Three Original Inequality Problems with Positive Variables

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Abstract

In this short note we present and solve three original inequality problems created by the author. The problems involve positive real variables and are based on classical constraints such as fixed product, ordering conditions and comparison estimates. The first result gives the sharp range of a cyclic sum under two symmetric constraints, while the other two inequalities are proved by elementary methods based on substitutions, monotonicity and direct estimates.

1 A sharp bound for a cyclic sum

Problem 1. *Let* $a, b, c > 0$*,* $abc = 1$ *and* $a + b + c = 5$*. Prove that*

$$\sum_{\text{cyc}} \frac{a^2 + a}{a^2 + 1} \in \left[\frac{202}{85}, \frac{5 + 2\sqrt{2}}{3}\right].$$

Proof. Denote

$$p = ab + bc + ca.$$

We first express the cyclic sum only in terms of p. Since

$$a + b + c = 5, \qquad abc = 1,$$

a direct computation gives

$$\sum_{\text{cyc}} \frac{a^2 + a}{a^2 + 1} = \frac{2p^2 + 4p + 10}{p^2 - 2p + 17}.$$

Indeed, after bringing the three fractions to the same denominator and using $a + b + c = 5$ and $abc = 1$, the numerator becomes

$$2p^2 + 4p + 10,$$

while the denominator becomes

$$p^2 - 2p + 17.$$

It remains to determine the possible range of p. By the standard *uvw* principle, the extrema of $p = ab + bc + ca$ under the constraints

$$a + b + c = 5, \qquad abc = 1$$

are attained when two variables are equal. Let

$$a = b = x, \qquad c = y.$$

Then

$$x^2 y = 1, \qquad 2x + y = 5.$$

Hence

$$y = \frac{1}{x^2},$$

and therefore

$$2x + \frac{1}{x^2} = 5.$$

Equivalently,

$$2x^3 - 5x^2 + 1 = 0.$$

We factor:

$$2x^3 - 5x^2 + 1 = \left(x - \frac{1}{2}\right)(2x^2 - 4x - 2).$$

Thus the positive solutions are

$$x = \frac{1}{2} \qquad \text{and} \qquad x = 1 + \sqrt{2}.$$

For $x = \frac{1}{2}$, we obtain

$$(a, b, c) = \left(\frac{1}{2}, \frac{1}{2}, 4\right),$$

and hence

$$p = ab + bc + ca = \frac{1}{4} + 2 + 2 = \frac{17}{4}.$$

For $x = 1 + \sqrt{2}$, we obtain

$$(a, b, c) = \left(1 + \sqrt{2}, 1 + \sqrt{2}, \frac{1}{(1 + \sqrt{2})^2}\right),$$

and hence

$$p = 1 + 4\sqrt{2}.$$

Therefore

$$\frac{17}{4} \leq p \leq 1 + 4\sqrt{2}.$$

Now define

$$F(p) = \frac{2p^2 + 4p + 10}{p^2 - 2p + 17}.$$

Then

$$F'(p) = -\frac{8(p^2 - 6p - 11)}{(p^2 - 2p + 17)^2}.$$

Since

$$1 + 4\sqrt{2} < 3 + 2\sqrt{5},$$

we have

$$p^2 - 6p - 11 < 0$$

for every

$$p \in \left[\frac{17}{4}, 1 + 4\sqrt{2}\right].$$

Thus F is increasing on this interval. Consequently,

$$F(p) \geq F\left(\frac{17}{4}\right) = \frac{202}{85},$$

and

$$F(p) \leq F(1 + 4\sqrt{2}) = \frac{5 + 2\sqrt{2}}{3}.$$

Therefore

$$\frac{202}{85} \leq \sum_{\text{cyc}} \frac{a^2 + a}{a^2 + 1} \leq \frac{5 + 2\sqrt{2}}{3}.$$

This proves the desired result. ☐

2 An inequality under ordering and product condition

Problem 2. *Let* $0 < a \leq b \leq c$*, such that* $abc = 1$*. Prove that*

$$\frac{8 - c - \frac{1}{c^2}}{a + b} \leq (a + b)^2 + c^2(c + 1 - a - b).$$

Proof. Put

$$x = a + b.$$

Since $abc = 1$, we have

$$ab = \frac{1}{c}.$$

Also, from $0 < a \leq b \leq c$ and $abc = 1$, it follows that $c \geq 1$. By AM-GM,

$$x = a + b \geq 2\sqrt{ab} = \frac{2}{\sqrt{c}}.$$

On the other hand, since $b \leq c$ and $ab = 1/c$, the largest possible value of $a + b$ is obtained for $b = c$, which gives

$$a = \frac{1}{c^2}.$$

Thus

$$x = a + b \leq c + \frac{1}{c^2}.$$

Hence

$$\frac{2}{\sqrt{c}} \leq x \leq c + \frac{1}{c^2}.$$

The desired inequality is equivalent, after multiplying by $x > 0$, to

$$8 - c - \frac{1}{c^2} \leq x\Big(x^2 + c^2(c + 1 - x)\Big).$$

Equivalently,

$$\Phi_c(x) \geq 0,$$

where

$$\Phi_c(x) = x^3 - c^2x^2 + (c^3 + c^2)x - 8 + c + \frac{1}{c^2}.$$

We shall prove that

$$\Phi_c(x) \geq 0$$

for every

$$c \geq 1, \qquad \frac{2}{\sqrt{c}} \leq x \leq c + \frac{1}{c^2}.$$

The derivative is

$$\Phi_c'(x) = 3x^2 - 2c^2x + c^3 + c^2.$$

This is a quadratic polynomial in x. If it has no real roots, then $\Phi_c'(x) > 0$ for every real x, so Φ_c is increasing and its minimum on the interval is attained at the left endpoint. Suppose now that Φ_c' has two real roots. They are

$$r_1 = \frac{c^2 - c\sqrt{c^2 - 3c - 3}}{3}, \qquad r_2 = \frac{c^2 + c\sqrt{c^2 - 3c - 3}}{3}.$$

