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4001. In ΔABC the following relationship holds:

$$w_a\cos\left(\frac{A}{2}\right)+w_b\cos\left(\frac{B}{2}\right)+w_c\cos\left(\frac{C}{2}\right)=\frac{sr}{R}\left(\frac{r_a+r_b}{h_a+h_b}+\frac{r_c+r_b}{h_c+h_b}+\frac{r_a+r_c}{h_a+h_c}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$r_c+r_b=s\left(\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)\right)=s.\frac{\sin\left(\frac{B+C}{2}\right)}{\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}=$$

$$s.\frac{\cos\left(\frac{A}{2}\right)}{\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}=s.\frac{cos^2\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}=s.\frac{cos^2\left(\frac{A}{2}\right)}{\frac{s}{4R}}=4R.cos^2\left(\frac{A}{2}\right)\quad(1)$$

$$h_c+h_b=\frac{2F(b+c)}{bc}\quad(2)$$

We have from (1) *and* (2):

$$\frac{r_c+r_b}{h_c+h_b}=\frac{4R.cos^2\left(\frac{A}{2}\right)}{\frac{2F(b+c)}{bc}}=\frac{4Rbc.cos^2\left(\frac{A}{2}\right)}{2sr(b+c)}=$$

$$\frac{2bc.\cos\left(\frac{A}{2}\right)}{b+c}.\cos\left(\frac{A}{2}\right)=w_a\cos\left(\frac{A}{2}\right).\frac{R}{sr}$$

$$w_a\cos\left(\frac{A}{2}\right)=\frac{sr}{R}.\frac{r_c+r_b}{h_c+h_b}$$

Let's summarize what we got from others by analogy:

$$w_a\cos\left(\frac{A}{2}\right)+w_b\cos\left(\frac{B}{2}\right)+w_c\cos\left(\frac{C}{2}\right)=\frac{sr}{R}\left(\frac{r_a+r_b}{h_a+h_b}+\frac{r_c+r_b}{h_c+h_b}+\frac{r_a+r_c}{h_a+h_c}\right)$$

4002. In *any* Δ ABC the following relationship holds :

$$\frac{g_a+g_b+g_c}{2r}+\frac{1}{2}\cdot\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\geq$$

$$\geq 2\sum_{cyc}\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum_{cyc}\frac{h_a}{n_a+s}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\sum_{cyc}\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum_{cyc}\frac{h_a}{n_a+s}=\frac{g_a+g_b+g_c+s}{2r}$$

(Reference : Identity in Triangle by Bogdan Fustei – 4;
published at www.ssmrmh.ro)

∴ the main inequality becomes : $\frac{g_a+g_b+g_c}{2r}+$

$$\frac{1}{2}\cdot\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\overset{?}{\geq}\frac{g_a+g_b+g_c+s}{2r}$$

$$\Leftrightarrow\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\boxed{\overset{?}{\underset{(*)}{\geq}}}\frac{s}{r}$$

$$\text{Now, }\left(\frac{s}{r}\right)^2\leq\sqrt{\frac{l_al_bl_c}{g_ag_bg_c}}\prod_{cyc}\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}$$

(Reference : Inequality in Triangle by Bogdan Fustei – 109;
published at www.ssmrmh.ro)

$$\Rightarrow\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\geq\frac{s}{r}\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{g_a+g_b+g_c}{2r}+\frac{1}{2}\cdot\sqrt[4]{\frac{l_al_bl_c}{g_ag_bg_c}}\cdot\prod_{cyc}\left(\frac{p_a}{p_a\cdot\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\geq$$

$$2\sum_{cyc}\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum_{cyc}\frac{h_a}{n_a+s}\ \forall\,\Delta\,ABC,$$

" = " iff Δ ABC is equilateral (QED)

4003. In ΔABC the following relationship holds:

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}\geq 4\sqrt{3}$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution 1 by Khaled Abd Imouti, Kasem Abotrabi-Syria

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}=(\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c})+(\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c})$$

$$l_2 \qquad\qquad l \qquad\qquad l_1$$

$$l_1=\frac{a^2}{a.h_a}+\frac{b^2}{b.h_b}+\frac{c^2}{c.h_c}=\frac{a^2}{2F}+\frac{b^2}{2F}+\frac{c^2}{2F}=\frac{a^2+b^2+c^2}{2F}\ \text{(I)}$$

In ΔABC and ΔA'B'C' : A'B'=B'C'=A'C' =1 (Daniel Pedoe 's)

$$a^2(1^2+1^2-1^2)+b^2(1^2+1^2-1^2)+C^2(1^2+1^2-1^2)\geq 16F\,.F'$$

$$F'=\frac{\sqrt{3}}{4}\Rightarrow a^2+b^2+c^2\geq 16F\times\frac{\sqrt{3}}{4}=4\sqrt{3}F\Rightarrow l_1\geq 2\sqrt{3}$$

$$l_2=\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}=\frac{ab}{a.h_a}+\frac{bc}{b.h_b}+\frac{ca}{c.h_c}=\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\qquad \text{(II)}$$

Let be the function $f(x)=\frac{1}{\sin x}$

By using Jensen's inequality:

$$\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\geq 3(\frac{1}{\sin\frac{A+B+C}{3}})=2\sqrt{3}$$

$$l=l_1+l_2\geq 4\sqrt{3}$$

Solution 2 by Khaled Abd Imouti, Kasem Abotrabi-Syria

$$P_\Delta=a+b+c\ ,\quad (a+b+c)^2\geq 12\sqrt{3}S_\Delta$$

$$\sqrt{\frac{a^2+b^2+c^2}{3}}\geq\frac{a+b+c}{3}\ \ldots.(QM-AM)\Rightarrow\frac{a^2+b^2+c^2}{3}\geq\frac{(a+b+c)^2}{9}\Rightarrow$$

$$a^2+b^2+c^2\geq\frac{(a+b+c)^2}{3}\Rightarrow a^2+b^2+c^2\geq 4\sqrt{3}S_\Delta\Rightarrow$$

$$\frac{a^2}{S_\Delta}+\frac{b^2}{S_\Delta}+\frac{c^2}{S_\Delta}=4\sqrt{3}\qquad (\,I\,)$$

$$l=\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}=\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}+\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}$$

$$=\frac{a^2}{a.h_a}+\frac{b^2}{b.h_b}+\frac{c^2}{c.h_c}+\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}$$

$$= \underbrace{\left(\frac{a^2}{2S_\Delta}+\frac{b^2}{2S_\Delta}+\frac{c^2}{2S_\Delta}\right)}_{l_1}+\underbrace{\left(\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}\right)}_{l_2}$$

$$l_2=\frac{ab}{a.h_a}+\frac{bc}{b.h_b}+\frac{ca}{c.h_c}=\frac{1}{2S_\Delta}(ab+bc+ca)=$$

$$=\frac{1}{2S_\Delta}\left(\frac{2S_\Delta}{\sin A}+\frac{2S_\Delta}{\sin B}+\frac{2S_\Delta}{\sin C}\right)=\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}$$

Let be the function $f(x)=\frac{1}{\sin x}$

By using Jensen's inequality:

$$l_2\ge 2\sqrt{3}\ \Rightarrow l_2\ge 3\left(\frac{1}{\sin\frac{A+B+C}{3}}\right)$$

So : $l=l_1+l_2\ge 2\sqrt{3}+2\sqrt{3}=4\sqrt{3}$

4004. **In any Δ ABC the following relationship holds :**

$$\frac{1}{3r^2}\le\sum_{cyc}\frac{1}{r_a^2}\le\frac{R^2}{12r^4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{r_a^2}=\sum_{cyc}\frac{(s-a)^2}{r^2s^2}=\frac{3s^2-2s.2s+2(s^2-4Rr-r^2)}{r^2s^2}=\frac{s^2-8Rr-2r^2}{r^2s^2}$$

$$\overset{?}{\le}\frac{R^2}{12r^4}\Leftrightarrow(R^2-12r^2)s^2+24r^3(4R+r)\underbrace{\overset{?}{\ge}}_{(*)}0\ \text{and it's trivially true if :}$$

$R^2-12r^2\ge 0$ and when : $R^2-12r^2<0$, then : LHS of $(*)\overset{\text{Gerretsen}}{\ge}$

$$(R^2-12r^2)(4R^2+4Rr+3r^2)+24r^3(4R+r)\overset{?}{\ge}0$$

$$\Leftrightarrow 4t^4+4t^3-45t^2+48t-12\overset{?}{\ge}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)(4t^3+12t^2-21t+6)\overset{?}{\ge}0$$

$\rightarrow$ true $\because t\overset{\text{Euler}}{\ge}2\Rightarrow(*)$ is true $\forall\,\Delta$ ABC $\therefore\sum_{cyc}\frac{1}{r_a^2}\le\frac{R^2}{12r^4}$ and again,

$$\sum_{cyc}\frac{1}{r_a^2}=\frac{s^2-8Rr-2r^2}{r^2s^2}\overset{?}{\ge}\frac{1}{3r^2}\Leftrightarrow s^2\overset{?}{\ge}12Rr+3r^2\rightarrow\text{true}$$

$$\because s^2 \overset{\text{Gerretsen}}{\geq} 12Rr+3r^2+4r(R-2r) \overset{\text{Euler}}{\geq} 12Rr+3r^2 \therefore \sum_{cyc}\frac{1}{r_a^2} \geq \frac{1}{3r^2} \text{ and so,}$$

$$\frac{1}{3r^2} \leq \sum_{cyc}\frac{1}{r_a^2} \leq \frac{R^2}{12r^4}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

4005. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{h_a}{h_a^2+r^2} \leq \frac{9}{10r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_a}{h_a^2+r^2} \overset{?}{\leq} \frac{9}{10r} \Leftrightarrow \sum_{cyc}\frac{\frac{h_a}{r}}{\frac{h_a^2}{r^2}+1} \overset{?}{\leq} \frac{9}{10} \Leftrightarrow \sum_{cyc}\frac{\frac{3}{x}}{\frac{9}{x^2}+1} \overset{?}{\leq} \frac{9}{10}$$

$$\left(x=\frac{3r}{h_a}, y=\frac{3r}{h_b}, y=\frac{3r}{h_c}\right) \Leftrightarrow \sum_{cyc}\frac{x}{9+x^2} \underset{(*)}{\underbrace{\overset{?}{\leq}}} \frac{3}{10}$$

$$\text{Now, } f(t)=\frac{t}{9+t^2}\ \forall\, t\in(0,3) \text{ is cocave as } f''(t)=\frac{2t(t^2-27)}{(t^2+9)^3}<0$$

$$(\because t^2<9<27) \therefore \text{as } \sum_{cyc}x=3 \Rightarrow 0<x,y,z<3, \text{hence} : \sum_{cyc}\frac{x}{9+x^2}$$

$$\overset{\text{Jensen}}{\leq} 3.\frac{\frac{\sum_{cyc}x}{3}}{9+\left(\frac{\sum_{cyc}x}{3}\right)^2} = \frac{3}{9+1} = \frac{3}{10} \Rightarrow (*) \text{ is true and so,}$$

$$\sum_{cyc}\frac{h_a}{h_a^2+r^2} \leq \frac{9}{10r}\ \forall\,\Delta\,ABC,''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

4006. In ΔABC the following relationship holds:

$$\sum\frac{h_a}{bc} \leq \frac{3}{4r}$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

$$\text{Let } \Delta \text{ donote the area of } \Delta ABC.$$

$$\text{Here, } \sum \frac{h_a}{bc} = \sum \frac{\frac{2\Delta}{a}}{bc} = \sum \frac{2\Delta}{abc} = \sum \frac{2\Delta}{4\Delta \cdot R} = \sum \frac{1}{2R} =$$
$$= \frac{3}{2R} \overset{Euler}{\overbrace{\leq}} \frac{3}{2 \cdot 2r} = \frac{3}{4r}$$

$$\text{Equality holds iff the triangle is equilateral.} \qquad (QED)$$

4007. In ΔABC the following relationship holds:

$$a\tan\left(\frac{A}{2}\right) + b\tan\left(\frac{B}{2}\right) + c\tan\left(\frac{C}{2}\right) \geq 3R$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$a\tan\left(\frac{A}{2}\right) + b\tan\left(\frac{B}{2}\right) + c\tan\left(\frac{C}{2}\right) = a.\frac{r_a}{s} + b.\frac{r_b}{s} + c.\frac{r_c}{s} =$$
$$= \frac{ar_a + br_b + cr_c}{s} = \frac{2s(2R-r)}{s} = 4R - 2r \overset{Euler}{\overbrace{\geq}} 3R$$
$$\text{Equality holds for } a = b = c.$$

4008.

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\cdot\frac{\sin^2\frac{C}{2}\cdot\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin^2\frac{B}{2}\cdot\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+2\sin\frac{C}{2}}\right)\geq 12$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, A', B', C', x', y', z' > 0,$$
$$\frac{x'}{y'+z'}(B'+C') + \frac{y'}{z'+x'}(C'+A') + \frac{z'}{x'+y'}(A'+B') \underbrace{\overset{\text{Walter Janous}}{\geq}}_{①} \sqrt{3\sum_{cyc} A'B'}$$

$$\text{Now, }\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\cdot\frac{\sin^2\frac{C}{2}\cdot\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin^2\frac{B}{2}\cdot\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+2\sin\frac{C}{2}}\right)=$$

$$\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)+\left(\sin\frac{C}{2}+\sin\frac{A}{2}\right)}\cdot\left(\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin^2\frac{B}{2}}+\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin^2\frac{C}{2}}\right)\right)$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c}x'=\sin\frac{A}{2}+\sin\frac{B}{2},y'=\sin\frac{B}{2}+\sin\frac{C}{2},z'=\sin\frac{C}{2}+\sin\frac{A}{2},\\ A'=\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin^2\frac{A}{2}},B'=\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin^2\frac{B}{2}},C'=\frac{\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin^2\frac{C}{2}}\end{array}\right)\overset{\text{via ①}}{\geq}$$

$$\sqrt{3\sum_{cyc}\left(\frac{\sqrt{\sin\frac{A}{2}+\sin\frac{B}{2}}}{\sin^2\frac{A}{2}}\cdot\frac{\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}}{\sin^2\frac{B}{2}}\right)}\overset{AM-GM}{\geq}3.\sqrt[6]{\prod_{cyc}\left(\frac{\sin\frac{B}{2}+\sin\frac{C}{2}}{\sin^4\frac{A}{2}}\right)}\overset{Cesaro}{\geq}$$

$$3.\sqrt[6]{\frac{8}{\sin^3\frac{A}{2}\sin^3\frac{B}{2}\sin^3\frac{C}{2}}}=\frac{3\sqrt{2}}{\sqrt{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}}\overset{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}}{\geq}\frac{3\sqrt{2}}{\sqrt{\frac{1}{8}}}=3\sqrt{2}.2\sqrt{2}=12$$

$$\text{and so, }\sum_{cyc}\left(\frac{\sin\frac{A}{2}+\sin\frac{B}{2}}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}\cdot\frac{\sin^2\frac{C}{2}\cdot\sqrt{\sin\frac{B}{2}+\sin\frac{C}{2}}+\sin^2\frac{B}{2}\cdot\sqrt{\sin\frac{C}{2}+\sin\frac{A}{2}}}{\sin\frac{A}{2}+\sin\frac{B}{2}+2\sin\frac{C}{2}}\right)\geq 12$$

$\forall\,\Delta\,ABC$, "=" iff $\Delta\,ABC$ is equilateral (QED)

4009. In any $\Delta\,ABC$ the following relationship holds :

$$s^2+16Rr+22r^2\leq\left(\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}\right)^2\leq\frac{1}{9}(18s^2+82Rr+79r^2)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\sqrt{r_br_c}\right)^2=s^2+2s.\sum_{cyc}\sqrt{(s-b)(s-c)}\overset{?}{\leq}\frac{1}{9}(18s^2+82Rr+79r^2)$$

$$\Leftrightarrow 2s.\sum_{cyc}\sqrt{(s-b)(s-c)}\overset{?}{\leq}\frac{9s^2+82Rr+79r^2}{9}$$

$$\Leftrightarrow 2\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}yz\right)\overset{?}{\leq}\left(\sum_{cyc}x^2\right)^2+\frac{82}{9}\cdot\frac{1}{4\sum_{cyc}x^2}\cdot\prod_{cyc}(y^2+z^2)+\frac{79}{9}\cdot\frac{x^2y^2z^2}{\sum_{cyc}x^2}$$

$$\left(\begin{array}{c}x=\sqrt{s-a},y=\sqrt{s-b},z=\sqrt{s-c}\Rightarrow s=\sum_{cyc}x^2\\ \Rightarrow a=y^2+z^2,b=z^2+x^2,c=x^2+y^2\end{array}\right)\Leftrightarrow 18\left(\sum_{cyc}x^2\right)^3+$$

$$41\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x^2y^2\right)+117x^2y^2z^2\boxed{\underbrace{\overset{?}{\geq}}_{(*)}}36\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}xy\right)$$

Assigning $y+z=X, z+x=Y, x+y=Z\Rightarrow X+Y-Z=2z>0, Y+Z-X=2x>0$ *and* $Z+X-Y=2y>0\Rightarrow X+Y>Z, Y+Z>X, Z+X>Y\Rightarrow X,Y,Z$ *form* sides of *a* triangle with semiperimeter, circumradius & inradius $=s_0,R_0,r_0$ (*say*)

and then : $\sum_{cyc}x=s_0, xyz=r_0^2s_0, \sum_{cyc}xy=4R_0r_0+r_0^2, \sum_{cyc}x^2=s_0^2-8R_0r_0-2r_0^2$,

$\sum_{cyc}x^2y^2=r_0^2\left((4R_0+r_0)^2-2s_0^2\right)$, & then, following simplification, $(*)$ becomes :

$$9s_0^6-(288R_0r_0+133r_0^2)s_0^4+r_0^2(3208R_0^2+1932R_0r_0+341r_0^2)s_0^2-$$

$$185r_0^3(4R_0+r_0)^3\boxed{\underbrace{\overset{?}{\geq}}_{(**)}}0$$ *and* we now prove : $9s^6-(288Rr+133r^2)s^4+$

$$r^2(3208R^2+1932Rr+341r^2)s^2-185r^3(4R+r)^3\boxed{\underbrace{\overset{?}{\geq}}_{①}}0\ \&\ \because 9(s^2-16Rr+5r^2)^3$$

$\overset{\text{Gerretsen}}{\geq} 0\ \therefore$ to prove ①, it suffices to prove : LHS of ① $\overset{?}{\geq}(s^2-16Rr+5r^2)^3$

$$\Leftrightarrow (72R-124r)s^4-r(1852R^2-3126Rr+167r^2)s^2+$$

$$r^2(12512R^3-21720R^2r+4290Rr^2-655r^3)\boxed{\underbrace{\overset{?}{\geq}}_{②}}0$$ *and again*, since

$(72R-124r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\ \therefore$ in order to prove ②,

it suffices to prove : LHS of ② $\overset{?}{\geq}(72R-124r)(s^2-16Rr+5r^2)^2\Leftrightarrow$

$$(452R^2-1562Rr+1073r^2)s^2\boxed{\underbrace{\overset{?}{\geq}}_{③}}r(5920R^3-21544R^2r+17350Rr^2-2445r^3)$$

Case 1 $452R^2-1562Rr+1073r^2\geq 0$ *and* then : LHS of ③ $\overset{\text{Gerretsen}}{\geq}$

$(452R^2-1562Rr+1073r^2)(16Rr-5r^2)\overset{?}{\geq}$ RHS of ③ $\Leftrightarrow$

$$(t-2)\big((t-2)(328t-115)+135\big)\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\to \text{true} \because t\overset{\text{Euler}}{\geq}2\Rightarrow ③ \text{ is true}$$

$$\boxed{\textit{Case 2}}\ 452R^2-1562Rr+1073r^2<0 \textit{ and then} : \text{LHS of } ③ \overset{\text{Rouche}}{\geq}$$

$$(452R^2-1562Rr+1073r^2)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)\overset{?}{\geq}③_{\text{RHS}}$$

$$\Leftrightarrow 2(R-2r)\overbrace{(452R^3-1358R^2r+1093Rr^2-343r^3)}^{\sigma>0}\overset{?}{\geq}$$

$$2(R-2r).\sqrt{R^2-2Rr}.\overbrace{\big(-(452R^2-1562Rr+1073r^2)\big)}^{\mu>0}$$

$$\Leftrightarrow \sigma^2\overset{?}{\geq}(R^2-2Rr)\mu^2\ \left(\because R\overset{\text{Euler}}{\geq}2r\right)\Leftrightarrow (t-2)(32t^3+8t^2+36t-28)+25+$$

$$64\left((9266t^2+2444t+11177)(t-2)^3+22343t^2-90307t+91253\right)\overset{?}{\geq}0$$

$$\to \text{true (strict inequality)} \because t\overset{\text{Euler}}{\geq}2 \textit{ and} \because (90307)^2-4(22343)(91253)<0$$

$$\Rightarrow ③ \text{ is true and combining, } ③\Rightarrow ②\Rightarrow ①\Rightarrow (**)\Rightarrow (*) \text{ is true and again,}$$

$$\left(\sum_{cyc}\sqrt{r_br_c}\right)^2\overset{\text{GM-HM}}{\geq}s^2+\sum_{cyc}\frac{4s(s-b)(s-c)}{s-b+s-c}=\sum_{cyc}\frac{4sbc(s-b)(s-c)}{4Rrs}$$

$$=s^2+\frac{r(s^2+(4R+r)^2)}{R}\overset{\text{Gerretsen + Euler}}{\geq}s^2+\frac{r(14Rr-r^2+(4R+r)^2)}{R}$$

$$=s^2+16Rr+22r^2 \textit{ and so, } s^2+16Rr+22r^2\leq\left(\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}\right)^2\leq$$

$$\frac{1}{9}(18s^2+82Rr+79r^2)\ \forall\,\Delta ABC,''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4010. In ΔABC the following relationship holds:

$$\frac{w_a}{h_a}+\frac{h_a}{w_a}\leq\sqrt{\frac{R+6r}{2r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\frac{w_a}{h_a}=\frac{2\sqrt{bc.s(s-a)}}{b+c}.\frac{2R}{bc}=\frac{4R\sqrt{s(s-a)}}{(2(s-a)+a)\sqrt{bc}}\overset{AM-GM}{\leq}\frac{4R\sqrt{s(s-a)}}{2\sqrt{2(s-a)abc}}=$$

$$=2R\sqrt{\frac{s}{8Rsr}}=\sqrt{\frac{R}{2r}}\ (1)$$

$$\left(\frac{w_a}{h_a}+\frac{h_a}{w_a}\right)^2=\left(\frac{w_a}{h_a}\right)^2+\left(\frac{h_a}{w_a}\right)^2+2\overset{\substack{(1)\\ w_a\ge h_a}}{\le}\left(\sqrt{\frac{R}{2r}}\right)^2+\left(\frac{w_a}{w_a}\right)^2+2=\frac{R+6r}{2r}$$

$$\frac{w_a}{h_a}+\frac{h_a}{w_a}\le\sqrt{\frac{R+6r}{2r}}$$

Equality holds for an equilateral triangle.

