\right)-\sum_{\text{cyc}}a\mathrm{b}^3\overset{\text{Schur}}{\leq}\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a^3\mathrm{b}+\sum_{\text{cyc}}a\mathrm{b}\right)+$$

$$\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}a\mathrm{b}\right)-\left(a\mathrm{bc}\left(\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}a\mathrm{b}\right)-3a\mathrm{bc}\right)\right)-\sum_{\text{cyc}}a\mathrm{b}^3\overset{\text{AM-GM}}{\leq}$$

$$\left(\sum_{\text{cyc}}a^2\right)\left(\sum_{\text{cyc}}a^3\mathrm{b}+\sum_{\text{cyc}}a\mathrm{b}\right)+\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}a\mathrm{b}\right)-\left(\sum_{\text{cyc}}a\right)\left(\sum_{\text{cyc}}a\mathrm{b}\right)+3-$$

$$3.\sqrt[3]{a^4\mathrm{b}^4\mathrm{c}^4}\ (\because a\mathrm{bc}\ =\ 1)\overset{\text{Vasc}}{\leq}\frac{1}{3}\left(\sum_{\text{cyc}}a^2\right)^3+\left(\sum_{\text{cyc}}a^2\right)^2+3-3\ (\because a\mathrm{bc}\ =\ 1)$$

$$\le \frac{1}{3}\left(\sum_{cyc} a^2\right).3\sum_{cyc} a^4 + 3\sum_{cyc} a^4 \therefore P \le \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^4\right) + 3\sum_{cyc} a^4 \to (**)$$

$$\text{Now,}\left(\sum_{cyc} a^4\right)^2 \ge \frac{1}{3}\left(\sum_{cyc} a^2\right)^2\left(\sum_{cyc} a^4\right) \overset{\text{AM-GM}}{\ge} \frac{1}{3}.3\sqrt[3]{a^2b^2c^2}.\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^4\right)$$

$$\overset{abc=1}{=}\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^4\right) \therefore \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^4\right) \overset{1}{\le} \left(\sum_{cyc} a^4\right)^2 \text{ and again,}$$

$$\left(\sum_{cyc} a^4\right)^2 \ge \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^4\right) \overset{\text{AM-GM}}{\ge} 3\sqrt[3]{a^2b^2c^2}.\left(\sum_{cyc} a^4\right) \overset{abc=1}{=} 3\sum_{cyc} a^4$$

$$\therefore 3\sum_{cyc} a^4 \overset{2}{\le} \left(\sum_{cyc} a^4\right)^2 \therefore 1, 2 \text{ and } (**) \Rightarrow P \le 2\left(\sum_{cyc} a^4\right)^2 \to (***)$$

$$\therefore (*) \text{ and } (***) \Rightarrow \sum_{cyc} \frac{a^7}{(b^2+c^2)(c^3+c+1)} \ge \frac{1}{2} \text{ and so,}$$

$$\frac{a^7}{(b^2+c^2)(c^3+c+1)} + \frac{b^7}{(c^2+a^2)(a^3+a+1)} + \frac{c^7}{(a^2+b^2)(b^3+b+1)} \ge \frac{1}{2}$$

$$\forall\, a,b,c > 0 \,\middle|\, abc = 1, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$