

# ROMANIAN MATHEMATICAL MAGAZINE

*Let  $a, b, c$  be positive real numbers. Prove that*

$$\frac{(a-b)^2}{ab} + \frac{(b-c)^2}{bc} + \frac{(c-a)^2}{ca} \geq 2\sqrt{2} \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{a^2 + b^2 + c^2}$$

*Proposed by Cheang Chantria-Cambodia*

*Solution by Jenish Rijal-Bardiya-Nepal*

*Here, the given inequality can be written as:*

$$\sum_{cyc} (a-b)^2 \left[ \frac{a^2 + b^2 + c^2}{ab} - 2\sqrt{2} \right] \geq 0$$

$$\text{Let } S_c := \frac{a^2 + b^2 + c^2}{ab} - 2\sqrt{2}, S_b := \frac{a^2 + b^2 + c^2}{ca} - 2\sqrt{2}, \text{ and } S_a := \frac{a^2 + b^2 + c^2}{bc} - 2\sqrt{2}$$

*The inequality is then equivalent to:  $(a-b)^2 S_c + (c-a)^2 S_b + (b-c)^2 S_a \geq 0$*

*W. L. O. G. assume that:  $c \geq b \geq a$*

$$* S_c := \frac{a^2 + b^2 + c^2}{ab} - 2\sqrt{2} \stackrel{c \geq b}{\geq} \frac{a^2 + b^2 + b^2}{ab} - 2\sqrt{2} \stackrel{AM-GM}{\geq} \frac{2\sqrt{a^2 \cdot 2b^2}}{ab} - 2\sqrt{2} = 0.$$

$$\therefore S_c > 0$$

$$* S_b := \frac{a^2 + b^2 + c^2}{ca} - 2\sqrt{2} \stackrel{b \geq a}{\geq} \frac{a^2 + a^2 + c^2}{ca} - 2\sqrt{2} \stackrel{AM-GM}{\geq} \frac{2\sqrt{2a^2 \cdot c^2}}{ca} - 2\sqrt{2} = 0.$$

$$\therefore S_b \geq 0$$

$$\begin{aligned} * S_b + S_a &= \frac{a^2 + b^2 + c^2}{ca} - 2\sqrt{2} + \frac{a^2 + b^2 + c^2}{bc} - 2\sqrt{2} = \\ &= \frac{(a+b)(a^2 + b^2 + c^2)}{abc} - 4\sqrt{2} \stackrel{AM-GM}{\geq} \frac{(2\sqrt{ab})(2ab + c^2)}{abc} - 4\sqrt{2} \stackrel{AM-GM}{\geq} \\ &\geq \frac{(2\sqrt{ab})(2\sqrt{2ab \cdot c^2})}{abc} - 4\sqrt{2} = 4\sqrt{2} - 4\sqrt{2} = 0. \quad \therefore S_b + S_a \geq 0 \end{aligned}$$

*We also have:*

$$(c-a)^2 = [(c-b) + (b-a)]^2 \geq (c-b)^2 = (b-c)^2$$

# ROMANIAN MATHEMATICAL MAGAZINE

*Now:*

$$\begin{aligned}
 (a-b)^2 S_c + (c-a)^2 S_b + (b-c)^2 S_a & \stackrel{(a-b)^2 S_c \geq 0}{\geq} (c-a)^2 S_b + (b-c)^2 S_a \stackrel{(c-a)^2 \geq (b-c)^2 \text{ \& } S_b \geq 0}{\geq} \\
 & \geq (b-c)^2 S_b + (b-c)^2 S_a = (b-c)^2 (S_b + S_a) \stackrel{S_b + S_a \geq 0}{\geq} 0
 \end{aligned}$$

*Equality holds for  $\{a = b = c\}$  or  $\left\{a = b = \frac{c}{\sqrt{2}} \text{ and permutations}\right\}$ .*

*Remark:  $S_c > 0$  because the equality is not possible  
in that case due to our assumption  $b \geq a$ .*