The roots are real only when

$$c^2 - 3c - 3 \geq 0,$$

which implies

$$c \geq \frac{3 + \sqrt{21}}{2}.$$

For such c, we have

$$r_2 \geq \frac{c^2}{3} > c + \frac{1}{c^2}.$$

Therefore the local minimum of Φ_c lies to the right of the interval

$$\left[\frac{2}{\sqrt{c}}, c + \frac{1}{c^2}\right].$$

Hence the minimum of Φ_c on this interval is attained at one of the two endpoints. It remains to check the endpoints. First,

$$\Phi_c\left(c + \frac{1}{c^2}\right) = \frac{2c^9 - 4c^6 + 3c^3 + 1}{c^6}.$$

Let

$$u = c^3.$$

Since $c \geq 1$, we have $u \geq 1$. Then

$$2c^9 - 4c^6 + 3c^3 + 1 = 2u^3 - 4u^2 + 3u + 1.$$

Now

$$(2u^3 - 4u^2 + 3u + 1)' = 6u^2 - 8u + 3.$$

The discriminant of this quadratic is

$$64 - 72 < 0,$$

so

$$6u^2 - 8u + 3 > 0$$

for all real u. Hence

$$2u^3 - 4u^2 + 3u + 1$$

is increasing. Since at $u = 1$ its value is

$$2 - 4 + 3 + 1 = 2 > 0,$$

we get

$$\Phi_c\left(c + \frac{1}{c^2}\right) > 0.$$

Second, write

$$c = t^2, \qquad t \geq 1.$$

Then

$$\Phi_c\left(\frac{2}{\sqrt{c}}\right) = \frac{2t^9 + 2t^7 - 3t^6 - 8t^4 + 8t + 1}{t^4}.$$

Let

$$t = 1 + v, \qquad v \geq 0.$$

The numerator becomes

$$2v^9 + 18v^8 + 74v^7 + 179v^6 + 276v^5 + 269v^4 + 146v^3 + 21v^2 - 10v + 2.$$

But

$$21v^2 - 10v + 2 > 0,$$

because its discriminant is

$$100 - 168 < 0.$$

All the remaining terms are non-negative for $v \geq 0$. Hence

$$\Phi_c\left(\frac{2}{\sqrt{c}}\right) > 0.$$

Thus

$$\Phi_c(x) \geq 0$$

on the whole admissible interval, and the desired inequality follows. ☐

3 A simple comparison inequality

Problem 3. *Let*

$$a \geq 1 \geq b \geq c > 0,$$

such that

$$b + c > 1.$$

Prove that

$$\frac{2a^2}{a^3 - 2a^2 + 3a - 1} \leq \frac{b^2 + c^2}{b + c - 1}.$$

Proof. First, we prove that the left-hand side is at most 2. Since $a \geq 1$, we have

$$a^3 - 2a^2 + 3a - 1 - a^2 = a^3 - 3a^2 + 3a - 1 = (a-1)^3 \geq 0.$$

Therefore

$$a^3 - 2a^2 + 3a - 1 \geq a^2.$$

Hence

$$\frac{2a^2}{a^3 - 2a^2 + 3a - 1} \leq 2.$$

It remains to prove that the right-hand side is at least 2. Let

$$s = b + c.$$

Since $b + c > 1$, we have

$$s > 1.$$

Also, because

$$1 \geq b \geq c > 0,$$

we get

$$s = b + c \leq 2.$$

By the elementary inequality

$$b^2 + c^2 \geq \frac{(b+c)^2}{2},$$

we obtain

$$\frac{b^2 + c^2}{b + c - 1} \geq \frac{s^2}{2(s-1)}.$$

Now

$$\frac{s^2}{2(s-1)} \geq 2$$

is equivalent to

$$s^2 \geq 4s - 4,$$

that is,

$$(s-2)^2 \geq 0.$$

Therefore

$$\frac{b^2 + c^2}{b + c - 1} \geq 2.$$

Combining the two estimates, we obtain

$$\frac{2a^2}{a^3 - 2a^2 + 3a - 1} \leq 2 \leq \frac{b^2 + c^2}{b + c - 1}.$$

Hence

$$\frac{2a^2}{a^3 - 2a^2 + 3a - 1} \leq \frac{b^2 + c^2}{b + c - 1}.$$

□

4 Conclusion

The three inequalities proved above illustrate how elementary methods can lead to precise and effective results when the structure of the constraints is used carefully. In the first problem, the symmetric conditions reduce the cyclic sum to a one-variable expression depending on $ab+bc+ca$, allowing the exact range to be obtained. In the second problem, the ordering condition and the fixed product lead naturally to a useful substitution and an interval estimate. The third problem is based on a direct comparison, reducing the desired inequality to two simple bounds. Although the arguments are elementary, each problem highlights a different technique: reduction by symmetry, interval analysis and comparison through standard inequalities. Together, they form a compact collection of original inequality problems with complete and self-contained solutions.