4011. In any Δ ABC the following relationship holds :

$$max\left\{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}},\frac{a+b}{a+c}+\frac{a+c}{a+b}\right\}\le\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\le\frac{w_b}{w_c}+\frac{w_c}{w_b}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{B}{2}}-2=\frac{\left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)^2}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}}=\frac{\left(\frac{(s-c)(s-a)}{ca}-\frac{(s-a)(s-b)}{ab}\right)^2}{\frac{(s-c)(s-a)}{ca}\cdot\frac{(s-a)(s-b)}{ab}}$$

$$=\frac{\left(b(s-c)-c(s-b)\right)^2}{bc(s-b)(s-c)}\Rightarrow\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{B}{2}}-2=\frac{s^2(b-c)^2}{bc(s-b)(s-c)}\to(m)\ \&\ \text{now,}$$

$$\frac{a+b}{a+c}+\frac{a+c}{a+b}\overset{?}{\le}\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\overset{\text{squaring}}{\Leftrightarrow}\left(\frac{a+b}{a+c}\right)^2+\left(\frac{a+c}{a+b}\right)^2\overset{?}{\le}\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}$$

$$\overset{\text{squaring}}{\Leftrightarrow}\left(\frac{a+b}{a+c}\right)^4+\left(\frac{a+c}{a+b}\right)^4-2\overset{?}{\le}\frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}+\frac{\sin^2\frac{C}{2}}{\sin^2\frac{B}{2}}-2\overset{\text{via (m)}}{\Leftrightarrow}$$

$$\frac{\left((a+b)^4-(c+a)^4\right)^2}{(a+b)^4(c+a)^4}\overset{?}{\le}\frac{s^2(b-c)^2}{bc(s-b)(s-c)}$$

$$\Leftrightarrow\frac{\left((a+b)^2+(c+a)^2\right)^2(2a+b+c)^2(b-c)^2}{(a+b)^4(c+a)^4}\overset{?}{\le}\frac{s^2(b-c)^2}{bc(s-b)(s-c)}\Leftrightarrow$$

$$bc(s-b)(s-c)\left((a+b)^2+(c+a)^2\right)^2(2a+b+c)^2\overset{?}{\le}s^2(a+b)^4(c+a)^4$$

$$(\because(b-c)^2\ge 0)\text{ and }\because bc(s-b)(s-c)\overset{\text{AM-GM}}{\le}\frac{(b+c)^2a^2}{16}\ \therefore\text{ it suffices to prove :}$$

$$4s(a+b)^2(c+a)^2\overset{?}{\ge}a(b+c)\left((a+b)^2+(c+a)^2\right)(2a+b+c)\Leftrightarrow$$

$$4\left(\sum_{\text{cyc}} x\right)(2z+x+y)^2(2y+z+x)^2 \overset{?}{\geq}$$

$$(y+z)(2x+y+z)((2z+x+y)^2+(2y+z+x)^2)(2(y+z)+2x+y+z)$$

$$(x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y) \Leftrightarrow$$

$$4x^5+20x^4(y+z)+36x^3(y+z)^2+8x^3yz+26x^2(y+z)^3+40x^2yz(y+z)+$$

$$6x(y+z)^4+56xyz(y+z)^2+4xy^2z^2+(y+z)^5+22yz(y+z)^3+4y^2z^2(y+z)$$

$$\overset{?}{\geq} 0 \to \text{true (strict inequality)} \Rightarrow \frac{a+b}{a+c}+\frac{a+c}{a+b} \overset{①}{\leq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

$$\text{Again, } \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \overset{?}{\leq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \overset{\substack{\text{squaring twice}\\ \text{and via (m)}}}{\Leftrightarrow} \frac{s^2(b-c)^2}{bc(s-b)(s-c)} \overset{?}{\geq} \frac{(b^2-c^2)^2}{b^2c^2}$$

$$\Leftrightarrow \left(\sum_{\text{cyc}} x\right)^2 (z+x)(x+y) \overset{?}{\geq} yz(2x+y+z)^2 \ (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow x^4+3x^3(y+z)+3x^2(y^2+yz+z^2)+x(y^3+z^3+y^2z+yz^2) \overset{?}{\geq} 0$$

$$\to \text{true (strict inequality)} \therefore \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \overset{②}{\leq} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}$$

$$\text{Finally, } \frac{w_b}{w_c}+\frac{w_c}{w_b} = \frac{\frac{2a.4R\cos\frac{C}{2}\sin\frac{C}{2}}{c+a}.\cos\frac{B}{2}}{\frac{2a.4R\cos\frac{B}{2}\sin\frac{B}{2}}{a+b}.\cos\frac{C}{2}} + \frac{\frac{2a.4R\cos\frac{B}{2}\sin\frac{B}{2}}{a+b}.\cos\frac{C}{2}}{\frac{2a.4R\cos\frac{C}{2}\sin\frac{C}{2}}{c+a}.\cos\frac{B}{2}}$$

$$= \frac{\left(\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right)^2}{\frac{c+a}{a+b}} + \frac{\left(\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}\right)^2}{\frac{a+b}{c+a}} \overset{\text{Bergstrom}}{\geq} \frac{\left(\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right)^2}{\frac{c+a}{a+b}+\frac{a+b}{c+a}} \overset{\text{via ①}}{\geq}$$

$$\frac{\left(\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right)\left(\frac{c+a}{a+b}+\frac{a+b}{c+a}\right)}{\frac{c+a}{a+b}+\frac{a+b}{c+a}}$$

$$\Rightarrow \frac{w_b}{w_c}+\frac{w_c}{w_b}\overset{③}{\geq}\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\text{ and so, ①, ② and ③}$$

$$\Rightarrow max\left\{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}},\frac{a+b}{a+c}+\frac{a+c}{a+b}\right\}\leq\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\leq\frac{w_b}{w_c}+\frac{w_c}{w_b}\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } b=c$$

4012. In any Δ ABC the following relationship holds :

$$\frac{r_b+r_c}{w_a}\leq\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_b+r_c}{w_a}\overset{?}{\leq}\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}\Leftrightarrow\frac{4R.s(s-a)}{bc.\frac{2bc}{b+c}.\cos^2\frac{A}{2}}\overset{?}{\leq}\frac{\frac{(s-c)(s-a)}{ca}+\frac{(s-a)(s-b)}{ab}}{\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$\Leftrightarrow\frac{s(s-a)}{bc.\frac{2bc}{b+c}.\frac{s(s-a)}{bc}}\overset{?}{\leq}\frac{b(s-c)(s-a)+c(s-a)(s-b)}{abc.(s-a)}$$

$$\Leftrightarrow 2\big(zx(z+x)+xy(x+y)\big)\overset{?}{\geq}x(y+z)(2x+y+z)$$

$$(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$

$$\Leftrightarrow x(y-z)^2\overset{?}{\geq}0\rightarrow\text{true}\because x>0\therefore\frac{r_b+r_c}{w_a}\leq\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}+\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } b=c$$

4013. In any Δ ABC with $\omega \to$

Brocard's angle the following relationship holds :

$$\frac{1}{\sin\omega} \geq 2.\frac{m_a + m_b + m_c}{h_a + h_b + h_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Since } \frac{1}{\sin\omega} = \frac{\sqrt{\sum_{cyc} a^2b^2}}{2rs} \text{ and } \because \left(\sum_{cyc} m_a\right)^2 \overset{\text{Chu-Yang}}{\leq} 4s^2 - 16Rr + 5r^2$$

$$\therefore \text{ in order to prove : } \frac{1}{\sin\omega} \overset{?}{\geq} 2.\frac{m_a + m_b + m_c}{h_a + h_b + h_c}, \text{ it suffices to prove :}$$

$$((s^2 + 4Rr + r^2)^2 - 16Rrs^2)(s^2 + 4Rr + r^2)^2 \overset{?}{\geq} 64R^2r^2s^2(4s^2 - 16Rr + 5r^2)$$

$$\Leftrightarrow s^8 + 4r^2s^6 - r^2(288R^2 - 16Rr - 6r^2)s^4 +$$

$$r^3(1024R^3 - 256R^2r + 32Rr^2 + 4r^3)s^2 + r^4(4R + r)^4 \boxed{\overset{?}{\underset{\textcircled{1}}{\geq}}} 0 \text{ and } \because$$

$$(s^2 - 16Rr + 5r^2)^4 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } \textcircled{1}, \text{ it suffices to prove :}$$

$$\text{LHS of } \textcircled{1} \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^4 \Leftrightarrow (4R - r)s^6 - r(114R^2 - 61Rr + 9r^2)s^4 +$$

$$r^2(1088R^3 - 976R^2r + 302Rr^2 - 31r^3)s^2 -$$

$$r^3(4080R^4 - 5136R^3r + 2394R^2r^2 - 501Rr^3 + 39r^4) \boxed{\overset{?}{\underset{\textcircled{2}}{\geq}}} 0 \text{ and } \because$$

$$(4R - r)(s^2 - 16Rr + 5r^2)^3 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } \textcircled{2}, \text{ it suffices to prove :}$$

$$\text{LHS of } \textcircled{2} \overset{?}{\geq} (4R - r)(s^2 - 16Rr + 5r^2)^3 \Leftrightarrow (78R^2 - 47Rr + 6r^2)s^4 -$$

$$r(1984R^3 - 1712R^2r + 478Rr^2 - 44r^3)s^2 +$$

$$r^2(12304R^4 - 14320R^3r + 6246R^2r^2 - 1199Rr^3 + 86r^4) \boxed{\overset{?}{\underset{\textcircled{3}}{\geq}}} 0 \text{ and } \because$$

$$(78R^2 - 47Rr + 6r^2)(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } \textcircled{3},$$

$$\text{it suffices to prove : LHS of } \textcircled{3} \overset{?}{\geq} (78R^2 - 47Rr + 6r^2)(s^2 - 16Rr + 5r^2)^2$$

$$\Leftrightarrow (128R^3 - 143R^2r + 46Rr^2 - 4r^3)s^2 \boxed{\overset{?}{\underset{\textcircled{4}}{\geq}}} r\begin{pmatrix} 1916R^4 - 2548R^3r + \\ 1190R^2r^2 - 234Rr^3 + 16r^4 \end{pmatrix}$$

$$\text{Finally, LHS of } \textcircled{4} \overset{\text{Gerretsen}}{\geq} (128R^3 - 143R^2r + 46Rr^2 - 4r^3)(16Rr - 5r^2) \overset{?}{\geq}$$

$$\text{RHS of } \textcircled{4} \Leftrightarrow 132t^4 - 380t^3 + 261t^2 - 60t + 4 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)(132t^3-116t^2+29t-2) \overset{?}{\ge} 0 \to \text{true} \because t \overset{Euler}{\ge} 2 \Rightarrow ④ \Rightarrow ③ \Rightarrow ②$$

$$\Rightarrow ① \text{ is true } \therefore \frac{1}{\sin\omega} \ge 2.\frac{m_a+m_b+m_c}{h_a+h_b+h_c}\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4014. In ΔABC the following relationship holds:

$$\sqrt{\frac{m_a}{r_a}}+\sqrt{\frac{r_a}{m_a}} \le \sqrt{\frac{2R}{r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$n_a^2 = s^2-2r_ah_a (Bogdan\ Fustei)\ , r_a = s\tan\frac{A}{2}$$

$$\sqrt{\frac{m_a}{r_a}}+\sqrt{\frac{r_a}{m_a}} = \frac{m_a+r_a}{\sqrt{m_ar_a}} = \sqrt{\frac{(m_a+r_a)^2}{m_ar_a}} = \sqrt{\frac{m_a^2+r_a^2}{m_ar_a}+2} \overset{\substack{m_a\le n_a\\ m_a\ge h_a}}{\le} \sqrt{\frac{n_a^2+r_a^2}{h_ar_a}+2} =$$

$$= \sqrt{\frac{s^2-2r_ah_a+r_a^2}{r_ah_a}+2} = \sqrt{\frac{s^2+s^2\tan^2\left(\frac{A}{2}\right)}{\frac{2F^2}{a(s-a)}}} = \sqrt{\frac{s^2\sec^2\left(\frac{A}{2}\right)}{\frac{2F^2}{a(s-a)}}} = \sqrt{\frac{a(s-a)}{2r^2\cos^2\left(\frac{A}{2}\right)}} =$$

$$= \sqrt{\frac{abc}{2sr^2}} = \sqrt{\frac{4Rrs}{2sr^2}} = \sqrt{\frac{2R}{r}}$$

Equality holds for an equilateral triangle.

4015. In ΔABC the following relationship holds:

$$\frac{b}{c}+\frac{c}{b} \le \frac{s^2}{4Rr}-\frac{11}{8}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$We\ need\ to\ show:$$

$$\frac{b}{c}+\frac{c}{b} \le \frac{s^2}{4Rr}-\frac{11}{8}\quad or, \frac{s^2}{4Rr}-\left(\frac{b}{c}+\frac{c}{b}\right) \ge \frac{11}{8}$$

$$\text{now, we will show } (a+b+c)^3-8a(b^2+c^2)\ge 11abc$$
$$\text{or, } (a+b+c)^3-8a(b^2+c^2)-11abc\ge 0$$
$$a=\max\{a,b,c\}.$$
$$\text{Let } b=a-x,\ c=a-y \text{ where } x,y\ge 0$$

$$(a+b+c)^3-8a(b^2+c^2)-11abc=$$

$$=(3a-x-y)^3-8a((a-x)^2+(a-y)^2)-11a(a-x)(a-y)=$$

$$=ax^2+ay^2+7axy-x^3-3x^2y-3xy^2-y^3=a((x+y)^2+5xy)-(x+y)^3=$$
$$=(x+y)^2(a-x-y)+5axy=(x+y)^2(b+c-a)+5axy\ge 0 \text{ true}$$

$$(\text{as } b=a-x,c=a-y,x,y\ge 0 \text{ then } a-x-y=b+c-a>0 \text{ in any triangle})$$
$$\text{so } (a+b+c)^3-8a(b^2+c^2)\ge 11abc$$

$$R.H.S=\frac{s^2}{4Rr}-\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{s^3}{4Rrs}-\frac{b^2+c^2}{bc}=\frac{(a+b+c)^3}{8abc}-\frac{b^2+c^2}{bc}=$$
$$=\frac{(a+b+c)^3-8a(b^2+c^2)}{8abc}\ge\frac{11abc}{8abc}=\frac{11}{8}$$

Equality holds for an equilateral triangle.

4016. In any Δ ABC the following relationship holds :

$$\max\left\{\frac{2w_a}{h_a},\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right\}\le\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}}\le\frac{r_b+r_c}{w_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2w_a}{h_a}\boxed{\le}\frac{2.\sqrt{s(s-a)}.a}{2.\sqrt{s(s-a)(s-b)(s-c)}}=\frac{s-b+s-c}{\sqrt{(s-b)(s-c)}}=\sqrt{\frac{s-b}{s-c}}+\sqrt{\frac{s-c}{s-b}}$$
$$=\sqrt{\frac{r_c}{r_b}}+\sqrt{\frac{r_b}{r_c}}=\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}}+\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}\text{ and again, }\sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\overset{?}{\le}\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}}$$

$$\overset{\text{squaring}}{\Leftrightarrow} \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}} + \frac{\sin\frac{C}{2}}{\sin\frac{B}{2}} \overset{?}{\leq} \frac{\tan\frac{B}{2}}{\tan\frac{C}{2}} + \frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}$$

$$\Leftrightarrow \frac{\sqrt{\frac{(s-c)(s-a)}{ca}}}{\sqrt{\frac{(s-a)(s-b)}{ab}}} + \frac{\sqrt{\frac{(s-a)(s-b)}{ab}}}{\sqrt{\frac{(s-c)(s-a)}{ca}}} \overset{?}{\leq} \frac{s-c}{s-b} + \frac{s-b}{s-c}$$

$$\Leftrightarrow \sqrt{\frac{b}{c}\cdot\frac{s-c}{s-b}} + \sqrt{\frac{c}{b}\cdot\frac{s-b}{s-c}} \boxed{\underset{(*)}{\overset{?}{\leq}}} \frac{s-c}{s-b} + \frac{s-b}{s-c}$$

$$\text{Now, LHS of } (*) \overset{\text{CBS}}{\leq} \sqrt{\left(\frac{b}{c}+\frac{c}{b}\right)\cdot\left(\frac{s-c}{s-b}+\frac{s-b}{s-c}\right)} \overset{?}{\leq} \frac{s-c}{s-b} + \frac{s-b}{s-c}$$

$$\Leftrightarrow \frac{b}{c} + \frac{c}{b} - 2 \overset{?}{\leq} \frac{s-c}{s-b} - 1 + \frac{s-b}{s-c} - 1 \Leftrightarrow \frac{(b-c)^2}{bc} \overset{?}{\leq} \frac{b-c}{s-b} - \frac{b-c}{s-c}$$

$$\Leftrightarrow \frac{(b-c)^2}{bc} \overset{?}{\leq} \frac{(b-c)^2}{(s-b)(s-c)} \Leftrightarrow \frac{(b-c)^2}{bc(s-b)(s-c)}\cdot\left(-s^2 + sa + bc - bc\right) \overset{?}{\leq} 0$$

$$\Leftrightarrow \frac{-s(s-a)(b-c)^2}{bc(s-b)(s-c)} \overset{?}{\leq} 0 \rightarrow \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} + \sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \boxed{\leq} \sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}} + \sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}} \text{ and finally,}$$

$$w_a\cdot\left(\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}} + \sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}}\right) \leq \sqrt{s(s-a)}\cdot\left(\sqrt{\frac{s-c}{s-b}} + \sqrt{\frac{s-b}{s-c}}\right)$$

$$= \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-b} + \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-c} = r_b + r_c$$

$$\therefore \sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}} + \sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}} \boxed{\leq} \frac{r_b + r_c}{w_a} \text{ and so, } max\left\{\frac{2w_a}{h_a}, \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} + \sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}}\right\} \leq$$

$$\sqrt{\frac{\tan\frac{B}{2}}{\tan\frac{C}{2}}} + \sqrt{\frac{\tan\frac{C}{2}}{\tan\frac{B}{2}}} \leq \frac{r_b + r_c}{w_a} \ \forall \Delta\, ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$

4017.In any Δ ABC the following relationship holds :

$$\frac{w_a}{h_a}+\frac{h_a}{w_a}\le\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\le\min\left\{\frac{w_b}{w_c}+\frac{w_c}{w_b},\frac{2w_a}{h_a}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}=\frac{\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}-1+\sin^2\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=\frac{2+\frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{2R}-1+\sin^2\frac{A}{2}}{\frac{1}{2}(C+S)}\left(C=\cos\frac{B-C}{2},S=\sin\frac{A}{2}\right)$$

$$=\frac{1+S^2+S(C-S)}{\frac{1}{2}(C+S)}$$

$$\therefore\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}=\frac{2(1+SC)}{C+S}\rightarrow ① \text{ and so, via } ①,\frac{w_a}{h_a}+\frac{h_a}{w_a}\overset{?}{\le}\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}$$

$$\Leftrightarrow C+\frac{1}{C}\overset{?}{\le}\frac{2(1+SC)}{C+S}\Leftrightarrow 2C+2SC^2\overset{?}{\ge}C+S+C^3+SC^2\Leftrightarrow C-C^3\overset{?}{\ge}S-SC^2$$

$$\Leftrightarrow(1-C^2)(C-S)\overset{?}{\ge}0\rightarrow\text{true}\because C^2=\cos^2\frac{B-C}{2}\le1\ \text{and}\because C=\cos\frac{B-C}{2}=$$

$$\frac{b+c}{a}.\sin\frac{A}{2}\overset{b+c>a}{>}\sin\frac{A}{2}=S\therefore\boxed{\frac{w_a}{h_a}+\frac{h_a}{w_a}\le\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}}\ \text{and again,}$$

$$\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\overset{?}{\le}\frac{w_b}{w_c}+\frac{w_c}{w_b}$$

$$\Leftrightarrow\frac{\frac{s(s-b)}{ca}+\frac{\cos^2\frac{C}{2}}{ab}}{\cos\frac{B}{2}\cos\frac{C}{2}}\overset{?}{\le}\frac{s(s-b)-\frac{s(s-b)(c-a)^2}{(c+a)^2}+s(s-c)-\frac{s(s-c)(a-b)^2}{(a+b)^2}}{\frac{2ca}{c+a}.\cos\frac{B}{2}.\frac{2ab}{a+b}.\cos\frac{C}{2}}$$

$$\Leftrightarrow 4as\left(s(b+c)-(b^2+c^2)\right)(a+b)(c+a)\overset{?}{\le}$$

$$sa(a+b)^2(c+a)^2-s(s-b)(c-a)^2(a+b)^2-s(s-c)(a-b)^2(c+a)^2$$

$$\Leftrightarrow 4(y+z)\begin{pmatrix}(x+y+z)(2x+y+z)-\\((z+x)^2+(x+y)^2)\end{pmatrix}(2z+x+y)(2y+z+x)\overset{?}{\le}$$

$$(y+z)(2z+x+y)^2(2y+z+x)^2-y(z-x)^2(2z+x+y)^2-$$

$$z(x-y)^2(2y+z+x)^2\ \begin{pmatrix} x=s-a,y=s-b,z=s-c\Rightarrow \\ a=y+z,b=z+x,c=x+y,s=x+y+z \end{pmatrix}$$

$$\Leftrightarrow x(y^2-z^2)^2+y^5+z^5+y^4z+yz^4\overset{?}{\geq}2y^3z^2+2y^2z^3\rightarrow \text{true}$$

$$\because y^5+yz^4\overset{\text{AM}-\text{GM}}{\geq}2y^3z^2\ and\ z^5+y^4z\overset{\text{AM}-\text{GM}}{\geq}2y^2z^3$$

$$\therefore \boxed{\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\leq\frac{w_b}{w_c}+\frac{w_c}{w_b}}\ and\ finally, via\ ①, \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\overset{?}{\leq}\frac{2w_a}{h_a}$$

$$\Leftrightarrow\frac{2(1+SC)}{C+S}\overset{?}{\leq}\frac{2}{C}\Leftrightarrow SC^2\overset{?}{\leq}S\rightarrow \text{true}\because C^2=\cos^2\frac{B-C}{2}\leq 1$$

$$\therefore \boxed{\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\leq\frac{2w_a}{h_a}}\ and\ so, \frac{w_a}{h_a}+\frac{h_a}{w_a}\leq\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}\leq$$

$$\min\left\{\frac{w_b}{w_c}+\frac{w_c}{w_b},2\frac{w_a}{h_a}\right\}\ \forall\,\Delta\,ABC,''=''\text{ iff } b=c\ (\text{QED})$$

4018. In ΔABC the following relationship holds:

$$\sqrt{\frac{w_a}{r_a}}+\sqrt{\frac{r_a}{w_a}}\leq\sqrt{\frac{2R}{r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$n_a^2=s^2-2r_ah_a(Bogdan\ Fustei)\ ,r_a=s\tan\frac{A}{2}$$

$$\sqrt{\frac{w_a}{r_a}}+\sqrt{\frac{r_a}{w_a}}=\frac{w_a+r_a}{\sqrt{w_ar_a}}=\sqrt{\frac{(w_a+r_a)^2}{w_ar_a}}=\sqrt{\frac{w_a^2+r_a^2}{w_ar_a}+2}\overset{\substack{w_a\leq n_a\\ w_a\geq ha}}{\leq}\sqrt{\frac{n_a^2+r_a^2}{h_ar_a}+2}=$$

$$=\sqrt{\frac{s^2-2r_ah_a+r_a^2}{r_ah_a}+2}=\sqrt{\frac{s^2+s^2\tan^2\left(\frac{A}{2}\right)}{\frac{2F^2}{a(s-a)}}}=\sqrt{\frac{s^2\sec^2\left(\frac{A}{2}\right)}{\frac{2F^2}{a(s-a)}}}=\sqrt{\frac{a(s-a)}{2r^2\cos^2\left(\frac{A}{2}\right)}}=$$

$$=\sqrt{\frac{abc}{2sr^2}}=\sqrt{\frac{4Rrs}{2sr^2}}=\sqrt{\frac{2R}{r}}$$

Equality holds for an equilateral triangle.

