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Let $a,b,c>0$ *such that* $abc=1$; then prove that :

$$\frac{a^5}{(b+1)(c^2+c+1)}+\frac{b^5}{(c+1)(a^2+a+1)}+\frac{c^5}{(a+1)(b^2+b+1)}\ge\frac{1}{2}$$

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$$\sum_{cyc}\frac{a^5}{(b+1)(c^2+c+1)}=\sum_{cyc}\frac{a^6}{a(b+1)(c^2+c+1)}\overset{\text{Bergstrom}}{\ge}$$

$$\frac{\left(\sum_{cyc}a^3\right)^2}{abc\sum_{cyc}a+3abc+2\sum_{cyc}ab+\sum_{cyc}a^2b+\sum_{cyc}a}$$

$$\therefore \text{LHS}\overset{(*)}{\ge}\frac{\left(\sum_{cyc}a^3\right)^2}{abc\sum_{cyc}a+3abc+2\sum_{cyc}ab+\sum_{cyc}a^2b+\sum_{cyc}a}$$

$$\text{Now, }\left(\sum_{cyc}a^3\right)^2\overset{\text{AM-GM}}{\ge}3abc\left(\sum_{cyc}a^3\right)\overset{\text{Holder}}{\ge}\frac{abc}{3}\left(\sum_{cyc}a\right)^3\overset{\text{AM-GM}}{\ge}\frac{abc}{3}.\left(3.\sqrt[3]{abc}\right)^2.\left(\sum_{cyc}a\right)$$

$$\overset{abc=1}{=}3abc\left(\sum_{cyc}a\right)\Rightarrow abc\sum_{cyc}a\ \boxed{\overset{(1)}{\le}}\ \frac{1}{3}\left(\sum_{cyc}a^3\right)^2\ \text{and again,}$$

$$\left(\sum_{cyc}a^3\right)^2\overset{\text{AM-GM}}{\ge}9a^2b^2c^2\overset{abc=1}{=}9abc\Rightarrow 3abc\ \boxed{\overset{(2)}{\le}}\ \frac{1}{3}\left(\sum_{cyc}a^3\right)^2$$

$$\text{Also, }\left(\sum_{cyc}a^3\right)^2\overset{\text{AM-GM}}{\ge}3abc\sum_{cyc}a^3\overset{abc=1}{=}3\sum_{cyc}a^3\overset{\text{Chebyshev}}{\ge}\left(\sum_{cyc}a\right)\left(\sum_{cyc}a^2\right)\ge$$

$$\left(\sum_{cyc}a\right)\left(\sum_{cyc}ab\right)\overset{\text{AM-GM}}{\ge}\left(3.\sqrt[3]{abc}\right)\left(\sum_{cyc}ab\right)\overset{abc=1}{=}3\sum_{cyc}ab$$

$$\Rightarrow\sum_{cyc}ab\ \boxed{\overset{(3)}{\le}}\ \frac{1}{3}\left(\sum_{cyc}a^3\right)^2$$

$$\text{Moreover, }a^3+a^3+b^3\overset{\text{AM-GM}}{\ge}3a^2b;\ b^3+b^3+c^3\overset{\text{AM-GM}}{\ge}3b^2c;$$

$$c^3+c^3+a^3\overset{\text{AM-GM}}{\ge}3c^2a\ \text{and via summation, }3\sum_{cyc}a^3\ge3\sum_{cyc}a^2b\ \text{and}\ \because$$

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$$\sum_{cyc} a^3 \overset{AM-GM}{\geq} 3abc = 3 \therefore \left(\sum_{cyc} a^3\right)^2 \geq 3\sum_{cyc} a^2b \Rightarrow \sum_{cyc} a^2b \boxed{\overset{④}{\leq}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\textbf{Finally}, \sum_{cyc} a \overset{abc=1}{=} abc\sum_{cyc} a \overset{\text{via } ①}{\leq} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2 \Rightarrow \sum_{cyc} a \boxed{\overset{⑤}{\leq}} \frac{1}{3}\left(\sum_{cyc} a^3\right)^2$$

$$\therefore ①, ②, ③, ④, ⑤, (*) \Rightarrow \sum_{cyc} \frac{a^5}{(b+1)(c^2+c+1)} \geq \frac{\left(\sum_{cyc} a^3\right)^2}{\left(\frac{1}{3}+\frac{1}{3}+\frac{2}{3}+\frac{1}{3}+\frac{1}{3}\right)\left(\sum_{cyc} a^3\right)^2}$$

$$= \frac{1}{2} \text{ and so}, \frac{a^5}{(b+1)(c^2+c+1)} + \frac{b^5}{(c+1)(a^2+a+1)} + \frac{c^5}{(a+1)(b^2+b+1)} \geq \frac{1}{2}$$

$$\forall a, b, c > 0 \,\big|\, abc = 1, '' = '' \text{ iff } a = b = c = 1 \text{ (QED)}$$