ROMANIAN MATHEMATICAL MAGAZINE

Let $a, b, c > 0$ *such that* $abc = 1$; then prove that :

$$\frac{a^6}{(b+c)(c^2+c+1)} + \frac{b^6}{(c+a)(a^2+a+1)} + \frac{c^6}{(a+b)(b^2+b+1)} \ge \frac{1}{2}$$

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$$\sum_{cyc} \frac{a^6}{(b+c)(c^2+c+1)} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{\left(\sum_{cyc} a^3\right)^2}{\sum_{cyc} ab^2 + \sum_{cyc} ab + 2\sum_{cyc} a + \sum_{cyc} a^3 + \sum_{cyc} a^2} \to (*)$$

$$a^3 + b^3 + b^3 \overset{\text{AM-GM}}{\ge} 3ab^2; \; b^3 + c^3 + c^3 \overset{\text{AM-GM}}{\ge} 3bc^2; \; c^3 + a^3 + a^3 \overset{\text{AM-GM}}{\ge} 3ca^2$$

$$\text{and via summation, } 3\sum_{cyc} a^3 \ge 3\sum_{cyc} ab^2 \text{ and } \because \sum_{cyc} a^3 \underbrace{\overset{\text{AM-GM}}{\ge}}_{(**)} 3abc = 3$$

$$\therefore \left(\sum_{cyc} a^3\right)^2 \ge 3\sum_{cyc} ab^2 \Rightarrow \sum_{cyc} ab^2 \boxed{\overset{①}{\le}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\text{Again, } \left(\sum_{cyc} a^3\right)^2 \overset{\text{via } (**)}{\ge} 3\sum_{cyc} a^3 \overset{\text{Chebyshev}}{\ge} \left(\sum_{cyc} a\right)\left(\sum_{cyc} a^2\right) \overset{\text{AM-GM}}{\ge} \left(3.\sqrt[3]{abc}\right)\left(\sum_{cyc} a^2\right)$$

$$\overset{abc=1}{=} 3\sum_{cyc} a^2 \Rightarrow \sum_{cyc} a^2 \boxed{\overset{②}{\le}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \text{ and } \because \sum_{cyc} ab \le \sum_{cyc} a^2$$

$$\therefore \text{ via } ②, \sum_{cyc} ab \boxed{\overset{③}{\le}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\text{Moreover, } \left(\sum_{cyc} a^3\right)^2 \overset{\text{via } (**)}{\ge} 3\sum_{cyc} a^3 \overset{\text{Holder}}{\ge} \frac{1}{3}\left(\sum_{cyc} a\right)^3 \overset{\text{AM-GM}}{\ge} \frac{1}{3}.\left(3.\sqrt[3]{abc}\right)^2.\left(\sum_{cyc} a\right)$$

$$\overset{abc=1}{=} 3\left(\sum_{cyc} a\right) \Rightarrow \sum_{cyc} a \boxed{\overset{④}{\le}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \text{ and finally, } \left(\sum_{cyc} a^3\right)^2 \overset{\text{via } (**)}{\ge} 3\sum_{cyc} a^3$$

$$\Rightarrow \sum_{cyc} a^3 \boxed{\overset{⑤}{\le}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \; \therefore ①, ②, ③, ④, ⑤, (*) \Rightarrow$$

$$\sum_{cyc} \frac{a^6}{(b+c)(c^2+c+1)} \ge \frac{\left(\sum_{cyc} a^3\right)^2}{\left(\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{1}{3}+\frac{1}{3}\right)\left(\sum_{cyc} a^3\right)^2} = \frac{1}{2} \text{ and so,}$$

$$\frac{a^6}{(b+c)(c^2+c+1)}+\frac{b^6}{(c+a)(a^2+a+1)}+\frac{c^6}{(a+b)(b^2+b+1)}\geq\frac{1}{2}$$
$$\forall\ a,b,c>0\,\Big|\,abc=1,''=''\text{ iff }\ a=b=c=1\ (\text{QED})$$