4019.

In any Δ ABC with $\omega \to$ Brocard's angle, the following relationship holds :

$$\frac{b+c}{2a}+\frac{2a}{b+c} \le \min\left\{\frac{1}{\sin\omega}, \frac{s^2}{4Rr}-\frac{11}{8}, \frac{s^2}{27r^2}+1\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s-a, y = s-b, z = s-c$; then : $a = y+z, b = z+x, c = x+y$ and $s = x+y+z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and then, we have the following set S of relations : $y^2+z^2 = x^2(m^2-2n), y^3+z^3 = x^3(m^3-3nn), y^4+z^4 = x^4((m^2-2n)^2-2n^2), y^5+z^5 = x^5\left(m((m^2-2n)^2+n^2-nm^2)\right), y^6+z^6 = x^6((m^3-3nn)^2-2n^3), y^7+z^7 = x^7\left(m((m^3-3nn)^2-2n^3)-mn((m^2-2n)^2+n^2-nm^2)\right)$, $y^8+z^8 = x^8(((m^2-2n)^2-2n^2)^2-2n^4)$ and now, $\frac{b+c}{2a}+\frac{2a}{b+c} \overset{?}{\le} \frac{1}{\sin\omega}$

$$\Leftrightarrow \frac{4\sum_{\text{cyc}} a^2b^2}{2\sum_{\text{cyc}} a^2b^2-\sum_{\text{cyc}} a^4}-2 \overset{?}{\ge} \frac{(b+c)^4+16a^4}{4a^2(b+c)^2}$$

$$\Leftrightarrow \frac{(y+z)^4+(z+x)^4+(x+y)^4}{2xyz(x+y+z)} \overset{?}{\ge} \frac{(2x+y+z)^4+16(y+z)^4}{(y+z)^2(2x+y+z)^2}$$

$$\Leftrightarrow 4x^6(y-z)^2+12x^5(y-z)^2(y+z)+21x^4(y^2+z^2)^2-76x^4y^2z^2+4x^4yz(y^2+z^2)+22x^3(y^5+z^5)+30x^3yz(y^3+z^3)-20x^3y^2z^2(y+z)+15x^2(y^6+z^6)+27x^2yz(y^4+z^4)-23x^2y^2z^2(y+z)^2-24y^3z^3+6x(y^7+z^7)+11xyz(y^5+z^5)-25xy^2z^2(y^3+z^3)-88xy^3z^3(y+z)+y^8+z^8+6yz(y^6+z^6)+17y^2z^2(y^4+z^4)+30y^3z^3(y^2+z^2)+36y^4z^4 \boxed{\underset{\textcircled{1}}{\underbrace{\overset{?}{\geq}}}} 0$$

and $\because 4x^6(y-z)^2+12x^5(y-z)^2(y+z) \ge 0$ & via set of relations "S", to prove ①, suffices to prove, following simplification : $(m+2)^2n^2-(2m^4+31m^3+63m^2+80m+80)n+m^2(m^4+6m^3+15m^2+22m+21) \boxed{\underset{\textcircled{2}}{\underbrace{\overset{?}{\geq}}}} 0$ and discriminant δ of LHS of ② =

$$(2m^4+31m^3+63m^2+80m+80)^2-4m^2(m+2)^2\left(m^4+6m^3+15m^2+22m+21\right)$$
$$= 84m^7+1041m^6+3802m^5+8573m^4+14352m^3+16144m^2+12800m+6400 > 0$$

$\therefore$ in order to prove ②, it suffices to prove :

$$2(m+2)^2n \overset{?}{\le} (2m^4+31m^3+63m^2+80m+80)-\sqrt{\delta} \text{ and } \because n \overset{\text{AM-GM}}{\le} \frac{m^2}{4}$$

$$\therefore \text{ it suffices to prove : } 2\sqrt{\delta} \overset{?}{\le} 2(2m^4+31m^3+63m^2+80m+80)-m^2(m+2)^2$$

$$\Leftrightarrow 2\sqrt{\delta} \overset{?}{\le} 3m^4+58m^3+122m^2+160m+160 \overset{\text{squaring}}{\Leftrightarrow}$$

$$\boxed{m^2(m+2)^2(3m+2)^2(m-2)^2 \overset{?}{\ge} 0} \to \text{true} \Rightarrow ① \text{ is true} \therefore \frac{b+c}{2a}+\frac{2a}{b+c} \le \frac{1}{\sin\omega}$$

$$\text{Again, } \frac{b+c}{2a}+\frac{2a}{b+c} \overset{?}{\le} \frac{s^2}{4Rr}-\frac{11}{8} \overset{\text{via earlier substitutions}}{\Leftrightarrow}$$

$$\left(8(x+y+z)^3-11(y+z)(z+x)(x+y)\right)(2x+y+z) \overset{?}{\ge}$$

$$4(z+x)(x+y)\left((2x+y+z)^2+4(y+z)^2\right) \Leftrightarrow 2x^3(y+z)+3x^2(y+z)^2-$$

$$16x^2yz+9x(y^3+z^3)-11xyz(y+z)+8(y^2+z^2)^2-30y^2z^2+yz(y^2+z^2) \overset{?}{\ge} 0$$

$$\overset{\substack{\text{via set of relations S} \\ \text{and following simplification}}}{\Leftrightarrow} 8m^4+9m^3+3m^2+2m-n(31m^2+38m+16) \overset{?}{\ge} 0 \text{ and}$$

$$\because n \overset{\text{AM-GM}}{\le} \frac{m^2}{4} \therefore \text{ it suffices to prove :}$$

$$4(8m^4+9m^3+3m^2+2m)-m^2(31m^2+38m+16) \overset{?}{\ge} 0$$

$$\Leftrightarrow \boxed{m(m+2)(m-2)^2 \overset{?}{\ge} 0} \to \text{true} \therefore \frac{b+c}{2a}+\frac{2a}{b+c} \le \frac{s^2}{4Rr}-\frac{11}{8}$$

$$\text{Finally, } \frac{b+c}{2a}+\frac{2a}{b+c} \overset{?}{\le} \frac{s^2}{27r^2}+1 \overset{\text{via earlier substitutions}}{\Leftrightarrow}$$

$$2(y+z)(2x+y+z)\left((x+y+z)^3+27xyz\right) \overset{?}{\ge} 27xyz\left((2x+y+z)^2+4(y+z)^2\right)$$

$$\Leftrightarrow 4x^4(y+z)+14x^3(y+z)^2-108x^3yz+18x^2(y+z)^3+10x(y^2+z^2)^2-$$

$$122xy^2z^2-41xyz(y^2+z^2)+2(y+z)^5 \overset{?}{\ge} 0 \overset{\substack{\text{via set of relations S} \\ \text{and following simplification}}}{\Leftrightarrow}$$

$$2m^5+10m^4+18m^3+14m^2+4m \overset{?}{\ge} (81m^2+108)n \text{ and} \because n \overset{\text{AM-GM}}{\le} \frac{m^2}{4}$$

$$\therefore \text{ it suffices to prove : } 4\left(2m^5+10m^4+18m^3+14m^2+4m\right) \overset{?}{\ge} m^2(81m^2+108)$$

$$\Leftrightarrow \boxed{m(8m^2-9m+4)(m-2)^2 \overset{?}{\ge} 0} \to \text{true} \because \text{discriminant of } 8m^2-9m+4=$$

$$81-128<0 \Rightarrow 8m^2-9m+4>0 \therefore \frac{b+c}{2a}+\frac{2a}{b+c} \le \frac{s^2}{27r^2}+1 \therefore$$

$$\frac{b+c}{2a}+\frac{2a}{b+c} \le \min\left\{\frac{1}{\sin\omega}, \frac{s^2}{4Rr}-\frac{11}{8}, \frac{s^2}{27r^2}+1\right\} \forall \Delta ABC,$$

$'' ='' \text{ iff } \Delta ABC$ is equilateral (QED)$'' ='' \text{ iff } \Delta ABC$ is equilateral (QED)

4020. If in $\Delta ABC, a = \max\{a,b,c\}$ then:

$$\frac{b+c}{2a}+\frac{2a}{b+c} \le \frac{2s^2}{9Rr}-1$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\frac{2s^2}{9Rr}=\frac{2s^3}{9Rrs}=\frac{8s^3}{9(4Rrs)}=\frac{(a+b+c)^3}{9abc}$$

$$Let\ x=\frac{b+c}{a}>1\ (as\ in\ \Delta ABC\ b+c>a)$$

$$x=\frac{b+c}{a}=\frac{b+c}{\sqrt{a^2}}\overset{a=\max\{a,b,c\}}{\le}\frac{b+c}{\sqrt{bc}}\overset{AM-GM}{\le}\frac{b+c}{\frac{b+c}{2}}=2\ \ so\ 1<x\le 2$$

$$bc\overset{AM-GM}{\le}\left(\frac{b+c}{2}\right)^2\ or\ \ bc\overset{x=\frac{b+c}{a}}{\le}\frac{a^2x^2}{4}\ \&\ abc\le\frac{a^3x^2}{4}$$

$$\frac{(a+b+c)^3}{9abc}=\frac{a^3\left(1+\frac{b+c}{a}\right)}{9abc}\ge\frac{a^3(1+x)^3}{9.\frac{a^3x^2}{4}}=\frac{4(1+x)^3}{9x^2}$$

$$We\ need\ to\ show:$$

$$\frac{b+c}{2a}+\frac{2a}{b+c}\le\frac{2s^2}{9Rr}-1\ \ or\ \frac{2s^2}{9Rr}-\left(\frac{b+c}{2a}+\frac{2a}{b+c}\right)\ge 1\ or$$

$$\frac{2s^2}{9Rr}-\left(\frac{b+c}{2a}+\frac{2a}{b+c}\right)-1\ge 0\ or\ \frac{4(1+x)^3}{9x^2}-\frac{x}{2}-\frac{2}{x}-1\ge 0$$

$$L.H.S=\frac{4(1+x)^3}{9x^2}-\frac{x}{2}-\frac{2}{x}-1=\frac{8(1+x)^3-9x^3-36x-18x^2}{18x^2}=$$

$$=\frac{8-12x+6x^2-x^3}{18x^2}=\frac{(2-x)^3}{18x^2}\ge 0\ true\ as\ 1<x\le 2$$

Equality holds for an equilateral triangle.

4021. In any Δ ABC the following relationship holds :

$$max\left\{\frac{h_b+h_c}{2r_a}+\frac{2r_a}{h_b+h_c},\frac{r_b+r_c}{2h_a}+\frac{2h_a}{r_b+r_c}\right\}\le\frac{R}{r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_b+h_c}{2r_a}+\frac{2r_a}{h_b+h_c}\overset{?}{\le}\frac{R}{r}\Leftrightarrow\frac{\frac{1}{b}+\frac{1}{c}}{\frac{1}{s-a}}+\frac{\frac{1}{s-a}}{\frac{1}{b}+\frac{1}{c}}\overset{?}{\le}\frac{abc}{4(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow\frac{b^2c^2+(s-a)^2(b+c)^2}{bc(s-a)(b+c)}\overset{?}{\le}\frac{abc}{4(s-a)(s-b)(s-c)}\Leftrightarrow$$

$$(z+x)^2(x+y)^2(y+z)(2x+y+z)\overset{?}{\ge}4yz((z+x)^2(x+y)^2+x^2(2x+y+z)^2)$$

$$(x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y)$$

$$\Leftrightarrow 2x^5(y+z)+5x^4(y+z)^2-20x^4yz+4x^3(y+z)^3-20x^3yz(y+z)+$$
$$x^2((y+z)^2-2yz)^2-8x^2y^2z^2+2x^2yz((y+z)^2-2yz)+$$
$$2xyz\left((y+z)^3-3yz(y+z)\right)+y^2z^2(y-z)^2\overset{?}{\ge}0\text{ and }\because y^2z^2(y-z)^2\ge 0$$

$\therefore$ it suffices to prove : $2m+5m^2-20n+4m^3-20mn+(m^2-2n)^2-8n^2+$
$$2n(m^2-2n)+2n(m^3-3mn)\overset{?}{\ge}0\ \left(m=\frac{y+z}{x},n=\frac{yz}{x^2}\right)$$

$$\Leftrightarrow(6m+8)n^2-(2m^3-2m^2-20m-20)n-(m^4+4m^3+5m^2+2m)\boxed{\overset{?}{\underset{(*)}{\le}}}0$$

Now, LHS of $(*)$ is *a* quadratic polynomial in "n" with discriminant, $\delta=$
$$(2m^3-2m^2-20m-20)^2+4(6m+8)(m^4+4m^3+5m^2+2m)$$
$$=4\left(m^6+4m^5+13m^4+62m^3+172m^2+216m+100\right)>0\text{ and so,}$$
in order to prove $(*)$, it suffices to prove :

$$2(6m+8)n\boxed{\overset{?}{\underset{\textcircled{1}}{\le}}}2m^3-2m^2-20m-20+\sqrt{\delta}\text{ AND}$$

$$2(6m+8)n\boxed{\overset{?}{\underset{\textcircled{2}}{\ge}}}2m^3-2m^2-20m-20-\sqrt{\delta}$$

Now, since $n\overset{\text{AM-GM}}{\le}\frac{m^2}{4}$ $\therefore$ in order to prove ①, it suffices to prove :

$$(3m+4)m^2\overset{?}{\le}2m^3-2m^2-20m-20+\sqrt{\delta}\Leftrightarrow m^3+6m^2+20m+20\overset{?}{\le}\sqrt{\delta}$$
$$\Leftrightarrow 4\left(m^6+4m^5+13m^4+62m^3+172m^2+216m+100\right)\overset{?}{\ge}$$
$$(m^3+6m^2+20m+20)^2\Leftrightarrow\boxed{m(3m+4)(m+2)^2(m-2)^2\overset{?}{\ge}0}\to\text{true}\because m>0$$

$\Rightarrow$ ① is true *and also*, $2(6m+8)n+\sqrt{\delta}>\sqrt{\delta}$
$$=\sqrt{(2m^3-2m^2-20m-20)^2+4(6m+8)(m^4+4m^3+5m^2+2m)}>$$
$$\sqrt{(2m^3-2m^2-20m-20)^2}=\left|2m^3-2m^2-20m-20\right|\ge$$
$$2m^3-2m^2-20m-20\Rightarrow 2(6m+8)n>2m^3-2m^2-20m-20-\sqrt{\delta}$$

$\Rightarrow$ ② is true (strict inequ*a*lity) $\therefore$ ① *and* ② *are* both true $\Rightarrow$ $(*)$ is true

$$\therefore\frac{h_b+h_c}{2r_a}+\frac{2r_a}{h_b+h_c}\le\frac{R}{r}\text{ and again, }\frac{r_b+r_c}{2h_a}+\frac{2h_a}{r_b+r_c}\overset{?}{\le}\frac{R}{r}$$

$$\Leftrightarrow \frac{\frac{1}{s-b}+\frac{1}{s-c}}{\frac{4}{a}}+\frac{\frac{4}{a}}{\frac{1}{s-b}+\frac{1}{s-c}} \overset{?}{\leq} \frac{abc}{4(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow \frac{a^4+16(s-b)^2(s-c)^2}{4a^2(s-b)(s-c)} \overset{?}{\leq} \frac{abc}{4(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow (y+z)^3(x^2+x(y+z)+yz) \overset{?}{\geq} x((y+z)^4+16y^2z^2)$$

$$\Leftrightarrow (mx)^3(x^2+x(mx)+nx^2) \overset{?}{\geq} x((mx)^4+16n^2x^4) \Leftrightarrow m^3(n+1) \boxed{\underset{(**)}{\overset{?}{\geq}}} 16n^2$$

$$\text{Now, via AM} - \text{GM}, m^2 \geq 4n \therefore m^3(n+1) \geq 4n\left(2\sqrt{n}\right)\left(2\sqrt{n}\right) = 16n^2$$

$$\Rightarrow (**) \text{ is true } \therefore \frac{r_b+r_c}{2h_a}+\frac{2h_a}{r_b+r_c} \leq \frac{R}{r} \text{ and so,}$$

$$\max\left\{\frac{h_b+h_c}{2r_a}+\frac{2r_a}{h_b+h_c}, \frac{r_b+r_c}{2h_a}+\frac{2h_a}{r_b+r_c}\right\} \leq \frac{R}{r}\ \forall\, \Delta\, ABC,$$
$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4022. In ΔABC the following relationship holds:

$$\max\left\{\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}, \frac{a+b}{a+c}+\frac{a+c}{a+b}\right\} \leq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}+\sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \leq \frac{w_b}{w_c}+\frac{w_c}{w_b}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Nepal

Here, WLOG assume that $b \geq c \Rightarrow B \geq C \Longrightarrow All \left\{\sqrt{\frac{b}{c}}, \frac{a+b}{a+c}, \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}, \frac{w_c}{w_b}\right\} \geq 1$

The function $f(x) = x+\frac{1}{x}$ *is strictly increasing for* $x \geq 1$.

$\therefore$ *In order to prove the original inequality, it suffices to prove that*:

$$\max\left\{\sqrt{\frac{b}{c}}, \frac{a+b}{a+c}\right\} \leq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} \leq \frac{w_c}{w_b}$$

Via Sine Law: $\frac{\sin B}{\sin C} = \frac{b}{c} \Rightarrow \frac{2\sin\frac{B}{2}\cos\frac{B}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}} = \frac{b}{c} \Rightarrow \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}} = \frac{b}{c}\cdot\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}$

$$\text{Since } B \geq C \Rightarrow \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \geq 1, \Leftrightarrow \frac{\sin\frac{B}{2}}{\sin\frac{C}{2}} \geq \frac{b}{c} \cdot 1 \Leftrightarrow \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} \overbrace{\geq}^{①} \sqrt{\frac{b}{c}}$$

$$\text{Now, Via HAS Formula: } \sin^2\frac{B}{2} = \frac{(s-a)(s-c)}{ac} \Rightarrow \frac{\sin^2\frac{B}{2}}{\sin^2\frac{C}{2}}$$

$$= \frac{b(a+b-c)}{c(a+c-b)} \overbrace{\geq}^{?} \left(\frac{a+b}{a+c}\right)^4$$

$$\Leftrightarrow (b-c)\underbrace{\left[a^5 + a^4(b+c) + 2a^3bc + 2a^2bc(b+c) + abc(3b^2 - bc + 3c^2) + bc(b^3 + c^3)\right]}_{>0} \overbrace{\geq}^{?} 0$$

$$\text{which is trivially true} \because b \geq c \quad \Rightarrow \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} \overbrace{\geq}^{②} \frac{a+b}{a+c}$$

$$\text{Finally, } \frac{w_c}{w_b} = \frac{b(a+c)\cos\frac{C}{2}}{c(a+b)\cos\frac{B}{2}} = \left(\frac{b \cdot \cos\frac{C}{2}}{c \cdot \cos\frac{B}{2}}\right) \cdot \frac{a+c}{a+b} = \left(\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}\right) \cdot \frac{a+c}{a+b} \overbrace{\geq}^{\text{Via } ②} \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}}$$

$$\longmapsto ③$$

$$\therefore \text{ Via } ①, ② \text{ and } ③: \ max\left\{\sqrt{\frac{b}{c}}, \frac{a+b}{a+c}\right\} \leq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} \leq \frac{w_c}{w_b}. \ \text{Which follows:}$$

$$max\left\{\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}}, \frac{a+b}{a+c} + \frac{a+c}{a+b}\right\} \leq \sqrt{\frac{\sin\frac{B}{2}}{\sin\frac{C}{2}}} + \sqrt{\frac{\sin\frac{C}{2}}{\sin\frac{B}{2}}} \leq \frac{w_b}{w_c} + \frac{w_c}{w_b}$$

Equality holds if the triangle is isosceles $(b = c)$.

