ROMANIAN MATHEMATICAL MAGAZINE

Let a,b,c be positive real numbers. Prove that

$$\frac{(a-b)^2}{ab}+\frac{(b-c)^2}{bc}+\frac{(c-a)^2}{ca}\ge 2\sqrt{2}\cdot\frac{(a-b)^2+(b-c)^2+(c-a)^2}{a^2+b^2+c^2}$$

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Here, the given inequality can be written as:

$$\sum_{cyc}(a-b)^2\left[\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}\right]\ge 0$$

$$\text{Let } S_c:=\frac{a^2+b^2+c^2}{ab}-2\sqrt{2},\ S_b:=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2},\text{ and } S_a:=\frac{a^2+b^2+c^2}{bc}-2\sqrt{2}$$

The inequality is then equivalent to: $(a-b)^2S_c+(c-a)^2S_b+(b-c)^2S_a\ge 0$

W.L.O.G. assume that: $c\ge b\ge a$

$$*\ S_c:=\frac{a^2+b^2+c^2}{ab}-2\sqrt{2}\overset{c\ge b}{\ge}\frac{a^2+b^2+b^2}{ab}-2\sqrt{2}\overset{AM-GM}{\ge}\frac{2\sqrt{a^2\cdot 2b^2}}{ab}-2\sqrt{2}=0.\ \therefore S_c>0$$

$$*\ S_b:=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2}\overset{b\ge a}{\ge}\frac{a^2+a^2+c^2}{ca}-2\sqrt{2}\overset{AM-GM}{\ge}\frac{2\sqrt{2a^2\cdot c^2}}{ca}-2\sqrt{2}=0.\ \therefore S_b\ge 0$$

$$*\ S_b+S_a=\frac{a^2+b^2+c^2}{ca}-2\sqrt{2}+\frac{a^2+b^2+c^2}{bc}-2\sqrt{2}=\frac{(a+b)(a^2+b^2+c^2)}{abc}-4\sqrt{2}\overset{AM-GM}{\ge}$$

$$\frac{(2\sqrt{ab})(2ab+c^2)}{abc}-4\sqrt{2}\overset{AM-GM}{\ge}\frac{(2\sqrt{ab})(2\sqrt{2ab\cdot c^2})}{abc}-4\sqrt{2}=4\sqrt{2}-4\sqrt{2}=0.\quad\therefore S_b+S_a\ge 0$$

We also have: $(c-a)^2=[(c-b)+(b-a)]^2\ge(c-b)^2=(b-c)^2$

$$(a-b)^2S_c+(c-a)^2S_b+(b-c)^2S_a\overset{(a-b)^2S_c\ge 0}{\ge}(c-a)^2S_b+(b-c)^2S_a\overset{(c-a)^2\ge(b-c)^2\ \&\ S_b\ge 0}{\ge}$$

$$(b-c)^2S_b+(b-c)^2S_a=(b-c)^2(S_b+S_a)\overset{S_b+S_a\ge 0}{\ge}0$$

Equality holds for $a=b=c$ *or* $a=b=\frac{c}{\sqrt{2}}$ *and permutations.*