4023. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{r_b + r_c}{2r_a}} + \sqrt{\frac{2r_a}{r_b + r_c}} \leq \sqrt{\frac{2R}{r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{r_b + r_c}{2r_a} = \frac{4R\cos^2\frac{A}{2}}{2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.\tan\frac{A}{2}} = \frac{1 - S^2}{S(C+S)}$$

$$\left(C=\cos\frac{B-C}{2} \text{ and } S=\sin\frac{A}{2}\right) \therefore \frac{r_b+r_c}{2r_a} \overset{①}{=} \frac{1-S^2}{S(C+S)} \text{ and } \frac{2R}{r} = \frac{2R}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}$$

$$= \frac{1}{S(C-S)} \therefore \frac{2R}{r} \overset{②}{=} \frac{1}{S(C-S)} \text{ and now, } \sqrt{\frac{r_b+r_c}{2r_a}}+\sqrt{\frac{2r_a}{r_b+r_c}} \overset{?}{\le} \sqrt{\frac{2R}{r}}$$

$$\Leftrightarrow \frac{r_b+r_c}{2r_a}+\frac{2r_a}{r_b+r_c}+2 \overset{?}{\le} \frac{2R}{r} \overset{\text{via ① and ②}}{\Leftrightarrow} \frac{1-S^2}{S(C+S)}+\frac{S(C+S)}{1-S^2}+2\overset{?}{\le}\frac{1}{S(C-S)}$$

$$\Leftrightarrow 2-2C^2-S^2+C^2S^2+CS-C^3S \overset{?}{\ge} 0$$

$$\Leftrightarrow 2(1-C^2)-S^2(1-C^2)+CS(1-C^2)\overset{?}{\ge}0 \Leftrightarrow (1-C^2)\big(2+S(C-S)\big)\overset{?}{\ge}0$$

$$\rightarrow \text{true} \because C^2=\cos^2\frac{B-C}{2}\le 1 \text{ and } C-S=\frac{b+c}{a}.\sin\frac{A}{2}-\sin\frac{A}{2}>\sin\frac{A}{2}-\sin\frac{A}{2}=0$$

$$\therefore \sqrt{\frac{r_b+r_c}{2r_a}}+\sqrt{\frac{2r_a}{r_b+r_c}}\le\sqrt{\frac{2R}{r}}\ \forall\,\Delta\,ABC, ''='' \text{ iff } b=c \text{ (QED)}$$

4024. For $x=(4r^2+(b-c)^2)^{\frac{1}{2}}$ and analogs,

in any Δ ABC, the following relationship holds :

$$\frac{AI+BI+CI}{r}\le\sum_{cyc}\sqrt{\frac{yz}{\left(p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}}-y\right)\left(p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}}-z\right)}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a \overset{?}{\ge} p_a^2+\frac{(b-c)^2}{18} \overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\ge}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$
$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\ge}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$
$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\geq}0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\Leftrightarrow\frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left(\begin{matrix}(s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+\\1701a^4\end{matrix}\right)}{324a(2s+a)^4}.(b-c)^2\overset{?}{\geq}0$$

$$\rightarrow \text{true (strict inequality)}\ \therefore m_a n_a\geq p_a^2+\frac{(b-c)^2}{18}\geq p_a^2\Rightarrow m_a n_a\geq p_a^2\rightarrow ①$$

$$\text{and } n_a g_a\overset{\text{Bogdan Fustei}}{\geq} m_a l_a\rightarrow ②\ \therefore ①\bullet② \Rightarrow (m_a n_a)(n_a g_a)\geq p_a^2.m_a l_a$$

$$\Rightarrow n_a\geq p_a.\sqrt{\frac{l_a}{g_a}}\ \text{and analogs and also, } x=(4r^2+(b-c)^2)^{\frac{1}{2}}$$

$$\overset{\text{Bogdan Fustei}}{=}\frac{an_a}{s}\ \text{and analogs}\ \therefore\sum_{cyc}\sqrt{\frac{yz}{\left(p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}}-y\right)\left(p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}}-z\right)}}=$$

$$\sum_{cyc}\frac{1}{\sqrt{\left(\frac{p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}}}{y}-1\right)\left(\frac{p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}}}{z}-1\right)}}\geq\sum_{cyc}\frac{1}{\sqrt{\left(\frac{n_b}{\left(\frac{bn_b}{s}\right)}-1\right)\left(\frac{n_c}{\left(\frac{cn_c}{s}\right)}-1\right)}}=$$

$$\sum_{cyc}\sqrt{\frac{bc}{(s-b)(s-c)}}=\sum_{cyc}\frac{1}{\sin\frac{A}{2}}=\frac{1}{r}.\sum_{cyc}\frac{r}{\sin\frac{A}{2}}=\frac{1}{r}.\sum_{cyc}AI\ \therefore\frac{AI+BI+CI}{r}\leq$$

$$\sum_{cyc}\sqrt{\frac{yz}{\left(p_b\left(\frac{l_b}{g_b}\right)^{\frac{1}{2}}-y\right)\left(p_c\left(\frac{l_c}{g_c}\right)^{\frac{1}{2}}-z\right)}}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

4025. In any Δ ABC the fol$l$owing rel$a$tionship holds :

$$\frac{r_b+r_c}{2w_a}+\frac{2w_a}{r_b+r_c}\le\frac{2s}{3\sqrt{3}r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$ & then : $a = y + z, b = z + x, c = x + y$

and $s = x + y + z$ and furthermore, we denote : $\frac{y+z}{x} = m$ and $\frac{yz}{x^2} = n$ and

then, we have the fol$l$owing set "S" of rel$a$tions : $y^2+z^2 = x^2(m^2-2n)$,

$$y^3+z^3 = x^3(m^3-3nn), y^4+z^4 = x^4((m^2-2n)^2-2n^2),$$

$$y^5+z^5 = x^5\left(m((m^2-2n)^2+n^2-nm^2)\right), y^6+z^6 = x^6((m^3-3nn)^2-2n^3),$$

$$y^7+z^7 = x^7\left(m((m^3-3nn)^2-2n^3)-mn((m^2-2n)^2+n^2-nm^2)\right),$$

$$y^8+z^8 = x^8(((m^2-2n)^2-2n^2)^2-2n^4) \text{ and now, } \frac{r_b+r_c}{2w_a} = \frac{\frac{rsa(s-a)}{(s-b)(s-c)(s-a)}}{\frac{4\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}}$$

$$= \frac{rs.a(s-a)}{r^2s.\frac{4\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)}} \Rightarrow \left(\frac{r_b+r_c}{2w_a}\right)^2 = \frac{a^2(s-a)^2}{r^2s.\frac{16bc}{(b+c)^2}\cdot(s-a)}$$

$$= \frac{a^2(s-a)^2}{(s-a)(s-b)(s-c).\frac{16bc}{(b+c)^2}\cdot(s-a)} \Rightarrow \left(\frac{r_b+r_c}{2w_a}\right)^2 = \frac{a^2(b+c)^2}{16bc(s-b)(s-c)}$$

$$\text{and so, } \frac{r_b+r_c}{2w_a}+\frac{2w_a}{r_b+r_c} \overset{?}{\le} \frac{2s}{3\sqrt{3}r} \overset{\text{squaring}}{\Leftrightarrow} \left(\frac{r_b+r_c}{2w_a}\right)^2+\left(\frac{2w_a}{r_b+r_c}\right)^2+2$$

$$\overset{?}{\le} \frac{4s^2}{27r^2} = \frac{4s^3}{27(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow \frac{a^2(b+c)^2}{16bc(s-b)(s-c)}+\frac{16bc(s-b)(s-c)}{a^2(b+c)^2}+2 \overset{?}{\le} \frac{4s^3}{27(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow \frac{(y+z)^2(2x+y+z)^2}{16yz(z+x)(x+y)}+\frac{16yz(z+x)(x+y)}{(y+z)^2(2x+y+z)^2}+2 \overset{?}{\le} \frac{4(x+y+z)^3}{27xyz}$$

$$\Leftrightarrow 256x^7(y+z)^2+1280x^6(y+z)^3+2192x^5(y^2+z^2)^2-4544x^5y^2z^2+$$

$$5568x^5yz(y^2+z^2)+1952x^4\left(y^5+z^5\right)+3872x^4yz(y^3+z^3)-$$

$$11968x^4y^2z^2(y+z)+1016x^3(y^6+z^6)+3376x^3yz(y^4+z^4)-$$

$$6008x^3y^2z^2(y^2+z^2)-30560x^3y^3z^3+296x^2(y^7+z^7)+2424x^2yz\left(y^5+z^5\right)+$$

$$4520x^2y^2z^2(y^3+z^3)-10312x^2y^3z^3(y+z)+37x(y^8+z^8)+744xyz(y^6+z^6)$$

$$+2860xy^2z^2(y^4+z^4)+5336xy^3z^3(y^2+z^2)-546xy^4z^4+64yz(y+z)^7z^4\boxed{\overset{?}{\underbrace{\geq}_{\textcircled{1}}}}0$$

& via set of relations "S", to prove ①, suffices to prove, following simplification

$$\overbrace{(6912+13824m+10368m^2+3456m^3+864m^4)}^{\Omega_1}n^2-$$

$$\overbrace{(64m^7+448m^6+352m^5-2720m^4-5888m^3-3200m^2)}^{\Omega_2}n-$$

$$\overbrace{(37m^8+296m^7+1016m^6+1952m^5+2192m^4+1280m^3+256m^2)}^{\Omega_3}+$$

$$6912n^4+13824(m+1)n^3\boxed{\overset{?}{\underbrace{\leq}_{\textcircled{2}}}}0 \text{ and } \because 6912n^4+13824(m+1)n^3$$

$$\overset{AM-GM}{\leq}6912n^2\left(\frac{m^4}{16}\right)+13824(m+1)n^2\left(\frac{m^2}{4}\right) \therefore \text{in order to prove } \textcircled{2},$$

$$\text{it suffices to prove}: \Omega_1.n^2-\Omega_2.n-\Omega_3+432n^2m^4+3456(m+1)m^2n^2\overset{?}{\leq}0$$

$$\Leftrightarrow\overbrace{(6912+13824m+13824m^2+6912m^3+1296m^4)}^{\sigma_1}n^2-$$

$$\overbrace{(64m^7+448m^6+352m^5-2720m^4-5888m^3-3200m^2)}^{\sigma_2}n-$$

$$\overbrace{(37m^8+296m^7+1016m^6+1952m^5+2192m^4+1280m^3+256m^2)}^{\sigma_3}\boxed{\overset{?}{\underbrace{\leq}_{\textcircled{3}}}}0$$

Now, LHS of ③ is a quadratic polynomial in "n" with discriminant,

$\boxed{\delta}=\sigma_2^2+4\sigma_1\sigma_3\,\boxed{>0}$ ($\because \sigma_1,\sigma_3>0$) and so, in order to prove ③,

$$\text{it suffices to prove}: 2\sigma_1.n\boxed{\overset{?}{\underbrace{\leq}_{(m)}}}\sigma_2+\sqrt{\delta} \text{ AND } 2\sigma_1.n\boxed{\overset{?}{\underbrace{\geq}_{(n)}}}\sigma_2-\sqrt{\delta}$$

Now, since $n\overset{AM-GM}{\leq}\frac{m^2}{4}$ ∴ in order to prove (m), it suffices to prove :

$$\sigma_1 . \frac{m^2}{2} \overset{?}{\leq} \sigma_2 + \sqrt{\delta} \Leftrightarrow -8m^2(m+2)^2(8m^3 - 57m^2 - 192m - 208) \overset{?}{\leq} \sqrt{\delta}$$

and it's trivi$ally$ true if : $8m^3 - 57m^2 - 192m - 208 \geq 0$ and when :

$8m^3 - 57m^2 - 192m - 208 < 0$, *then it suffices to prove* :

$$64m^4(m+2)^4(8m^3 - 57m^2 - 192m - 208)^2 \overset{?}{\leq} \sigma_2^2 + 4\sigma_1\sigma_3$$

$$\Leftrightarrow \boxed{6912m^2(4m+1)(3m^2+4m+4)(m+2)^6(m-2)^2 \overset{?}{\geq} 0} \to \text{true} \because m > 0$$

$\Rightarrow$ (m) is true and $also$, $2\sigma_1 . n + \sqrt{\delta} > \sqrt{\delta} = \sqrt{\sigma_2^2 + 4\sigma_1\sigma_3} > \sqrt{\sigma_2^2} = |\sigma_2| \geq \sigma_2$

$\Rightarrow 2\sigma_1 . n > \sigma_2 - \sqrt{\delta} \Rightarrow$ (n) is true (strict inequality) $\therefore$ (m) and (n) are both true

$\Rightarrow$ ③ $\Rightarrow$ ② $\Rightarrow$ ① is true $\therefore \frac{r_b + r_c}{2w_a} + \frac{2w_a}{r_b + r_c} \leq \frac{2s}{3\sqrt{3}r}$ $\forall \Delta$ ABC,

" =" iff Δ ABC is equilateral (QED)

4026. In any Δ ABC the following relationship ho$l$ds :

$$3r\sqrt{\frac{3R}{2}} \leq w_a . \sqrt{r_a} \leq m_a . \sqrt{r_a} \leq \frac{s^2}{3\sqrt{3r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a . \sqrt{r_a} \overset{?}{\leq} \frac{s^2}{3\sqrt{3r}} \overset{\text{squaring}}{\Leftrightarrow} \left(\frac{(b-c)^2 + 4s(s-a)}{4}\right) . \frac{rs}{s-a} \overset{?}{\leq} \frac{s^4}{27r}$$

$$\Leftrightarrow \frac{(b-c)^2 + 4s(s-a)}{4} \overset{?}{\leq} \frac{s^4(s-a)}{27(s-a)(s-b)(s-c)}$$

$$\Leftrightarrow 4(x+y+z)^4 \overset{?}{\geq} 27yz\left((y-z)^2 + 4x(x+y+z)\right)$$

$$\left(\begin{matrix} x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \\ and\ s = x+y+z \end{matrix}\right)$$

$$\Leftrightarrow 4x^4 + 16x^3(y+z) + 24x^2(y+z)^2 - 108x^2 . yz + 16x(y+z)^3 - 108xyz(y+z)$$

$$+4((y+z)^2 - 2yz)^2 + 70x^2y^2z^2 - 11yz((y+z)^2 - 2yz) \overset{?}{\geq} 0$$

$$\Leftrightarrow 4x^4 + 16x^3(mx) + 24x^2(mx)^2 - 108x^2 . nx^2 + 16x(mx)^3 - 108x . nx^2(mx)$$

$$+4((mx)^2 - 2nx^2)^2 + 70x^2 . n^2x^4 - 11nx^2((mx)^2 - 2nx^2) \overset{?}{\geq} 0$$

$$\left(\frac{y+z}{x} = m \text{ and } \frac{yz}{x^2} = n\right) \Leftrightarrow 108n^2 - 27(m+2)^2 . n + 4(m+1)^4 \boxed{\underset{\textcircled{1}}{\underbrace{\overset{?}{\geq}}}} 0$$

Now, LHS of ① is a quadratic polynomial in "n" with discriminant, $\delta = 729(m+2)^4 - (16)(108)(m+1)^4$

$$= -27(37m^4 + 40m^3 - 264m^2 - 608m - 368) \le 0 \text{ if :}$$

$37m^4 + 40m^3 - 264m^2 - 608m - 368 \ge 0$ and then ① is definitely true and when : $\boxed{37m^4 + 40m^3 - 264m^2 - 608m - 368 < 0}$, then : $\delta > 0$

and then, in order to prove ①, it suffices to prove : $216n \overset{?}{\le} 27(m+2)^2 - \sqrt{\delta}$

and since $n \overset{\text{AM-GM}}{\le} \frac{m^2}{4}$ ∴ it suffices to prove : $54m^2 \overset{?}{\le} 27(m+2)^2 - \sqrt{\delta}$

$$\Leftrightarrow \sqrt{\delta}\ \boxed{\overset{?}{\le}\ ②}\ \overset{?}{\le} 27(4 + 4m - m^2)$$

Now, $37m^4 + 40m^3 - 264m^2 - 608m - 368$

$$= (m^2 - 4m - 4)(37m^2 + 188m + 636) + 2688m + 2176$$

$$\Rightarrow m^2 - 4m - 4 = \frac{\overbrace{(37m^4 + 40m^3 - 264m^2 - 608m - 368)}^{<0} - 2688m - 2176}{37m^2 + 188m + 636}$$

$< 0 \Rightarrow 4 + 4m - m^2 > 0$ and so, following squaring, ② becomes :

$$729(4 + 4m - m^2)^2 \overset{?}{\ge} -27(37m^4 + 40m^3 - 264m^2 - 608m - 368)$$

$$\Leftrightarrow \boxed{432(m+1)(4m+1)(m-2)^2 \overset{?}{\ge} 0} \to \text{true} \because m > 0 \Rightarrow ② \Rightarrow ① \text{ is true}$$

$$\therefore \boxed{m_a \cdot \sqrt{r_a} \le \frac{s^2}{3\sqrt{3r}}} \text{ and again, } w_a \cdot \sqrt{r_a} \overset{?}{\ge} 3r.\sqrt{\frac{3R}{2}} \overset{\text{squaring}}{\Leftrightarrow}$$

$$\left(s(s-a).\frac{4bc}{(b+c)^2}\right).\frac{rs}{s-a} \overset{?}{\ge} 9r^2.\frac{3abc}{8rs}$$

$$\Leftrightarrow 32(x+y+z)^3 \overset{?}{\ge} 27(y+z)(2x+y+z)^2$$

$$\Leftrightarrow 32x^3 - 12x^2(y+z) - 12x(y+z)^2 + 5(y+z)^3 \overset{?}{\ge} 0$$

$$\Leftrightarrow 32x^3 - 12x^2(mx) - 12x(mx)^2 + 5(mx)^3 \overset{?}{\ge} 0 \Leftrightarrow \boxed{(5m+8)(m-2)^2 \overset{?}{\ge} 0}$$

$$\to \text{true} \because m > 0 \therefore \boxed{w_a \cdot \sqrt{r_a} \ge 3r.\sqrt{\frac{3R}{2}}} \text{ and } \because w_a \le m_a, \text{ hence :}$$

$$3r.\sqrt{\frac{3R}{2}} \le w_a \cdot \sqrt{r_a} \le m_a \cdot \sqrt{r_a} \le \frac{s^2}{3\sqrt{3r}},$$

" = " for $m_a \cdot \sqrt{r_a} \le \frac{s^2}{3\sqrt{3r}}$ iff Δ ABC is equilateral, " = " for $w_a \cdot \sqrt{r_a} \ge 3r.\sqrt{\frac{3R}{2}}$ iff $b + c = 2a$ and " = " for $w_a \cdot \sqrt{r_a} \le m_a \cdot \sqrt{r_a}$ iff $b = c$ (QED)

4027. In any Δ ABC the following relationship holds :

$$g_a \leq \sqrt{\frac{R+2r}{4r}}h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$g_a^2 \overset{?}{\leq} \left(\frac{R}{4r}+\frac{1}{2}\right).h_a^2 \overset{\text{via Bogdan Fustei}}{\Leftrightarrow} s(s-a)-\frac{s-a}{a}.(b-c)^2 \overset{?}{\leq}$$

$$\left(\frac{abc}{16(s-a)(s-b)(s-c)}+\frac{1}{2}\right)\left(s(s-a)-\frac{s(s-a)}{a^2}.(b-c)^2\right)$$

$$\Leftrightarrow \frac{sa-(b-c)^2}{a} \overset{?}{\leq} s.\frac{abc+8(s-a)(s-b)(s-c)}{16(s-a)(s-b)(s-c)}.\frac{4(s-b)(s-c)}{a^2}$$

$$\Leftrightarrow \big((y+z)(z+x)(x+y)+8xyz\big)(x+y+z) \overset{?}{\geq} 4(y+z)x\left(\begin{matrix}(y+z)(x+y+z)-\\(y-z)^2\end{matrix}\right)$$

$$\left(\begin{matrix}x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \text{ and}\\ s=x+y+z\end{matrix}\right)$$

$$\Leftrightarrow x^3(y+z)-2x^2(y+z)^2+8x^2.yz+x\left((y+z)^3-3yz(y+z)\right)-$$

$$4x.yz(y+z)+yz(y+z)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow x^3(mx)-2x^2(mx)^2+8x^2.nx^2+x\left((mx)^3-3nx^2(mx)\right)-$$

$$4x.nx^2(mx)+nx^2(mx)^2 \overset{?}{\geq} 0 \left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right)$$

$$\Leftrightarrow n(m^2-7m+8)+m(m-1)^2 \boxed{\underbrace{\overset{?}{\geq}}_{(*)}} 0 \text{ and it's trivially true if :}$$

$$m^2-7m+8 \geq 0 \text{ and when : } m^2-7m+8<0, \text{then : since } n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$$

$$\therefore \text{ in order to prove } (*), \text{it suffices to prove : } \frac{m^2}{4}.(m^2-7m+8)+m(m-1)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (m+1)(m-2)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \because m>0 \text{ as } x,y,z>0 \Rightarrow (*) \text{ is true}$$

$$\therefore g_a \leq \sqrt{\frac{R+2r}{4r}}.h_a, ''='' \text{ iff } y=z \text{ and } y+z=2x \Rightarrow$$

$$''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

4028. In any Δ ABC the following relationship holds :

$$p_a \leq \frac{\sqrt{9s^2-48Rr-66r^2}}{9r}.h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\mathrm{p}_a^2 \overset{?}{\leq} \frac{9\mathrm{s}^2 - 48\mathrm{Rr} - 66\mathrm{r}^2}{81\mathrm{r}^2}.\mathrm{h}_a^2 \overset{\substack{\text{via Bogdan Fustei} \\ \text{and} \\ \text{Mohamed Amine Ben Ajiba}}}{\Leftrightarrow}$$

$$\mathrm{s}(\mathrm{s}-a) + \frac{\mathrm{s}(3\mathrm{s}+a)}{(2\mathrm{s}+a)^2}.(\mathrm{b}-\mathrm{c})^2 \overset{?}{\leq}$$

$$\left(\frac{\dfrac{9\mathrm{s}^3}{81(\mathrm{s}-a)(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})} - }{\dfrac{48a\mathrm{bc}}{81.4(\mathrm{s}-a)(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})} - \dfrac{66}{81}}\right)\left(\frac{4\mathrm{s}(\mathrm{s}-a)(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})}{a^2}\right)$$

$$\Leftrightarrow \frac{x\big(\mathrm{y}+\mathrm{z}+2(x+\mathrm{y}+\mathrm{z})\big)^2 + \big(\mathrm{y}+\mathrm{z}+3(x+\mathrm{y}+\mathrm{z})\big)(\mathrm{y}-\mathrm{z})^2}{\big(\mathrm{y}+\mathrm{z}+2(x+\mathrm{y}+\mathrm{z})\big)^2}$$

$$\overset{?}{\leq} \frac{12(x+\mathrm{y}+\mathrm{z})^3 - 16(\mathrm{y}+\mathrm{z})(\mathrm{z}+x)(x+\mathrm{y}) - 88x\mathrm{yz}}{27(\mathrm{y}+\mathrm{z})^2}$$

$$\left(\begin{matrix} x = \mathrm{s}-a, \mathrm{y} = \mathrm{s}-\mathrm{b}, \mathrm{z} = \mathrm{s}-\mathrm{c} \Rightarrow a = \mathrm{y}+\mathrm{z}, \mathrm{b} = \mathrm{z}+x, \mathrm{c} = x+\mathrm{y} \ and \\ \mathrm{s} = x+\mathrm{y}+\mathrm{z} \end{matrix}\right)$$

$$\Leftrightarrow 12x^5 + 56x^4(\mathrm{y}+\mathrm{z}) + 80x^3(\mathrm{y}+\mathrm{z})^2 - 88x^3.\mathrm{n}x^2 +$$

$$36x^2\left((\mathrm{m}x)^3 - 3\mathrm{n}x^2(\mathrm{m}x)\right) - 172x^2.\mathrm{n}x^2.\mathrm{m}x - 165x.\mathrm{n}x^2.(\mathrm{m}x)^2 +$$

$$72.\mathrm{n}x^2.\left((\mathrm{m}x)^3 - 3\mathrm{n}x^2(\mathrm{m}x)\right) + 216\mathrm{n}^2x^4.\mathrm{m}x \overset{?}{\geq} 0 \ \left(\mathrm{m} = \frac{\mathrm{y}+\mathrm{z}}{x}, \mathrm{n} = \frac{\mathrm{yz}}{x^2}\right)$$

$$\Leftrightarrow \mathrm{n}(72\mathrm{m}^3 - 165\mathrm{m}^2 - 280\mathrm{m} - 88) + 36\mathrm{m}^3 + 80\mathrm{m}^2 + 56\mathrm{m} + 12 \boxed{\underbrace{\overset{?}{\geq}}_{(*)}} 0 \ and \ \text{it's}$$

$$\text{trivially true if} : 72\mathrm{m}^3 - 165\mathrm{m}^2 - 280\mathrm{m} - 88 \geq 0 \ \ and \ \text{when} :$$

$$72\mathrm{m}^3 - 165\mathrm{m}^2 - 280\mathrm{m} - 88 < 0, then : \text{since n} \overset{\text{AM-GM}}{\leq} \frac{\mathrm{m}^2}{4}$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove} :$$

$$\frac{\mathrm{m}^2}{4}.(72\mathrm{m}^3 - 165\mathrm{m}^2 - 280\mathrm{m} - 88) + 36\mathrm{m}^3 + 80\mathrm{m}^2 + 56\mathrm{m} + 12 \overset{?}{\geq} 0$$

$$\Leftrightarrow (8\mathrm{m}+3)(3\mathrm{m}+2)^2(\mathrm{m}-2)^2 \overset{?}{\geq} 0 \to \text{true} \because \mathrm{m} > 0 \ as \ x, \mathrm{y}, \mathrm{z} > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore \mathrm{p}_a \leq \frac{\sqrt{9\mathrm{s}^2 - 48\mathrm{Rr} - 66\mathrm{r}^2}}{9\mathrm{r}}.\mathrm{h}_a \, '' ='' \ \text{iff } \mathrm{y} = \mathrm{z} \ and \ \mathrm{y}+\mathrm{z} = 2x \Rightarrow$$

$$'' ='' \ \text{iff } \Delta\,\mathrm{ABC} \text{ is equilateral (QED)}$$

4029. In any Δ ABC the following relationship holds :

$$g_{\mathrm{b}}g_{\mathrm{c}} \leq \mathrm{r}\left(\mathrm{r}_a + 2a\cos\frac{\mathrm{A}}{2}\right)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$g_a \overset{\text{Bogdan Fustei}}{=} \sqrt{(s-a)^2+\frac{4(s-a)(s-b)(s-c)}{a}} > s-a \text{ and analogs}$$

$$\text{and so, } g_b g_c > (s-b)(s-c) = rr_a \therefore g_b g_c \overset{?}{\le} r\left(r_a + 2a.\cos\frac{A}{2}\right)$$

$$\Leftrightarrow (g_b g_c - rr_a)^2 \overset{?}{\le} 4r^2a^2.\frac{s(s-a)}{bc} \overset{\text{via Bogdan Fustei}}{\Leftrightarrow}$$

$$\left((s-b)^2+\frac{4(s-a)(s-b)(s-c)}{b}\right)\left((s-c)^2+\frac{4(s-a)(s-b)(s-c)}{c}\right)+$$

$$(s-b)^2(s-c)^2-\frac{4(s-a)^2(s-b)(s-c)a^2}{bc} \overset{?}{\le} 2g_bg_c(s-b)(s-c)$$

$$\Leftrightarrow \frac{(y(z+x)+4zx)(z(x+y)+4xy)+yz(z+x)(x+y)-4x^2(y+z)^2}{(z+x)(x+y)} \overset{?}{\le} 2g_bg_c$$

$$(x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y)$$

$$\Leftrightarrow \frac{2yz(5x^2+3xy+3xz+yz)}{(z+x)(x+y)} \overset{?}{\le} 2g_bg_c \overset{\text{via Bogdan Fustei}}{\Leftrightarrow}$$

$$\frac{y^2z^2(5x^2+3xy+3xz+yz)^2}{(z+x)^2(x+y)^2} \overset{?}{\le} \frac{yz(y(z+x)+4zx)(z(x+y)+4xy)}{(z+x)(x+y)}$$

$$\Leftrightarrow 4x^3\left(x(y-z)^2+y^3+z^3-yz(y+z)\right) \overset{?}{\ge} 0 \Leftrightarrow 4x^3(x+y+z)(y-z)^2 \overset{?}{\ge} 0$$

$$\rightarrow \text{true} \because x,y,z>0 \therefore g_bg_c \le r\left(r_a+2a.\cos\frac{A}{2}\right)\ \forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4030. In any Δ ABC the following relationship holds :

$$\frac{m_b}{h_c}+\frac{m_c}{h_b}\le\frac{\sqrt{s^2-11r^2}}{2r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_am_b \overset{?}{\le} \frac{2c^2+ab}{4} \Leftrightarrow \left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right)\overset{?}{\le}\frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2\overset{?}{\ge}0 \Leftrightarrow (a-b)^2(a+b+c)(a+b-c)\overset{?}{\ge}0$$

$$\rightarrow \text{true} \Rightarrow m_am_b\le\frac{2c^2+ab}{4} \text{ and analogs } \therefore \left(\frac{m_b}{h_c}+\frac{m_c}{h_b}\right)^2 = \frac{m_b^2}{h_c^2}+\frac{m_c^2}{h_b^2}+\frac{2m_bm_c}{h_bh_c}\le$$

$$\frac{c^2(4s(s-b)+(c-a)^2)}{16r^2s^2}+\frac{b^2(4s(s-c)+(a-b)^2)}{16r^2s^2}+\frac{2bc\left(\frac{2a^2+bc}{4}\right)}{4r^2s^2}\overset{?}{\leq}\frac{s^2-11r^2}{4r^2}$$

$$\Leftrightarrow 4\left(\sum_{cyc}x\right)^4-44xyz\left(\sum_{cyc}x\right)\overset{?}{\geq}4\left(\sum_{cyc}x\right)y(x+y)^2+(x+y)^2(z-x)^2+$$

$$4\left(\sum_{cyc}x\right)z(z+x)^2+(z+x)^2(x-y)^2+4(y+z)^2(z+x)(x+y)+$$

$$2(z+x)^2(x+y)^2\ \left(\begin{matrix}x=s-a,y=s-b,z=s-c\Rightarrow\\ a=y+z,b=z+x,c=x+y\text{ and }s=x+y+z\end{matrix}\right)$$

$$\Leftrightarrow 2x^3(y+z)+x^2(y+z)^2-5x^2.yz-5x.yz(y+z)+2yz(y+z)^2-y^2z^2\overset{?}{\geq}0$$

$$\Leftrightarrow 2x^3(mx)+x^2(mx)^2-5x^2.nx^2-5x.nx^2.mx+2nx^2(mx)^2-n^2x^4\overset{?}{\geq}0$$

$$\left(\frac{y+z}{x}=m\text{ and }\frac{yz}{x^2}=n\right)\Leftrightarrow n^2-(2m^2-5m-5)n-(m^2+2m)\boxed{\overset{?}{\underset{①}{\leq}}}0$$

Now, LHS of ① is a quadratic polynomial in "n" with discriminant, $\delta=(2m^2-5m-5)^2+4(m^2+2m)>0$ and so, in order to prove ①, it suffices to prove :

$$2n\boxed{\overset{?}{\underset{(*)}{\leq}}}2m^2-5m-5+\sqrt{\delta}\text{ AND }2n\boxed{\overset{?}{\underset{(**)}{\geq}}}2m^2-5m-5-\sqrt{\delta}$$

Since $n\overset{AM-GM}{\leq}\frac{m^2}{4}$ ∴ in order to prove $(*)$, it suffices to prove :

$\frac{m^2}{2}\overset{?}{\leq}2m^2-5m-5+\sqrt{\delta}\Leftrightarrow 2\sqrt{\delta}\overset{?}{\geq}10m+10-3m^2$ and it's trivially true if : $10m+10-3m^2\leq 0$ and when : $10m+10-3m^2>0$, then it suffices to prove :

$$4(2m^2-5m-5)^2+16(m^2+2m)\overset{?}{\geq}(10m+10-3m^2)^2$$

$$\Leftrightarrow\boxed{m(7m+8)(m-2)^2\overset{?}{\geq}0}\rightarrow\text{true}\because m>0\Rightarrow(*)\text{ is true}$$

Again, $2n+\sqrt{\delta}>\sqrt{(2m^2-5m-5)^2+4(m^2+2m)}>\sqrt{(2m^2-5m-5)^2}=|2m^2-5m-5|\geq 2m^2-5m-5$ and so, $2n>2m^2-5m-5-\sqrt{\delta}$ $\Rightarrow(**)$ is true (strict inequality) ∴ both $(*)$ and $(**)$ are true ⇒ ① is true

$$\therefore\frac{m_b}{h_c}+\frac{m_c}{h_b}\leq\frac{\sqrt{s^2-11r^2}}{2r}\ \forall\Delta ABC,\text{"="}\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

4031. In any Δ ABC the following relationship holds :

$$m_bh_c+m_ch_b\leq 2m_ar_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a m_b \overset{?}{\le} \frac{2c^2+ab}{4} \Leftrightarrow \left(\frac{2b^2+2c^2-a^2}{4}\right)\left(\frac{2c^2+2a^2-b^2}{4}\right) \overset{?}{\le} \frac{(2c^2+ab)^2}{16}$$

$$\Leftrightarrow a^4+b^4-2a^2b^2-a^2c^2+2abc^2-b^2c^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow (a+b)^2(a-b)^2-c^2(a-b)^2 \overset{?}{\ge} 0 \Leftrightarrow (a-b)^2(a+b+c)(a+b-c) \overset{?}{\ge} 0$$

$$\to \text{true} \Rightarrow m_a m_b \le \frac{2c^2+ab}{4} \text{ and analogs}$$

$$\therefore (m_b h_c + m_c h_b)^2 = m_b^2 h_c^2 + m_c^2 h_b^2 + 2m_b m_c h_b h_c \le$$

$$\frac{(4s(s-b)+(c-a)^2)}{4}.\frac{4r^2s^2}{c^2}+\frac{b^2(4s(s-c)+(a-b)^2)}{4}.\frac{4r^2s^2}{b^2}+$$

$$2\left(\frac{2a^2+bc}{4}\right).\frac{4r^2s^2}{bc} \overset{?}{\le} 4m_a^2 r_a^2 = (4s(s-a)+(b-c)^2).\frac{r^2s^2}{(s-a)^2}$$

$$\Leftrightarrow (z+x)^2(x+y)^2\left(4x\left(\sum_{cyc} x\right)+(y-z)^2\right) \overset{?}{\ge}$$

$$x^2(z+x)^2\left(4y\left(\sum_{cyc} x\right)+(z-x)^2\right)+x^2(x+y)^2\left(4z\left(\sum_{cyc} x\right)+(x-y)^2\right)+$$

$$2(z+x)(x+y)x^2\left(2(y+z)^2+(z+x)(x+y)\right)$$

$$\left(\begin{matrix} x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \\ \text{and } s=x+y+z \end{matrix}\right)$$

$$\Leftrightarrow 4x^5(y+z)+5x^4(y+z)^2-20x^4.yz+2x^3\left((y+z)^3-3yz(y+z)\right)-$$

$$10x^3.yz(y+z)+2x^2.yz((y+z)^2-4yz)+2xyz(y+z)^3-$$

$$4xy^2z^2(y+z)+y^2z^2(y-z)^2 \overset{?}{\ge} 0 \text{ and } \because y^2z^2(y-z)^2 \ge 0 \therefore \text{it suffices to prove :}$$

$$4x^5(mx)+5x^4(mx)^2-20x^4.nx^2+2x^3\left((mx)^3-3nx^2(mx)\right)-10x^3.nx^2(mx)$$

$$+2x^2.nx^2((mx)^2-4nx^2)+2xnx^2(mx)^3-4x.n^2x^4(mx) \overset{?}{\ge} 0 \left(\begin{matrix} \frac{y+z}{x}=m \text{ and} \\ \frac{yz}{x^2}=n \end{matrix}\right)$$

$$\Leftrightarrow (8+4m)n^2-(2m^3+2m^2-16m-20)n-(2m^3+5m^2+4m) \boxed{\underbrace{\overset{?}{\le}}_{①}} 0$$

Now, LHS of ① is a quadratic polynomial in "n" with discriminant, $\delta = (2m^3+2m^2-16m-20)^2+4(8+4m)(2m^3+5m^2+4m) > 0$ and so, in order to prove ①, it suffices to prove :

$$2(8+4m)n \boxed{\underbrace{\overset{?}{\le}}_{(*)}} 2m^3+2m^2-16m-20+\sqrt{\delta} \text{ AND}$$

$$2(8+4m)n \boxed{\underset{(**)}{\overset{?}{\geq}}} 2m^3+2m^2-16m-20-\sqrt{\delta}$$

$$\text{Since } n \overset{AM-GM}{\leq} \frac{m^2}{4} \therefore \text{in order to prove } (*), \text{it suffices to prove:}$$

$$2(2+m)m^2 \overset{?}{\leq} 2m^3+2m^2-16m-20+\sqrt{\delta}$$

$$\Leftrightarrow 2m^2+16m+20 \overset{?}{\leq} \sqrt{\delta} \overset{\text{squaring}}{\Leftrightarrow}$$

$$(2m^3+2m^2-16m-20)^2+4(8+4m)(2m^3+5m^2+4m) \overset{?}{\geq}$$

$$(2m^2+16m+20)^2 \Leftrightarrow \boxed{4m(m+2)^3(m-2)^2 \overset{?}{\geq} 0} \to \text{true} \because m>0$$

$$\Rightarrow (*) \text{ is true and again}, 2(8+4m)n+\sqrt{\delta} >$$

$$\sqrt{(2m^3+2m^2-16m-20)^2+4(8+4m)(2m^3+5m^2+4m)} >$$

$$\sqrt{(2m^3+2m^2-16m-20)^2} = \left|2m^3+2m^2-16m-20\right| \geq$$

$$2m^3+2m^2-16m-20 \text{ and so}, 2(8+4m)n > 2m^3+2m^2-16m-20-\sqrt{\delta}$$

$$\Rightarrow (**) \text{ is true (strict inequality)} \therefore \text{both } (*) \text{ and } (**) \text{ are true} \Rightarrow ① \text{ is true}$$

$$\therefore m_b h_c + m_c h_b \leq 2m_a r_a \ \forall \Delta ABC, \text{"} = \text{" iff } \Delta ABC \text{ is equilateral (QED)}$$

4032. In ΔABC the following relationship holds:

$$\sum_{cyc} \sqrt[3]{\left(1+\frac{6r}{h_a}\right)\left(\frac{r}{h_b}\right)} \geq 3$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

By applying Hölder's *Inequality, we obtain*:

$$\sum_{cyc} \sqrt[3]{\left(1+\frac{6r}{h_a}\right)\left(\frac{r}{h_b}\right)} = \sum_{cyc}\left(\sqrt[3]{\left(1+\frac{6r}{h_a}\right)}\sqrt[3]{\left(\frac{r}{h_b}\right)}\cdot\sqrt[3]{1}\right) \leq \sqrt[3]{\sum_{cyc}\left(1+\frac{6r}{h_a}\right)\cdot\sum_{cyc}\left(\frac{r}{h_b}\right)\cdot\sum_{cyc}(1)}$$

Now,

$$\sqrt[3]{\sum_{cyc}\left(1+\frac{6r}{h_a}\right)\cdot\sum_{cyc}\left(\frac{r}{h_b}\right)\cdot\sum_{cyc}(1)} = \sqrt[3]{\left(3+\sum_{cyc}\frac{6r}{h_a}\right)\cdot\left(r\cdot\sum_{cyc}\frac{1}{h_b}\right)\cdot 3} =$$

$$= \sqrt[3]{\left(3+6r\cdot\sum_{cyc}\frac{1}{h_a}\right)\left(r\cdot\sum_{cyc}\frac{1}{h_b}\right)\cdot 3} = \sqrt[3]{\left(3+6r\cdot\frac{1}{r}\right)\left(r\cdot\frac{1}{r}\right)\cdot 3} = \sqrt[3]{27} = 3$$

$$\therefore \sum_{cyc} \sqrt[3]{\left(1+\frac{6r}{h_a}\right)\left(\frac{r}{h_b}\right)} \leq 3$$

Equality holds iff ΔABC is equilateral.

4033. **In any Δ ABC the following relationship holds :**

$$\left(\sum_{cyc}\frac{1}{h_a^3}+\frac{R}{2r^2s^2}\right)\left(\sum_{cyc}\frac{1}{h_bh_c}\right) \geq \frac{4}{81r^5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{1}{h_a^3}+\frac{R}{2r^2s^2}\right)\left(\sum_{cyc}\frac{1}{h_bh_c}\right) =$$

$$= \left(\frac{2s(s^2-6Rr-3r^2)}{8r^3s^3}+\frac{R}{2r^2s^2}\right)\left(\frac{s^2+4Rr+r^2}{2R.\frac{2r^2s^2}{R}}\right) =$$

$$= \frac{(s^2-4Rr-3r^2)(s^2+4Rr+r^2)}{16r^5s^4} \overset{?}{\geq} \frac{4}{81r^5}$$

$$\Leftrightarrow 17s^4-162r^2s^2-r^2(1296R^2+1296Rr+243r^2) \underbrace{\overset{?}{\geq}}_{①} 0$$

$$\text{Now, via Gerretsen, LHS of } ① \geq$$

$$17s^2(16Rr-5r^2)-162r^2s^2-r^2(1296R^2+1296Rr+243r^2) \overset{?}{\geq} 0$$

$$\Leftrightarrow (272R-247r)s^2 \underbrace{\overset{?}{\geq}}_{②} r(1296R^2+1296Rr+243r^2) \text{ and indeed,}$$

$$(272R-247r)s^2 \overset{\text{Gerretsen}}{\geq} (272R-247r)(16Rr-5r^2) \overset{?}{\geq}$$

$$r(1296R^2+1296Rr+243r^2) \Leftrightarrow 16r(R-2r)(191R-31r) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \overset{\text{Euler}}{\geq} 2r \Rightarrow ② \Rightarrow ① \text{ is true and so,}$$

$$\left(\sum_{cyc}\frac{1}{h_a^3}+\frac{R}{2r^2s^2}\right)\left(\sum_{cyc}\frac{1}{h_bh_c}\right) \geq \frac{4}{81r^5}\ \forall\,\Delta\,\text{ABC},$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

4034. In ΔABC the following relationship holds:

$$9-\frac{2r}{R}\le\prod_{cyc}\frac{r_b+r_c}{r_a}\le\frac{4R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{r_b+r_c}{r_a}=\frac{p\left(\tan\left(\frac{B}{2}\right)+\tan\left(\frac{C}{2}\right)\right)}{p\tan\left(\frac{A}{2}\right)}=\frac{\frac{\sin\left(\frac{B}{2}+\frac{C}{2}\right)}{\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}}{\tan\left(\frac{A}{2}\right)}=\frac{\cos\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}$$

$$\prod_{cyc}\frac{r_b+r_c}{r_a}=\frac{\cos\left(\frac{A}{2}\right)}{\tan\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}\cdot\frac{\cos\left(\frac{B}{2}\right)}{\tan\left(\frac{B}{2}\right).\cos\left(\frac{A}{2}\right).\cos\left(\frac{C}{2}\right)}\cdot$$

$$\frac{\cos\left(\frac{C}{2}\right)}{\tan\left(\frac{C}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{A}{2}\right)}=\frac{1}{\tan\left(\frac{A}{2}\right).\tan\left(\frac{B}{2}\right).\tan\left(\frac{C}{2}\right).\cos\left(\frac{A}{2}\right).\cos\left(\frac{B}{2}\right).\cos\left(\frac{C}{2}\right)}=$$

$$=\frac{1}{\prod_{cyc}\sin\left(\frac{A}{2}\right)}=\frac{4R}{r}$$

$$\frac{r_b+r_c}{r_a}=\frac{\frac{F}{p-b}+\frac{F}{p-c}}{\frac{F}{p-a}}=\frac{a(p-a)}{(p-b)(p-c)}$$

$$\prod_{cyc}\frac{r_b+r_c}{r_a}=\frac{a(p-a)}{(p-b)(p-c)}\cdot\frac{b(p-b)}{(p-a)(p-c)}\cdot\frac{c(p-c)}{(p-a)(p-b)}=$$

$$=\frac{abc}{(p-a)(p-b)(p-c)}=\frac{4RF.p}{p(p-a)(p-b)(p-c)}=$$

$$\frac{4RF.p}{F^2}=\frac{4Rp}{F}=\frac{4R.p}{pr}=\frac{4R}{r}$$

$$\frac{4R}{r}\ge 9-\frac{2r}{R};\ \ Let\ \ \frac{R}{r}=x,\ \ Then:\ \ 4x\ge 9-\frac{2}{x}\ \ \rightarrow$$

$$4x^2-9x+2\ge 0\ \rightarrow\ \ (4x-1)(x-2)\ge 0\ \ x\rightarrow\ \ x\ge 2,\frac{R}{r}\ge 2\ \ (True)$$

Equality holds if $a=b=c$

4035. In any Δ ABC the following relationship holds:

$$108\mathrm{r}^2(\mathrm{R}-\mathrm{r})\le\sum_{\mathrm{cyc}}a^2\mathrm{r}_a\le 27\mathrm{R}^2(\mathrm{R}-\mathrm{r})$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} a^2 r_a = \sum_{cyc}\left(a^2.\frac{rs}{s-a}\right) = \frac{rs}{r^2 s}.\sum_{cyc}\left(a^2(-s^2+sa+bc)\right)$$

$$= \frac{-2s^2(s^2-4Rr-r^2)+2s^2(s^2-6Rr-3r^2)+8Rrs^2}{r} = \frac{4s^2(R-r)}{r} \text{ and } \because$$

$$108r^2 \overset{\text{Mitrinovic}}{\le} 4s^2 \overset{\text{Mitrinovic}}{\le} 27R^2 \therefore 108r^2(R-r) \le \sum_{cyc} a^2 r_a \le 27R^2(R-r)$$

$$\forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

4036. In any acute ΔABC the following relationship holds :

$$3(p^2-(2R+r)^2) \le \sum_{cyc}\frac{AH.BH^2}{AH+BH} \le 6(R-r)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\Delta ABC \text{ is acute} \Rightarrow AH = 2R\cos A > 0 \text{ and analogs}$$

$$\therefore \sum_{cyc}\frac{AH.BH^2}{AH+BH} \overset{AM-GM}{\ge} 3.\sqrt[3]{\frac{\prod_{cyc}AH^3}{\prod_{cyc}(AH+BH)}} \overset{AM-GM}{\ge} 3\left(\prod_{cyc}AH\right).\frac{3}{\sum_{cyc}(AH+BH)}$$

$$= 9.8R^3.\frac{p^2-(2R+r)^2}{4R^2}.\frac{1}{2R}.\frac{1}{2\sum_{cyc}\cos A} \ge 9.8R^3.\frac{p^2-(2R+r)^2}{4R^2}.\frac{1}{2R}.\frac{1}{2.\frac{3}{2}}$$

$$\therefore \sum_{cyc}\frac{AH.BH^2}{AH+BH} \ge 3(p^2-(2R+r)^2) \text{ and again,}$$

$$\sum_{cyc}\frac{AH.BH^2}{AH+BH} \overset{AM-GM}{\le} \sum_{cyc}\frac{(AH+BH)^2.BH}{4(AH+BH)} = \frac{1}{8}\left(2\sum_{cyc}AH^2+\left(\sum_{cyc}AH\right)^2-\sum_{cyc}AH^2\right)$$

$$= \frac{1}{8}\left(4R^2\left(3-\sum_{cyc}\sin^2 A\right)+4R^2.\frac{(R+r)^2}{R^2}\right)$$

$$= \frac{1}{8}\left(12R^2+4(R+r)^2-2(p^2-4Rr-r^2)\right) \overset{\text{Gerretsen}}{\le}$$

$$\frac{1}{8}\left(12R^2+4(R+r)^2-2(16Rr-5r^2-4Rr-r^2)\right) = 2(R^2-Rr+r^2) \overset{?}{\le} 6(R-r)^2$$

$$\Leftrightarrow 2R^2-5Rr+2r^2 \overset{?}{\ge} 0 \Leftrightarrow (2R-r)(R-2r) \overset{?}{\ge} 0 \to \text{true} \because R \overset{\text{Euler}}{\ge} 2r$$

$$\therefore \sum_{cyc}\frac{AH.BH^2}{AH+BH}\le 6(R-r)^2 \text{ and so, } 3\left(p^2-(2R+r)^2\right)\le\sum_{cyc}\frac{AH.BH^2}{AH+BH}\le 6(R-r)^2$$

$\forall$ acute Δ ABC, "=" iff Δ ABC is equilateral (QED)

4037. In any Δ ABC the following relationship holds:

$$54r\left(1-\frac{r}{R}\right)\le\sum_{cyc}h_a\cot^2\frac{A}{2}\le\frac{27R}{2}\left(\frac{R}{r}-1\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}h_a\cot^2\frac{A}{2}=\sum_{cyc}\left(\frac{bc}{2R}\cdot\frac{s(s-a)^2}{(s-b)(s-c)(s-a)}\right)$$

$$=\frac{1}{2Rr^2}\cdot\sum_{cyc}\left(bc(s^2-2sa+a^2)\right)=\frac{s^2(s^2+4Rr+r^2)-4s.4Rrs}{2Rr^2}$$

$$=\frac{s^2(s^2-12Rr+r^2)}{2Rr^2}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen}}}{\le}\frac{\frac{27R^2}{4}(4R^2-8Rr+4r^2)}{2Rr^2}=\frac{27R(R-r)^2}{2r^2}$$

$$=\frac{27R}{2}\left(\frac{R}{r}-1\right)^2\ \therefore\sum_{cyc}h_a\cot^2\frac{A}{2}\le\frac{27R}{2}\left(\frac{R}{r}-1\right)^2 \text{ and again,}$$

$$\sum_{cyc}h_a\cot^2\frac{A}{2}=\frac{s^2(s^2-12Rr+r^2)}{2Rr^2}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen}}}{\ge}\frac{27r^2(4Rr-4r^2)}{2Rr^2}=54r\left(1-\frac{r}{R}\right)$$

$$\therefore\sum_{cyc}h_a\cot^2\frac{A}{2}\ge 54r\left(1-\frac{r}{R}\right) \text{ and so, } 54r\left(1-\frac{r}{R}\right)\le\sum_{cyc}h_a\cot^2\frac{A}{2}\le\frac{27R}{2}\left(\frac{R}{r}-1\right)^2$$

$\forall\ \Delta$ ABC, "=" iff Δ ABC is equilateral (QED)

4038.

In any Δ ABC the following relationship holds:

$$\frac{1}{3r}\le\sum_{cyc}\frac{\tan^2\frac{A}{2}}{h_a}\le\frac{8R^2-23r^2}{27r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan^2\frac{A}{2}}{h_a}=2R.\sum_{cyc}\frac{(s-b)(s-c)}{s(s-a)bc}=2R.\sum_{cyc}\frac{-s(s-a)+bc}{s(s-a)bc}$$

$$=2R\left(\frac{-2s}{4Rrs}+\frac{1}{r^2s^2}\sum_{cyc}(s-b)(s-c)\right)=2R\left(-\frac{1}{2Rr}+\frac{4R+r}{rs^2}\right)=\frac{8R^2+2Rr-s^2}{rs^2}$$

$$\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Mitrinovic}}}{\le}\frac{8R^2+2Rr-16Rr+5r^2}{27r^3}\overset{\text{Euler}}{\le}\frac{8R^2-28r^2+5r^2}{27r^3}\text{ and so,}$$

$$\sum_{cyc}\frac{\tan^2\frac{A}{2}}{h_a}\le\frac{8R^2-23r^2}{27r^3}\text{ and again,}\sum_{cyc}\frac{\tan^2\frac{A}{2}}{h_a}=\frac{8R^2+2Rr-s^2}{rs^2}\overset{?}{\ge}\frac{1}{3r}$$

$$\Leftrightarrow 2s^2\overset{?}{\le}12R^2+3Rr\text{ and indeed, }2s^2\overset{\text{Gerretsen}}{\le}8R^2+8Rr+6r^2\overset{?}{\le}12R^2+3Rr$$

$$\Leftrightarrow(4R+3r)(R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r\text{ and so,}\sum_{cyc}\frac{\tan^2\frac{A}{2}}{h_a}\ge\frac{1}{3r}$$

$$\therefore\frac{1}{3r}\le\sum_{cyc}\frac{\tan^2\frac{A}{2}}{h_a}\le\frac{8R^2-23r^2}{27r^3}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

4039. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}\ge\sum_{cyc}\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}\overset{?}{\ge}\sum_{cyc}\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}\Leftrightarrow\sum_{cyc}\frac{r_a}{r_b}\overset{?}{\ge}\frac{\frac{s(s-a)}{bc}}{\frac{s(s-b)}{ca}}\Leftrightarrow\sum_{cyc}\frac{s-b}{s-a}\overset{?}{\ge}\sum_{cyc}\frac{a(s-a)}{b(s-b)}$$

$$\Leftrightarrow\sum_{cyc}\frac{y}{x}\overset{?}{\ge}\sum_{cyc}\frac{(y+z)x}{(z+x)y}\left(\begin{matrix}x=s-a,y=s-b,z=s-c\Rightarrow\\ a=y+z,b=z+x,c=x+y\end{matrix}\right)$$

$$\Leftrightarrow\frac{1}{xyz}.\sum_{cyc}y^2z\overset{?}{\ge}\frac{1}{xyz(x+y)(y+z)(z+x)}.\sum_{cyc}\left((x+y)(y+z)^2.zx^2\right)$$

$$\Leftrightarrow \sum_{cyc} x^4y^2 + xyz\sum_{cyc} x^3 \underbrace{\overset{?}{\geq}}_{\text{①}} xyz\sum_{cyc} xy^2 + 3x^2y^2z^2$$

$$\text{Now}, \sum_{cyc} x^4y^2 = (x^2y)^2 + (y^2z)^2 + (z^2x)^2 \geq$$

$$(x^2y)(y^2z) + (y^2z)(z^2x) + (z^2x)(x^2y) = xyz\sum_{cyc} xy^2 \text{ and also},$$

$$xyz\sum_{cyc} x^3 \overset{AM-GM}{\geq} 3x^2y^2z^2 \text{ and so}, \sum_{cyc} x^4y^2 + xyz\sum_{cyc} x^3 \geq xyz\sum_{cyc} xy^2 + 3x^2y^2z^2$$

$$\Rightarrow \text{①} \text{ is true} \therefore \sum_{cyc}\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}} \geq \sum_{cyc}\frac{\cos^2\frac{A}{2}}{\cos^2\frac{B}{2}}\ \forall\, \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4040. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}} \geq \sum_{cyc}\frac{\sin A}{\sin B}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}} \overset{?}{\geq} \sum_{cyc}\frac{\sin A}{\sin B} \Leftrightarrow \sum_{cyc}\frac{r_b}{r_a} \overset{?}{\geq} \sum_{cyc}\frac{a}{b} \Leftrightarrow \sum_{cyc}\frac{s-a}{s-b} \overset{?}{\geq} \sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow \sum_{cyc}\frac{x}{y} \overset{?}{\geq} \sum_{cyc}\frac{y+z}{z+x} \begin{pmatrix} x = s-a, y = s-b, z = s-c \Rightarrow \\ a = y+z, b = z+x, c = x+y \end{pmatrix}$$

$$\Leftrightarrow \frac{1}{xyz}\cdot\sum_{cyc} zx^2 \overset{?}{\geq} \frac{1}{(x+y)(y+z)(z+x)}\cdot\sum_{cyc}\left((x+y)(y+z)^2\right)$$

$$\Leftrightarrow \sum_{cyc} x^2y^4 + \sum_{cyc} x^3y^3 \underbrace{\overset{?}{\geq}}_{\text{①}} xyz\sum_{cyc} x^2y + 3x^2y^2z^2$$

$$\text{Now}, \sum_{cyc} x^2y^4 = (xy^2)^2 + (yz^2)^2 + (zx^2)^2 \geq$$

$$(xy^2)(yz^2) + (yz^2)(zx^2) + (zx^2)(xy^2) = xyz\sum_{cyc} x^2y \text{ and also},$$

$$\sum_{cyc} x^3y^3 \overset{AM-GM}{\geq} 3x^2y^2z^2 \text{ and so}, \sum_{cyc} x^2y^4 + \sum_{cyc} x^3y^3 \geq xyz\sum_{cyc} x^2y + 3x^2y^2z^2$$

$$\Rightarrow ① \text{ is true } \therefore \sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}} \geq \sum_{cyc}\frac{\sin A}{\sin B}\ \forall\,\Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

4041. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\tan A}{\tan B} \geq \sum_{cyc}\frac{\sin 2A}{\sin 2B}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let us consider a $\Delta\,A'B'C'$ with angles $A' \equiv (\pi - 2A), B' \equiv (\pi - 2B)$ and $C' \equiv (\pi - 2C)$ & then : $\cos A'\cos B'\cos C' = \cos(\pi - 2A)\cos(\pi - 2B)\cos(\pi - 2C)$ $= -\cos 2A\cos 2B\cos 2C = 1 + 4\cos A\cos B\cos C > 0$ $(\because \Delta\,ABC$ being acute $\Rightarrow \cos A\cos B\cos C > 0) \Rightarrow \Delta\,A'B'C'$ is acute $\rightarrow (*)$

$$\text{Now, } \sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}} \overset{?}{\geq} \sum_{cyc}\frac{\sin A}{\sin B} \Leftrightarrow \sum_{cyc}\frac{r_b}{r_a} \overset{?}{\geq} \sum_{cyc}\frac{a}{b} \Leftrightarrow \sum_{cyc}\frac{s-a}{s-b} \overset{?}{\geq} \sum_{cyc}\frac{a}{b}$$

$$\Leftrightarrow \sum_{cyc}\frac{x}{y} \overset{?}{\geq} \sum_{cyc}\frac{y+z}{z+x}\ \begin{pmatrix} x = s-a, y = s-b, z = s-c \Rightarrow \\ a = y+z, b = z+x, c = x+y \end{pmatrix}$$

$$\Leftrightarrow \frac{1}{xyz}\cdot\sum_{cyc} zx^2 \overset{?}{\geq} \frac{1}{(x+y)(y+z)(z+x)}\cdot\sum_{cyc}\left((x+y)(y+z)^2\right)$$

$$\Leftrightarrow \sum_{cyc} x^2y^4 + \sum_{cyc} x^3y^3 \underset{①}{\underbrace{\overset{?}{\geq}}} xyz\sum_{cyc} x^2y + 3x^2y^2z^2$$

$$\text{Now, } \sum_{cyc} x^2y^4 = (xy^2)^2 + (yz^2)^2 + (zx^2)^2 \geq$$

$$(xy^2)(yz^2) + (yz^2)(zx^2) + (zx^2)(xy^2) = xyz\sum_{cyc} x^2y \text{ and also,}$$

$$\sum_{cyc} x^3y^3 \overset{AM-GM}{\geq} 3x^2y^2z^2 \text{ and so, } \sum_{cyc} x^2y^4 + \sum_{cyc} x^3y^3 \geq xyz\sum_{cyc} x^2y + 3x^2y^2z^2$$

$$\Rightarrow ① \text{ is true } \therefore \sum_{cyc}\frac{\cot\frac{A}{2}}{\cot\frac{B}{2}} \geq \sum_{cyc}\frac{\sin A}{\sin B}\ \forall\,\Delta\,ABC \text{ and implementing it}$$

$$\text{on } \Delta\,A'B'C', \text{ we get : } \sum_{cyc}\frac{\cot\frac{\pi-2A}{2}}{\cot\frac{\pi-2B}{2}} \geq \sum_{cyc}\frac{\sin(\pi-2A)}{\sin(\pi-2B)} \text{ on } \Delta\,A'B'C', \text{ we get :}$$

$$\sum_{cyc}\frac{\cot\frac{\pi-2A}{2}}{\cot\frac{\pi-2B}{2}}\geq\sum_{cyc}\frac{\sin(\pi-2A)}{\sin(\pi-2B)}\Rightarrow\sum_{cyc}\frac{\tan A}{\tan B}\geq\sum_{cyc}\frac{\sin 2A}{\sin 2B}\ \textit{and}\ \because$$

$$\Delta\, A'B'C'\overset{\text{via }(*)}{\text{is}}\ \textit{acute},\text{hence}:\sum_{cyc}\frac{\tan A}{\tan B}\geq\sum_{cyc}\frac{\sin 2A}{\sin 2B}\ \forall\ \textit{acute}\ \Delta\, ABC,$$

$$''=''\ \text{iff}\ \Delta\, ABC\ \text{is equil}\textit{ateral}\ (\text{QED})$$

4042. In ΔABC the following relationship holds:

$$(m_a)^{b^2+c^2}(m_b)^{a^2+c^2}(m_c)^{b^2+a^2}\geq(3r)^{72r^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$(m_a)^{b^2+c^2}(m_b)^{a^2+c^2}(m_c)^{b^2+a^2}\overset{\textit{Weighted}}{\underset{\sim}{\geq}}\left(\frac{2(a^2+b^2+c^2)}{\frac{a^2+b^2}{m_c}+\frac{c^2+b^2}{m_a}+\frac{a^2+c^2}{m_b}}\right)^{2\sum a^2}\overset{\textit{Chebyshev}}{\underset{\sim}{\geq}}$$

$$\geq\left(\frac{2(a^2+b^2+c^2)}{\frac{1}{3}\left(2(a^2+b^2+c^2)\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)\right)}\right)^{2\sum a^2}\geq\left(\frac{3}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^{2\sum a^2}\overset{\textit{Neuberg}}{\underset{\sim}{\geq}}$$

$$\geq\left(\frac{3}{\frac{1}{r}}\right)^{72r^2}=(3r)^{72r^2}.\ \textit{Equality holds if}\ a=b=c.$$

4043. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\sqrt{\sin A}}\geq\sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}=\sum_{cyc}\frac{\sqrt{2\cos\frac{B-C}{2}}}{\sqrt{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}}\leq\sum_{cyc}\frac{\sqrt{2}}{\sqrt{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}}$$

$$\left(\because 0<\cos\frac{B-C}{2}\le 1\right)=\sum_{cyc}\frac{\sqrt{2}}{\sqrt{\sin B+\sin C}}$$

$$\therefore \sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}\le\sum_{cyc}\frac{\sqrt{2}}{\sqrt{\sin B+\sin C}}\to ①$$

$$\text{Again, }\sum_{cyc}\frac{1}{\sqrt{\sin A}}=\frac{1}{2}\cdot\sum_{cyc}\left(\frac{1}{\sqrt{\sin B}}+\frac{1}{\sqrt{\sin C}}\right)\overset{\text{Bergstrom}}{\ge}\frac{1}{2}\cdot\sum_{cyc}\frac{4}{\sqrt{\sin B}+\sqrt{\sin C}}$$

$$\overset{\text{CBS}}{\ge}\sum_{cyc}\frac{2}{\sqrt{2(\sin B+\sin C)}}=\sum_{cyc}\frac{\sqrt{2}}{\sqrt{\sin B+\sin C}}\overset{\text{via }①}{\ge}\sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}\text{ and so,}$$

$$\sum_{cyc}\frac{1}{\sqrt{\sin A}}\ge\sum_{cyc}\frac{1}{\sqrt{\cos\frac{A}{2}}}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

4044. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\sqrt{\cos A}}\ge\sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}}=\sum_{cyc}\frac{\sqrt{2\cos\frac{B-C}{2}}}{\sqrt{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}}\le\sum_{cyc}\frac{\sqrt{2}}{\sqrt{2\cos\frac{B+C}{2}\cos\frac{B-C}{2}}}$$

$$\left(\because 0<\cos\frac{B-C}{2}\le 1\right)=\sum_{cyc}\frac{\sqrt{2}}{\sqrt{\cos B+\cos C}}$$

$$\therefore \sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}}\le\sum_{cyc}\frac{\sqrt{2}}{\sqrt{\cos B+\cos C}}\to ①$$

$$\text{Again, }\sum_{cyc}\frac{1}{\sqrt{\cos A}}=\frac{1}{2}\cdot\sum_{cyc}\left(\frac{1}{\sqrt{\cos B}}+\frac{1}{\sqrt{\cos C}}\right)\overset{\text{Bergstrom}}{\ge}\frac{1}{2}\cdot\sum_{cyc}\frac{4}{\sqrt{\cos B}+\sqrt{\cos C}}\overset{\text{CBS}}{\ge}$$

$$\sum_{cyc}\frac{2}{\sqrt{2(\cos B+\cos C)}}=\sum_{cyc}\frac{\sqrt{2}}{\sqrt{\cos B+\cos C}}\overset{\text{via }①}{\ge}\sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}}\text{ and so,}$$

$$\sum_{cyc}\frac{1}{\sqrt{\cos A}} \ge \sum_{cyc}\frac{1}{\sqrt{\sin\frac{A}{2}}}\ \forall \text{ acute } \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4045. In ΔABC the following relationship holds:

$$\frac{a^3}{{h_a}^3}+\frac{a^3}{{h_b}^3}+\frac{a^3}{{h_c}^3} \ge \frac{a^3}{{r_a}^3}+\frac{a^3}{{r_b}^3}+\frac{a^3}{{r_c}^3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Jenish Rijal-Nepal

$$\sum_{cyc}\frac{a^3}{{h_a}^3} \ge \sum_{cyc}\frac{a^3}{{r_a}^3} \Leftrightarrow \sum_{cyc}\frac{a^3}{{h_a}^3} = \sum_{cyc}\frac{a^6}{8\Delta^3} \ge \sum_{cyc}\frac{a^3}{{r_a}^3} = \sum_{cyc}\frac{a^3(b+c-a)^3}{8\Delta^3} \Leftrightarrow$$

$$\sum_{cyc}a^6 \ge \sum_{cyc}a^3(b+c-a)^3$$

Thus, it suffices to prove that: $\sum_{cyc}a^3(b+c-a)^3 \le \sum_{cyc}a^6$

By AM – GM Inequality,

$$a(b+c-a) \le \left(\frac{a+b+c-a}{2}\right)^2 = \left(\frac{b+c}{2}\right)^2 \Rightarrow a^3(b+c-a)^3 \le \left(\frac{b+c}{2}\right)^6$$

By symmetry, the same holds for b and c.

$$\Rightarrow \sum_{cyc}a^3(b+c-a)^3 \le \sum_{cyc}\left(\frac{b+c}{2}\right)^6 \overset{\text{Power Mean}}{\le} \sum_{cyc}\left(\frac{b^6+c^6}{2}\right) = \sum_{cyc}a^6.$$

Equality holds if and only if the triangle is equilateral.

4046. In any Δ ABC the following relationship holds :

$$m_b + m_c \le \frac{\sqrt{s^2 - 7Rr + 3r^2}}{4r}(h_b + h_c)$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a, y = s - b, z = s - c$; then : $a = y + z, b = z + x, c = x + y$

$$\text{and } s = x + y + z \text{ and furthermore, we denote : } \frac{y+z}{x} = m \text{ and } \frac{yz}{x^2} = n$$

$$\text{and then, we have the following set "S" of relations : } y^2 + z^2 = x^2(m^2 - 2n),$$

$$y^3 + z^3 = x^3(m^3 - 3nn), y^4 + z^4 = x^4((m^2 - 2n)^2 - 2n^2),$$

$$y^5 + z^5 = x^5\left(m((m^2 - 2n)^2 + n^2 - nm^2)\right), y^6 + z^6 = x^6((m^3 - 3nn)^2 - 2n^3),$$

$$\text{and now, } (m_b + m_c)^2 \le m_b^2 + m_c^2 + \frac{2a^2 + bc}{2}$$

(Reference : Solution to Inequality in Triangle – 316 by Dang Ngoc Minh; published at www.ssmrmh.ro)

$$= \frac{4s(s-b) + (c-a)^2 + 4s(s-c) + (a-b)^2 + 2(2a^2 + bc)}{4} \overset{?}{\le}$$

$$\frac{s^2 - 7Rr + 3r^2}{16r^2}.(h_b + h_c)^2 = \frac{s^3 - 7Rrs + 3r^2s}{16r^2}\left(4r^2s\left(\frac{1}{b^2} + \frac{1}{c^2} + \frac{2}{bc}\right)\right)$$

$$\Leftrightarrow 4(y+z)\left(\sum_{cyc} x\right) + (z-x)^2 + (x-y)^2 + 2\left(2(y+z)^2 + (z+x)(x+y)\right) \overset{?}{\le}$$

$$\frac{(x+y+z)(2x+y+z)^2}{(z+x)^2(x+y)^2}\left(\left(\sum_{cyc} x\right)^3 - \frac{7}{4}(y+z)(z+x)(x+y) + 3xyz\right)$$

$$\Leftrightarrow 4x^5(y+z) - 4x^4(y+z)^2 + 16x^4.yz - 3x^3(y+z)^3 + 4x^3.yz(y+z) +$$
$$26x^2(y^2+z^2)^2 - 112x^2.y^2z^2 + 4x^2.yz(y^2+z^2) + 25x(y^5+z^5) +$$
$$30xyz(y^3+z^3) - 51x.y^2z^2(y+z) + 4(y^6+z^6) + 17yz(y^4+z^4) - 4y^2z^2(y+z)^3$$
$$-26y^3z^3 \boxed{\overset{?}{\underbrace{\ge}_{①}}} 0 \text{ and via set of relations "S", in order to prove ①,}$$

it suffices to prove, following simplification :

$$\overbrace{(36m^2 + 16m + 16)}^{\sigma_1} n^2 + \overbrace{(7m^4 + 95m^3 + 100m^2 - 4m - 16)}^{\sigma_2} n -$$
$$\overbrace{(4m^6 + 25m^5 + 26m^4 - 3m^3 - 4m^2 + 4m)}^{\sigma_3} \boxed{\overset{?}{\underbrace{\le}_{②}}} 0$$

Now, LHS of ② is a quadratic polynomial in "n" with discriminant, $\boxed{\delta} =$ $\sigma_2^2 + 4\sigma_1\sigma_3 = (m+1)^2(m+2)^2(625m^4 + 1436m^3 - 716m^2 - 96m + 64)$

Now, $625m^4 + 1436m^3 - 716m^2 - 96m + 64 =$

$$\frac{1}{27}\left((1875m^2 + 5558m + 1349)(3m-1)^2 + 379 - 56m\right) > 0 \text{ whenever :}$$

$$0 < m \le \frac{379}{56} \text{ and } 625m^4 + 1436m^3 - 716m^2 - 96m + 64 =$$

$$(625m^2 + 2686m + 4031)(m-1)^2 + 5280m - 3967 > 0 \text{ when : } m > \frac{379}{56}$$

$$\text{and so, } \boxed{\delta} = (m+1)^2(m+2)^2(625m^4 + 1436m^3 - 716m^2 - 96m + 64)$$

$> 0\ \forall\ m > 0$ and so, in order to prove ②, it suffices to prove :

$$2\sigma_1.n\ \boxed{\underbrace{\overset{?}{\leq}}_{(m)}} - \sigma_2 + \sqrt{\delta}\ \text{AND}\ 2\sigma_1.n\ \boxed{\underbrace{\overset{?}{\geq}}_{(n)}} - \sigma_2 - \sqrt{\delta}$$

Now, since $n \overset{AM-GM}{\leq} \dfrac{m^2}{4}$ $\therefore$ in order to prove (m), it suffices to prove :

$$\sigma_1.\frac{m^2}{2} \overset{?}{\leq} -\sigma_2 + \sqrt{\delta} \Leftrightarrow (m+2)^2(25m^2+3m-4) \overset{?}{\leq} \sqrt{\delta}\ \text{and it's trivially true if :}$$

$25m^2 + 3m - 4 \leq 0$ and when : $25m^2 + 3m - 4 > 0$, *then it suffices to prove* :

$$(m+2)^4(25m^2+3m-4)^2 \overset{?}{\leq} (m+1)^2(m+2)^2\left(\begin{matrix}625m^4+1436m^3-716m^2-\\ 96m+64\end{matrix}\right)$$

$$\Leftrightarrow \boxed{4m(m+2)^2(m-2)^2(9m^2+4m+4) \overset{?}{\geq} 0} \to \text{true}$$

$$\because m > 0 \Rightarrow \text{(m) is true and also, } 2\sigma_1.n + \sqrt{\delta} > \sqrt{\delta} = \sqrt{\sigma_2^2 + 4\sigma_1\sigma_3}$$

$$= \sqrt{\sigma_2^2 + 4(36m^2+16m+16).m(m+1)^2(4m^3+17m^2-12m+4)}$$

$$> \sqrt{\sigma_2^2}\ (\because \Delta \text{ of } 17m^2-12m+4 = -128 < 0 \Rightarrow 4m^3+17m^2-12m+4 > 0)$$

$$= |\sigma_2| \geq -\sigma_2\ (\because |x| + x \geq 0\ \forall\ x \in \mathbb{R}) \Rightarrow 2\sigma_1.n > -\sigma_2 - \sqrt{\delta} \Rightarrow \text{(n) is true}$$

(strict inequality) $\therefore$ (m) and (n) are both true $\Rightarrow$ ② $\Rightarrow$ ① is true

$$\therefore m_b + m_c \leq \frac{\sqrt{s^2-7Rr+3r^2}}{4r}(h_b+h_c)\ \forall\ \Delta\ ABC,$$

" = " iff Δ ABC is equilateral (QED)

4047. In any Δ ABC holds :

$$max\left\{\frac{w_b+w_c}{h_b+h_c} + \frac{h_b+h_c}{w_b+w_c}, \frac{m_b+m_c}{w_b+w_c} + \frac{w_b+w_c}{m_b+m_c}\right\} \leq \frac{m_b+m_c}{h_b+h_c} + \frac{h_b+h_c}{m_b+m_c} \leq \frac{\sqrt{s^2-7Rr+51r^2}}{4r}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x = s - a$, $y = s - b$, $z = s - c$; then : $a = y + z$, $b = z + x$, $c = x + y$

and $s = x + y + z$ and furthermore, we denote : $\dfrac{y+z}{x} = m$ and $\dfrac{yz}{x^2} = n$

and then, we have the following set "S" of relations : $y^2 + z^2 = x^2(m^2 - 2n)$,

$y^3 + z^3 = x^3(m^3 - 3nn)$, $y^4 + z^4 = x^4((m^2-2n)^2 - 2n^2)$,

$y^5 + z^5 = x^5\left(m((m^2-2n)^2 + n^2 - nm^2)\right)$, $y^6 + z^6 = x^6((m^3-3nn)^2 - 2n^3)$,

$$\text{and now, } (m_b + m_c)^2 \leq m_b^2 + m_c^2 + \frac{2a^2 + bc}{2}$$

(Reference : Solution to Inequality in Triangle – 316 by Dang Ngoc Minh;)

$$= \frac{4s(s-b)+(c-a)^2+4s(s-c)+(a-b)^2+2(2a^2+bc)}{4} \overset{?}{\leq}$$

$$\frac{s^2-7Rr+3r^2}{16r^2}.(h_b+h_c)^2 = \frac{s^3-7Rrs+3r^2s}{16r^2}\left(4r^2s\left(\frac{1}{b^2}+\frac{1}{c^2}+\frac{2}{bc}\right)\right)$$

$$\Leftrightarrow 4(y+z)\left(\sum_{cyc} x\right)+(z-x)^2+(x-y)^2+2\left(2(y+z)^2+(z+x)(x+y)\right) \overset{?}{\leq}$$

$$\frac{(x+y+z)(2x+y+z)^2}{(z+x)^2(x+y)^2}\left((x+y+z)^3-\frac{7}{4}(y+z)(z+x)(x+y)+3xyz\right)$$

$$\Leftrightarrow 4x^5(y+z)-4x^4(y+z)^2+16x^4.yz-3x^3(y+z)^3+4x^3.yz(y+z)+$$
$$26x^2(y^2+z^2)^2-112x^2.y^2z^2+4x^2.yz(y^2+z^2)+25x(y^5+z^5)+$$
$$30xyz(y^3+z^3)-51x.y^2z^2(y+z)+4(y^6+z^6)+17yz(y^4+z^4)-4y^2z^2(y+z)^3$$

$-26y^3z^3 \boxed{\underset{\textcircled{1}}{\overset{?}{\geq}}}$ 0 *and* via set of rel*a*tions ''S'', to prove ①, suffices to prove,

$$\overbrace{(36m^2+16m+16)}^{\sigma_1}n^2+\overbrace{(7m^4+95m^3+100m^2-4m-16)}^{\sigma_2}n-$$
$$\overbrace{(4m^6+25m^5+26m^4-3m^3-4m^2+4m)}^{\sigma_3}\boxed{\underset{\textcircled{2}}{\overset{?}{\leq}}}0;\text{ now, discriminant, }\boxed{\delta}=$$

$$\sigma_2^2+4\sigma_1\sigma_3=(m+1)^2(m+2)^2\overbrace{(625m^4+1436m^3-716m^2-96m+64)}^{\mu}\ \&$$

$$\mu=\frac{1}{27}\left((1875m^2+5558m+1349)(3m-1)^2+379-56m\right)>0\text{ for }m\leq\frac{379}{56}$$

$$\&\ \mu=(625m^2+2686m+4031)(m-1)^2+5280m-3967>0\ when: m>\frac{379}{56}$$

$\boxed{\delta}>0\ \forall\ m>0$ *and* so, in order to prove ②, it suffices to prove :

$$2\sigma_1.n\boxed{\underset{(m)}{\overset{?}{\leq}}}-\sigma_2+\sqrt{\delta}\text{ AND }2\sigma_1.n\boxed{\underset{(n)}{\overset{?}{\geq}}}-\sigma_2-\sqrt{\delta}$$

Since $n \overset{AM-GM}{\leq} \frac{m^2}{4}$ ∴ to prove (m), it suffices to prove : $\sigma_1.\frac{m^2}{2}\overset{?}{\leq}-\sigma_2+\sqrt{\delta}$

$\Leftrightarrow (m+2)^2(25m^2+3m-4)\overset{?}{\leq}\sqrt{\delta}$ *and* it's trivi*ally* true if :

$25m^2+3m-4\leq 0$ *and* when : $25m^2+3m-4>0$, *then it suffices to prove* :

$$(m+2)^4(25m^2+3m-4)^2\overset{?}{\leq}(m+1)^2(m+2)^2\mu\Leftrightarrow$$

$\boxed{4m(m+2)^2(m-2)^2(9m^2+4m+4)\overset{?}{\geq}0}$ → true; *also*, $2\sigma_1.n+\sqrt{\delta}>\sqrt{\delta}$

$$=\sqrt{\sigma_2^2+4(36m^2+16m+16).m(m+1)^2\left(4m^3+\overbrace{17m^2-12m+4}^{\Delta=-128<0}\right)}>\sqrt{\sigma_2^2}$$

$$\geq-\sigma_2\Rightarrow 2\sigma_1.n>-\sigma_2-\sqrt{\delta}\ \therefore\ (m),(n)\text{ true}\ \therefore\ \frac{m_b+m_c}{h_b+h_c}\overset{(\blacksquare)}{\leq}\frac{\sqrt{s^2-7Rr+3r^2}}{4r}$$

$$\text{Let } \theta = \frac{\sqrt{s^2 - 7Rr + 51r^2}}{4r} \text{ and then}: \theta^2 \overset{?}{\ge} 4 \Leftrightarrow s^2 - 7Rr + 51r^2 \overset{?}{\ge} 64r^2$$

$$\to \text{true via Gerretsen and Euler} \therefore \theta \ge 2 \to (\blacksquare\blacksquare) \text{ and let } t = \frac{m_b + m_c}{h_b + h_c}$$

$$\text{and then}: t + \frac{1}{t} \overset{?}{\le} \theta \Leftrightarrow 2t \underbrace{\overset{?}{\le}}_{(i)} \theta + \sqrt{\theta^2 - 4} \text{ AND } 2t \underbrace{\overset{?}{\ge}}_{(ii)} \theta - \sqrt{\theta^2 - 4}$$

$$\text{Now, } 2t \overset{\text{via } (\blacksquare)}{\le} 2.\frac{\sqrt{s^2 - 7Rr + 3r^2}}{4r} \overset{?}{\le} \frac{\sqrt{s^2 - 7Rr + 51r^2}}{4r} + \frac{\sqrt{s^2 - 7Rr - 13r^2}}{4r}$$

$$\overset{\text{squaring}}{\Leftrightarrow} s^2 - 7Rr - 13r^2 \overset{?}{\le} \sqrt{(s^2 - 7Rr - 13r^2)(s^2 - 7Rr + 51r^2)}$$

$$\Leftrightarrow s^2 - 7Rr - 13r^2 \overset{?}{\le} s^2 - 7Rr + 51r^2 \left(\because s^2 - 7Rr - 13r^2 \overset{\text{Gerretsen and Euler}}{\ge} 0\right)$$

$$\to \text{true} \Rightarrow \text{(i) is true; again, } 2t \ge 2 \overset{?}{\ge} \theta - \sqrt{\theta^2 - 4} \Leftrightarrow \theta^2 - 4 \overset{?}{\ge} \theta^2 - 4\theta + 4 \to$$

$$\text{true via } (\blacksquare\blacksquare) \Rightarrow \text{(ii) true} \therefore t + \frac{1}{t} \le \theta \;\&\; \text{let } \frac{w_b + w_c}{h_b + h_c} = \alpha \ge 1 \;\&\; \frac{m_b + m_c}{w_b + w_c} = \beta \ge 1$$

$$\text{then}: \left(\alpha + \frac{1}{\alpha}\right), \left(\beta + \frac{1}{\beta}\right) \le t + \frac{1}{t} \left(\because \alpha, \beta \le t \text{ with } \alpha, \beta, t \ge 1 \;\& k + \frac{1}{k} \text{ is} \uparrow \text{on } [1, \infty)\right)$$

$$\therefore \left(\alpha + \frac{1}{\alpha}\right), \left(\beta + \frac{1}{\beta}\right) \le t + \frac{1}{t} \;\&\; \text{so, } max\left\{\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}\right\} \le t + \frac{1}{t} \le \frac{\sqrt{s^2 - 7Rr + 51r^2}}{4r}$$

$$\forall \Delta ABC, '' ='' \text{ iff } a = b = c \text{ (QED)}$$

4048. In ΔABC the following relationship holds:

$$1 + \sum_{cyc} \frac{r^2}{{h_a}^2} \ge 4 \sum_{cyc} \frac{r^2}{h_a h_b}$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

$$\frac{r}{h_a} + \frac{r}{h_b} + \frac{r}{h_c} = 1 \quad \left[\because \frac{r}{h_a} + \frac{r}{h_b} + \frac{r}{h_c} = r\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) = r \cdot \frac{1}{r} = 1\right]$$

$$\Rightarrow \left(\frac{r}{h_a} + \frac{r}{h_b} + \frac{r}{h_c}\right)^2 = 1^2 = 1 \Rightarrow \sum_{cyc} \frac{r^2}{{h_a}^2} + 2 \cdot \sum_{cyc} \frac{r^2}{h_a h_b} = 1$$

$$\Rightarrow 4 \sum_{cyc} \frac{r^2}{h_a h_b} = 2 - 2 \sum_{cyc} \frac{r^2}{{h_a}^2}$$

Substituting this into our original inequality:

$$1 + \sum_{cyc} \frac{r^2}{{h_a}^2} \ge 2 - 2 \sum_{cyc} \frac{r^2}{{h_a}^2} \Leftrightarrow 3 \sum_{cyc} \frac{r^2}{{h_a}^2} \ge 1 \Leftrightarrow \sum_{cyc} \frac{r^2}{{h_a}^2} \ge \frac{1}{3}$$

Thus it suffices to show that: $\sum_{cyc}\frac{r^2}{{h_a}^2}\ge\frac{1}{3}$

$$\sum_{cyc}\frac{r^2}{{h_a}^2}=\sum_{cyc}\frac{\left(\frac{r}{h_a}\right)^2}{1}\overset{BERGSTROM}{\overbrace{\ge}}\frac{\left(\sum_{cyc}\frac{r}{h_a}\right)^2}{3}=\frac{1^2}{3}=\frac{1}{3}$$

$$\therefore 1+\sum_{cyc}\frac{r^2}{{h_a}^2}\ge 4\sum_{cyc}\frac{r^2}{h_ah_b}$$

Equality holds if and only if the triangle is equilateral.

4049. In ΔABC the following relationship holds:

$$\frac{r_ar_b^2+r_br_c^2+r_cr_a^2}{(r_a+r_b+r_c)(r_ar_b+r_br_c+r_cr_a)}\ge\frac{16r^2-3R^2}{3R^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Jenish Rijal-Nepal

$$\sum_{cyc}r_ar_b^2=\sum_{cyc}\frac{(r_ar_b)^2}{r_a}\overset{\text{BERGSTROM}}{\overbrace{\ge}}\frac{(r_ar_b+r_br_c+r_cr_a)^2}{r_a+r_b+r_c}$$

$$\therefore\ LHS=\frac{r_ar_b^2+r_br_c^2+r_cr_a^2}{(r_a+r_b+r_c)(r_ar_b+r_br_c+r_cr_a)}\ge\frac{r_ar_b+r_br_c+r_cr_a}{(r_a+r_b+r_c)^2}$$

$$Via\ well-known\ identities\ \sum_{cyc}r_a=4R+r\ \ and\ \sum_{cyc}r_ar_b=s^2\ ,we\ have:$$

$$LHS\ge\frac{r_ar_b+r_br_c+r_cr_a}{(r_a+r_b+r_c)^2}=\frac{s^2}{(4R+r\ )^2}\overset{\text{Mitrinovic}}{\overbrace{\ge}}\frac{27r^2}{(4R+r\ )^2}$$

$$Thus,it\ suffices\ to\ prove\ that:\ \frac{27r^2}{(4R+r\ )^2}\ge\frac{16r^2-3R^2}{3R^2}$$

$$\Leftrightarrow\ 48R^4+24R^3r-172R^2r^2-128Rr^3-16r^4\ge 0$$

$$\Leftrightarrow\ 4(R-2r)(12R^3+30R^2r+17Rr^2+2r^3)\ge 0$$

which is true via Euler's Inequality $(R\ge 2r)$

Equality holds if and only if the triangle is equilateral.

4050. In ΔABC the following relationship holds:

$$\frac{a}{b}\sqrt{a^2+b^2}+\frac{b}{c}\sqrt{b^2+c^2}+\frac{c}{a}\sqrt{c^2+a^2}>\frac{1}{\sqrt{2}}\left(\frac{a^2-b^2}{a+3b}+\frac{b^2-c^2}{b+3c}+\frac{c^2-a^2}{c+3a}\right)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Jenish Rijal-Nepal

$$\sqrt{a^2+b^2}\overset{RMS-AM}{\overbrace{\geq}}\frac{a+b}{\sqrt{2}}\Leftrightarrow\frac{a}{b}\sqrt{a^2+b^2}\geq\frac{1}{\sqrt{2}}\cdot\frac{a(a+b)}{b}$$

$$\frac{a(a+b)}{b}-\frac{a^2-b^2}{a+3b}=(a+b)\left(\frac{a}{b}-\frac{a-b}{a+3b}\right)=\dots=\frac{(a+b)^3}{b(a+3b)}>0$$

$$\Leftrightarrow\frac{a(a+b)}{b}>\frac{a^2-b^2}{a+3b}\Leftrightarrow\frac{a}{b}\sqrt{a^2+b^2}\geq\frac{1}{\sqrt{2}}\cdot\frac{a(a+b)}{b}>\frac{1}{\sqrt{2}}\cdot\left(\frac{a^2-b^2}{a+3b}\right)$$

$$Thus,\qquad\sum_{cyc}\frac{a}{b}\sqrt{a^2+b^2}>\sum_{cyc}\frac{1}{\sqrt{2}}\cdot\left(\frac{a^2-b^2}{a+3b}\right)$$

$$\frac{a}{b}\sqrt{a^2+b^2}+\frac{b}{c}\sqrt{b^2+c^2}+\frac{c}{a}\sqrt{c^2+a^2}>\frac{1}{\sqrt{2}}\left(\frac{a^2-b^2}{a+3b}+\frac{b^2-c^2}{b+3c}+\frac{c^2-a^2}{c+3a}\right)$$

4051. In any Δ ABC the following relationship holds :

$$\frac{s-n_a-n_b-n_c}{2r}+\sum_{cyc}\frac{n_a}{h_a}\geq\sum_{cyc}\left(\left(\frac{n_a-\sqrt{4r^2+(b-c)^2}}{\sqrt{4r^2+(b-c)^2}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\left(\left(\frac{n_a-\sqrt{4r^2+(b-c)^2}}{\sqrt{4r^2+(b-c)^2}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)\overset{\text{Bogdan Fustei}}{=}$$

$$\sum_{cyc}\left(\left(\frac{n_a-\frac{an_a}{s}}{\frac{an_a}{s}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)=\sum_{cyc}\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)$$

$$=\sum_{cyc}\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}\right)-\sum_{cyc}\frac{sn_a}{2rs}+\sum_{cyc}\frac{n_a}{h_a}\overset{?}{\leq}\frac{s-n_a-n_b-n_c}{2r}+\sum_{cyc}\frac{n_a}{h_a}$$

$$\Leftrightarrow\frac{s}{2r}\underset{(*)}{\underbrace{\overset{?}{\geq}}}\sum_{cyc}\left(\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}\right)\right)$$

$$\text{Now, }\sum_{cyc}\left(\left(\frac{s-a}{a}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}\right)\right)=\frac{1}{abc}\sum_{cyc}((s-a)(bm_c+cm_b))$$

$$=\frac{2rs}{4Rrs}.\sum_{cyc}\left((s-a)\left(\frac{m_c}{h_b}+\frac{m_b}{h_c}\right)\right)\le\frac{1}{2R}.\sum_{cyc}\left((s-a)\left(\frac{R}{r}\right)\right)$$

($\because \frac{R}{r}\ge\frac{m_b}{h_c}+\frac{m_c}{h_b}$ and analogs → reference : article titled "New Triangle Inequalities With Brocard's Angle" by Bogdan Fustei, Mohamed Amine Ben Ajiba; Lemma 12, 6 − 7, published at : www.ssmrmh.ro)

$$=\frac{1}{2r}\sum_{cyc}(s-a)=\frac{s}{2r}\Rightarrow(*)\text{ is true }\therefore\frac{s-n_a-n_b-n_c}{2r}+\sum_{cyc}\frac{n_a}{h_a}$$

$$\ge\sum_{cyc}\left(\left(\frac{n_a-\sqrt{4r^2+(b-c)^2}}{\sqrt{4r^2+(b-c)^2}}\right).\left(\frac{m_b}{b}+\frac{m_c}{c}-\frac{n_a}{h_a}\right)\right)\ \forall\,\Delta\,ABC,$$

" =" iff Δ ABC is equilateral (QED)

4052. In any Δ ABC the following relationship holds :

$$16m_a\ge 9p_a+2g_a+5\sqrt{r_br_c}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$9p_a+2g_a+5\sqrt{r_br_c}\overset{CBS}{\le}\sqrt{(9+2+5)\left(9p_a^2+2g_a^2+5s(s-a)\right)}\overset{?}{\le}16m_a$$

$$\Leftrightarrow 16m_a^2\overset{?}{\ge}9p_a^2+2g_a^2+5s(s-a)$$

$$\Leftrightarrow 16s(s-a)+4(b-c)^2\overset{?}{\ge}9s(s-a)+\frac{9s(3s+a)(b-c)^2}{(2s+a)^2}+$$

$$2s(s-a)-\frac{2(s-a)(b-c)^2}{a}+5s(s-a)$$

(via Bogdan Fustei and Mohamed Amine Ben Ajiba and via Bogdan Fustei)

$$\Leftrightarrow 4+\frac{2(s-a)}{a}\overset{?}{\ge}\frac{9s(3s+a)}{(2s+a)^2}\ (\because (b-c)^2\ge 0)\Leftrightarrow 8s^3-11s^2a+sa^2+2a^3\overset{?}{\ge}0$$

$$\Leftrightarrow (s-a)\left((s-a)(8s+5a)+3a^2\right)\overset{?}{\ge}0\to\text{true}\because s>a$$

$$\therefore 16m_a\ge 9p_a+2g_a+5\sqrt{r_br_c}\ \forall\,\Delta\,ABC,\text{" =" iff } b=c\text{ (QED)}$$

4053. In any Δ ABC the following relationship holds :

$$18p_a\ge 7n_a+4m_a+7g_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$7n_a + 4m_a + 7g_a \overset{CBS}{\leq} \sqrt{(7+4+7)(7n_a^2+4m_a^2+7g_a^2)} \overset{?}{\leq} 18p_a$$

$$\Leftrightarrow 18p_a^2 \overset{?}{\geq} 7n_a^2 + 4m_a^2 + 7g_a^2$$

$$\Leftrightarrow 18s(s-a) + \frac{18s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} 7s(s-a) + \frac{7s}{a}(b-c)^2 + 4s(s-a) + (b-c)^2 + 7s(s-a) - \frac{7(s-a)(b-c)^2}{a}$$

(via Bogdan Fustei and Mohamed Amine Ben Ajiba and via Bogdan Fustei)

$$\Leftrightarrow \frac{18s(3s+a)}{(2s+a)^2} \overset{?}{\geq} \frac{7s}{a} + 1 - \frac{7(s-a)}{a} \; (\because (b-c)^2 \geq 0) \Leftrightarrow \frac{9s(3s+a)}{(2s+a)^2} \overset{?}{\geq} 4$$

$$\Leftrightarrow 11s^2 - 7sa - 4a^2 \overset{?}{\geq} 0 \Leftrightarrow (s-a)(11s+4a) \overset{?}{\geq} 0 \to \text{true} \because s > a$$

$$\therefore 18p_a \geq 7n_a + 4m_a + 7g_a \; \forall \, \Delta ABC, '' ='' \text{ iff } b = c \text{ (QED)}$$

4054. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc} \tan^2 A \geq \sum_{cyc} \cot^2 \frac{A}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \tan^2 A = \left(\sum_{cyc} \tan A\right)^2 - 2\sum_{cyc} \tan A \tan B$$

$$= \left(\frac{\prod_{cyc} \sin A}{\prod_{cyc} \cos A}\right)^2 - 2\left(\frac{\prod_{cyc} \sin A}{\prod_{cyc} \cos A}\right)\left(\sum_{cyc} \cot A\right)$$

$$= \left(\frac{\frac{4Rrs}{8R^3}}{\frac{s^2-4R^2-4Rr-r^2}{4R^2}}\right)^2 - 2\left(\frac{\frac{4Rrs}{8R^3}}{\frac{s^2-4R^2-4Rr-r^2}{4R^2}}\right)\left(\frac{s^2-4Rr-r^2}{2rs}\right)$$

$$= \frac{4r^2s^2 - 2(s^2-4R^2-4Rr-r^2)(s^2-4Rr-r^2)}{(s^2-4R^2-4Rr-r^2)^2} \overset{?}{\geq} \sum_{cyc} \cot^2 \frac{A}{2}$$

$$= \sum_{cyc} \frac{s^2}{r_a^2} = s^2\left(\frac{1}{r^2} - \frac{2(4R+r)}{s^2 r}\right) = \frac{s^2-8Rr-2r^2}{r^2}$$

$$\Leftrightarrow -s^6 + (8R^2+16Rr+2r^2)s^4 - (16R^4+96R^3r+96R^2r^2+24Rr^3-3r^4)s^2$$

$$+8Rr(16R^4+36R^3r+28R^2r^2+9Rr^3+r^4)\boxed{\overset{?}{\underset{(*)}{\geq}}}0$$

$$\text{and} \because P=-(s^2-4R^2-4Rr-r^2)(s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2)$$

$$\overset{\substack{\text{Walker}\\ \text{and}\\ \text{Gerretsen}}}{\geq} 0\ (\text{and} \because s>2R+r) \therefore \text{in order to prove } (*), \text{it suffices to prove :}$$

$$\text{LHS of } (*) \overset{?}{\geq} P \Leftrightarrow -(2R^2+5r^2)s^4+(16R^4+16R^3r+32R^2r^2+48Rr^3+18r^4)s^2-$$

$$\left(32R^6+64R^5r+80R^4r^2+160R^3r^3+158R^2r^4+64Rr^5+9r^6\right)\boxed{\overset{?}{\underset{(**)}{\geq}}}0$$

$$\text{and} \because Q=-(2R^2+5r^2)(s^2-4R^2-4Rr-r^2)(s^2-4R^2-4Rr-3r^2)\overset{\substack{s>2R+r\\ \text{and}\\ \text{Gerretsen}}}{\geq} 0$$

$$\therefore \text{in order to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} Q$$

$$\Leftrightarrow 32R^4+16R^3r+4R^2r^2+8Rr^3+3r^4\boxed{\overset{?}{\underset{(***)}{\geq}}}(8R^2-4Rr+r^2)s^2$$

$$\text{Finally}, (8R^2-4Rr+r^2)s^2\overset{\text{Rouche}}{\leq}$$

$$(8R^2-4Rr+r^2)\left(2R^2+10Rr-r^2+2(R-2r).\sqrt{R^2-2Rr}\right)\overset{?}{\leq}\text{LHS of } (***)$$

$$\Leftrightarrow 2(R-2r)(8R^3-12R^2r+Rr^2-r^3)\overset{?}{\geq}2(R-2r).\sqrt{R^2-2Rr}.(8R^2-4Rr+r^2)$$

$$\text{and} \because R-2r\overset{\text{Euler}}{\geq}0 \therefore \text{it suffices to prove :}$$

$$(8R^3-12R^2r+Rr^2-r^3)\overset{?}{>}(R^2-2Rr)(8R^2-4Rr+r^2)^2$$

$$\Leftrightarrow r^3(32R^3+8R^2r+r^3)\overset{?}{>}0 \to \text{trivially true} \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{\text{cyc}}\tan^2 A \geq \sum_{\text{cyc}}\cot^2\frac{A}{2}\ \forall \text{ acute } \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4055. In ΔABC the following relationship holds:

$$(r_a)^{a^2}.(r_b)^{b^2}.(r_c)^{c^2}\geq(3r)^{36r^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$(r_a)^{a^2}.(r_b)^{b^2}.(r_c)^{c^2}\overset{\text{Weighted GM-HM}}{\geq}\left(\frac{a^2+b^2+c^2}{\frac{a^2}{r_a}+\frac{b^2}{r_b}+\frac{c^2}{r_c}}\right)^{a^2+b^2+c^2}\overset{\text{Chebyshev}}{\geq}$$

$$\geq\left(\frac{(a^2+b^2+c^2)}{\frac{1}{3}\left((a^2+b^2+c^2)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\right)}\right)^{a^2+b^2+c^2}=(3r)^{a^2+b^2+c^2}\overset{Neuberg}{\overbrace{\geq}}(3r)^{36r^2}$$

$$Equality\ \ holds\ \ if\ \ a=b=c.$$

4056. **In ΔABC the following relationship holds:**

$$(r_a+r_b)^c.(r_b+r_c)^a.(r_a+r_c)^b\leq(3R)^{3R\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbajan

$$In\ \Delta ABC\ wlog:\ \ a\leq b\leq c,\quad\ r_a\leq r_b\leq r_c$$

$$\rightarrow\begin{cases}r_a+r_b\leq r_a+r_c\leq r_b+r_c\\ c\geq b\geq a\end{cases}$$

$$(r_a+r_b)^c.(r_b+r_c)^a.(r_a+r_c)^b\overset{Weighted\ \ GM-HM}{\overbrace{\leq}}$$

$$\leq\left(\frac{c(r_a+r_b)+a(r_b+r_c)+b(r_a+r_c)}{a+b+c}\right)^{\sum a}\overset{Chebyshev}{\overbrace{\leq}}$$

$$\leq\left(\frac{\frac{1}{3}(a+b+c)\big(2(r_a+r_b+r_c)\big)}{a+b+c}\right)^{a+b+c}=$$

$$=\left(\frac{2}{3}(4R+r)\right)^{a+b+c}\overset{Euler}{\overbrace{\leq}}\left(\frac{2}{3}\left(4R+\frac{R}{2}\right)\right)^{2p}=(3R)^{2p}\overset{Mitrinovici}{\overbrace{\leq}}(3R)^{3R\sqrt{3}}$$

$$Equality\ \ holds\ \ if\ \ a=b=c.$$

4057. **In ΔABC the following relationship holds:**

$$(m_a+m_b)^c.(m_c+m_b)^a.(m_a+m_c)^b\geq(6r)^{6r\sqrt{3}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$(m_a+m_b)^c.(m_c+m_b)^a.(m_a+m_c)^b\overset{Weighted\ \ GM-HM}{\overbrace{\geq}}$$

$$\geq\left(\frac{a+b+c}{\frac{a}{m_c+m_b}+\frac{b}{m_a+m_c}+\frac{c}{m_a+m_b}}\right)^{\sum a}\geq$$

$$In\ \Delta ABC\ wlog:\ \ a\leq b\leq c,\qquad m_c\leq m_b\leq m_a$$

$$\to\begin{cases}\frac{1}{m_a+m_b}\leq\frac{1}{m_a+m_c}\leq\frac{1}{m_c+m_b}\\ c\geq b\geq a\end{cases}$$

$$\overset{Chebyshev}{\geq}\left(\frac{a+b+c}{\frac{1}{3}(a+b+c)\left(\frac{1}{m_c+m_b}+\frac{1}{m_a+m_c}+\frac{1}{m_a+m_b}\right)}\right)^{\sum a}=$$

$$=\left(\frac{3}{\frac{1}{m_c+m_b}+\frac{1}{m_a+m_c}+\frac{1}{m_a+m_b}}\right)^{\sum a}\overset{\left\{\frac{1}{m_c+m_b}\leq\frac{1}{4}\left(\frac{1}{m_b}+\frac{1}{m_c}\right)\right\}}{\geq}$$

$$\left(\frac{3}{\frac{1}{4}\cdot 2\left(\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}\right)}\right)^{\sum a}=\left(\frac{6}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}}\right)^{\sum a}\geq\left(\frac{6}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}}\right)^{\sum a}=$$

$$=(6r)^{\sum a}\overset{Mitrinovici}{\geq}(6r)^{6r\sqrt{3}}$$

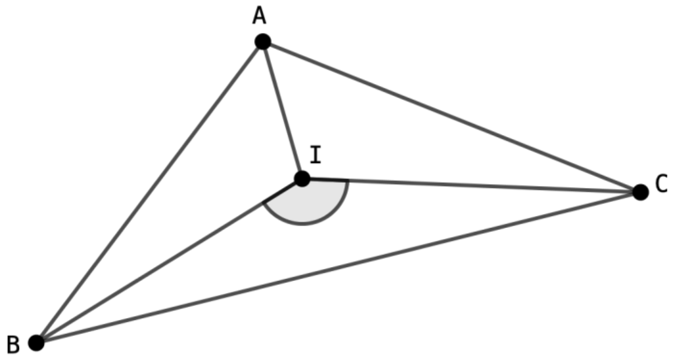
$$Equality\ holds\ if\ a=b=c$$

4058. In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\sin A+\sin B+\sin C=4\sin\left(\widehat{AIB}\right)\sin\left(\widehat{BIC}\right)\sin\left(\widehat{CIA}\right)$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania



$$\sphericalangle BIC=180^\circ-\frac{\widehat{B}}{2}-\frac{\widehat{C}}{2}=180^\circ-\frac{\widehat{B}+\widehat{C}}{2}=$$

$$=180^\circ-\frac{180^\circ-\widehat{A}}{2}=180^\circ-90^\circ+\frac{\widehat{A}}{2}=90^\circ+\frac{\widehat{A}}{2}$$

$$\sin(\widehat{BIC})=\sin\left(90^\circ+\frac{A}{2}\right)=\sin 90^\circ\cos\frac{A}{2}+\sin\frac{A}{2}\cos 90^\circ=\cos\frac{A}{2}$$

Analogous: $\sin(\widehat{AIB})=\cos\frac{C}{2};\sin(\widehat{CIA})=\sin\frac{B}{2}$

$$\sin A+\sin B+\sin C=2\sin\frac{A+B}{2}\cos\frac{A-B}{2}+\sin\left(2\cdot\frac{C}{2}\right)=$$

$$=2\sin\frac{180^\circ-C}{2}\cos\frac{A-B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}=$$

$$=2\cos\frac{C}{2}\cos\frac{A-B}{2}+2\cos\frac{C}{2}\cdot\cos\left(90^\circ-\frac{C}{2}\right)=$$

$$=2\cos\frac{C}{2}\left(\cos\frac{A-B}{2}+\cos\frac{180^\circ-C}{2}\right)=$$

$$=2\cos\frac{C}{2}\cdot 2\cos\frac{A-B+180^\circ-C}{4}\cos\frac{A-B-180^\circ+C}{4}=$$

$$=4\cos\frac{C}{2}\cos\frac{A-B-C+A+B+C}{4}\cos\frac{A-B-A-B-C+C}{4}=$$

$$=4\cos\frac{C}{2}\cos\frac{2A}{4}\cos\left(-\frac{2B}{4}\right)=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=4\sin(\widehat{AIB})\sin(\widehat{BIC})\sin(\widehat{CIA})$$

4059. **In ΔABC the following relationship holds:**

$$\sin 2026A+\sin 2026B+\sin 2026C=4\sin 1013A\sin 1013B\sin 1013C$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Denote:

$$1013A=x;1013B=y;1013C=z$$

$$x+y+z=1013(A+B+C)=1013\pi \quad (1)$$

$$\cos\left(\frac{1013\pi}{2}-y\right)=\cos\frac{1013\pi}{2}\cos y+\sin\frac{1013\pi}{2}\sin y=$$

$$=\cos\left(506\pi+\frac{\pi}{2}\right)\cos y+\sin\left(506\pi+\frac{\pi}{2}\right)\sin y=\cos\frac{\pi}{2}\cos y+\sin\frac{\pi}{2}\sin y=\sin y$$

$$\cos\left(\frac{1013\pi}{2}-y\right)=\sin y \quad (2)$$

Analogous:

$$\cos\left(x-\frac{1013\pi}{2}\right)=\cos\left(\frac{1013\pi}{2}-x\right)=\sin x \quad (3)$$

$$\sin 2026A+\sin 2026B+\sin 2026C=\sin 2x+\sin 2y+\sin 2z=$$

$$=2\sin\frac{2x+2y}{2}\cos\frac{2x-2y}{2}+2\sin z\cos z=$$

$$=2\sin(x+y)\cos(x-y)+2\sin z\cos z\overset{(1)}{=}$$

$$=2\sin(1013\pi-z)\cos(x-y)+2\sin z\cos z=$$

$$=2(\sin 1013\pi\cos z-\sin z\cos 1013\pi)\cos(x-y)+2\sin z\cos z=$$

$$=2(0\cdot\cos z-\sin z\cdot(-1)^{1013})\cos(x-y)+2\sin z\cos z=$$

$$=2\sin z\cos(x-y)+2\sin z\cos z=2\sin z\left(\cos(x-y)+\cos z\right)=$$

$$=2\sin z\cdot 2\cos\frac{x-y+z}{2}\cdot\cos\frac{x-y-z}{2}=$$

$$=4\sin z\cos\frac{x+z-y}{2}\cos\frac{y+z-x}{2}\overset{(1)}{=}$$

$$=4\sin z\cos\left(\frac{1013\pi-2y}{2}\right)\cos\left(\frac{1013\pi-2x}{2}\right)\overset{(2),(3)}{=}4\sin z\sin y\sin x=$$

$$=4\sin 1013A\sin 1013B\sin 1013C$$

4060. If in ΔABC we have: $\tan\frac{A}{2}\tan\frac{B}{2}=\frac{1}{3}$ then:

$$2\sin C=\sin A+\sin B$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{1}{3}\Rightarrow\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\frac{1}{3}$$

$$\Rightarrow\frac{s-c}{s}=\frac{1}{3}\Rightarrow 3s-3c=s\Rightarrow 2s-3c=0$$

$$\Rightarrow a+b+c-3c=0\Rightarrow 2c=a+b$$

$$2\cdot 2R\sin C=2R\sin A+2R\sin B$$

$$2\sin C=\sin A+\sin B$$

4061. In ΔABC the following relationship holds:

$$\cot A+\cot B+\cot C=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\cot A+\cot B+\cot C=\sum_{cyc}\cot A=\sum_{cyc}\frac{\cos A}{\sin A}=$$

$$=\sum_{cyc}\frac{\frac{b^2+c^2-a^2}{2bc}}{\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{4F}=$$

$$=\frac{1}{4F}\left(\sum_{cyc}b^2+\sum_{cyc}c^2-\sum_{cyc}a^2\right)=\frac{1}{4F}\left(\sum_{cyc}a^2+\sum_{cyc}a^2-\sum_{cyc}a^2\right)=\frac{1}{4F}\sum_{cyc}a^2=$$

$$=\frac{1}{4F}(a^2+b^2+c^2)=\frac{1}{4\cdot 2R^2\sin A\sin B\sin C}\sum_{cyc}4R^2\sin^2 A=$$

$$=\frac{4R^2}{8R^2}\cdot\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\sin A\sin B\sin C}=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}$$

4062. In any Δ ABC the following relationship holds :

$$g_a n_a\le s(s-a)+\frac{1}{2}(b-c)^2\le m_a n_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$g_a n_a\overset{?}{\le} s(s-a)+\frac{1}{2}(b-c)^2$$

$$\Leftrightarrow\left(s(s-a)-\frac{(s-a)(b-c)^2}{a}\right)\left(s(s-a)+\frac{s}{a}(b-c)^2\right)\overset{?}{\le}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow s^2(s-a)^2+\frac{s^2(s-a)}{a}(b-c)^2-\frac{s(s-a)^2}{a}(b-c)^2-\frac{s(s-a)}{a^2}(b-c)^4\overset{?}{\le}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow \left(\frac{1}{4}+\frac{s(s-a)}{a^2}\right)(b-c)^2+s(s-a)\left(1+\frac{s-a}{a}-\frac{s}{a}\right)\overset{?}{\geq} 0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow \frac{(2s-a)^2(b-c)^2}{4a^2}\overset{?}{\geq} 0 \to \text{true} \therefore g_a n_a \leq s(s-a)+\frac{1}{2}(b-c)^2$$

$$\text{and } m_a n_a \overset{?}{\geq} s(s-a)+\frac{1}{2}(b-c)^2$$

$$\Leftrightarrow \left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s}{a}(b-c)^2\right)\overset{?}{\geq}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow s^2(s-a)^2+\frac{s^2(s-a)}{a}(b-c)^2+\frac{s(s-a)}{4}(b-c)^2+\frac{s}{4a}(b-c)^4\overset{?}{\geq}$$

$$s^2(s-a)^2+\frac{(b-c)^4}{4}+s(s-a)(b-c)^2$$

$$\Leftrightarrow \frac{s-a}{4a}.(b-c)^2+s(s-a)\left(\frac{s}{a}+\frac{1}{4}-1\right)\overset{?}{\geq} 0\ (\because (b-c)^2\geq 0)$$

$$\Leftrightarrow \frac{s-a}{4a}.(b-c)^2+s(s-a)\left(\frac{4s-3a}{a}\right)\overset{?}{\geq} 0 \to \text{true} \because s>a$$

$$\therefore m_a n_a \geq s(s-a)+\frac{1}{2}(b-c)^2 \text{ and so,}$$

$$g_a n_a \leq s(s-a)+\frac{1}{2}(b-c)^2 \leq m_a n_a \forall \Delta ABC,''='' \text{ iff } b=c\ (\text{QED})$$

4063. In any Δ ABC the following relationship holds :

$$s(s-a)+\frac{2}{3}(b-c)^2 \leq p_a n_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$p_a n_a \overset{?}{\geq} s(s-a)+\frac{2}{3}(b-c)^2$$

$$\Leftrightarrow \left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)\left(s(s-a)+\frac{s}{a}(b-c)^2\right)\overset{?}{\geq}$$

$$s^2(s-a)^2+\frac{4(b-c)^4}{9}+\frac{4}{3}s(s-a)(b-c)^2$$

$$\Leftrightarrow s^2(s-a)^2+\frac{s^2(s-a)}{a}(b-c)^2+\frac{s^2(3s+a)(s-a)}{(2s+a)^2}(b-c)^2+$$

$$\frac{s^2(3s+a)}{a(2s+a)^2}(b-c)^4 \overset{?}{\geq} s^2(s-a)^2+\frac{4(b-c)^4}{9}+\frac{4}{3}s(s-a)(b-c)^2$$

$$\Leftrightarrow \left(\frac{s^2(3s+a)}{a(2s+a)^2}-\frac{4}{9}\right)(b-c)^2+s(s-a)\left(\frac{s}{a}+\frac{s(3s+a)}{(2s+a)^2}-\frac{4}{3}\right)\overset{?}{\geq} 0$$

$$(\because (b-c)^2\geq 0) \Leftrightarrow \frac{(s-a)(27s^2+20sa+4a^2)}{9a(2s+a)^2}.(b-c)^2+$$

$$s(s-a).\frac{(s-a)(12s^2+17sa+7a^2)+3a^3}{3a(2s+a)^2}\overset{?}{\geq} 0 \to \text{true} \because s>a$$

$$\therefore s(s-a)+\frac{2}{3}(b-c)^2\leq p_a n_a\ \forall\ \Delta\ ABC, ''='' \text{ iff } b=c\ (QED)$$

4064. In ΔABC the following relationship holds

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}\geq 8\sqrt{3}R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}=\sum_{cyc}\frac{a+b}{\sin^2 C}\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{\sin^2 A\sin^2 B\sin^2 C}}\overset{CESARO}{\geq}$$

$$\geq 3\sqrt[3]{\frac{8abc}{\frac{a^2}{4R^2}\cdot\frac{b^2}{4R^2}\cdot\frac{c^2}{4R^2}}}=3\cdot 4R^2\sqrt[3]{\frac{8abc}{a^2b^2c^2}}=12R^2\cdot 2\cdot\frac{1}{\sqrt[3]{abc}}=\frac{24R^2}{\sqrt[3]{4Rrs}}\geq$$

$$\overset{EULER}{\geq}\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot s}}=\frac{24R^2}{\sqrt[3]{2R^2\cdot\frac{3\sqrt{3}}{2}R}}=\frac{24R^2}{\sqrt[3]{\left(\sqrt{3}R\right)^3}}=\frac{24R^2}{\sqrt{3}R}=\frac{24R}{\sqrt{3}}=\frac{24\sqrt{3}R}{3}=8\sqrt{3}R$$

Equality holds for $a=b=c$.

4065. In ΔABC the following relationship holds:

$$\frac{a+b}{(\sin A+\sin B)^2}+\frac{b+c}{(\sin B+\sin C)^2}+\frac{c+a}{(\sin C+\sin A)^2}\geq 2\sqrt{3}R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{(\sin A+\sin B)^2}+\frac{b+c}{(\sin B+\sin C)^2}+\frac{c+a}{(\sin C+\sin A)^2}=$$

$$=\sum_{cyc}\frac{a+b}{(\sin A+\sin B)^2}=\sum_{cyc}\frac{a+b}{\left(\frac{a}{2R}+\frac{b}{2R}\right)^2}=4R^2\sum_{cyc}\frac{a+b}{(a+b)^2}=$$

$$=4R^2\cdot\sum_{cyc}\frac{1^2}{a+b}\overset{BERGSTROM}{\geq}4R^2\cdot\frac{(1+1+1)^2}{a+b+b+c+c+a}=$$

$$=4R^2\cdot\frac{9}{4s}=\frac{9R^2}{s}\overset{MITRINOVIC}{\geq}\frac{9R^2}{\frac{3\sqrt{3}}{2}R}=\frac{3R\cdot 2}{\sqrt{3}}=2\sqrt{3}R$$

Equality holds for $a=b=c$.

4066. In ΔABC the following relationship holds:

$$\frac{a+b}{(\sin A+\sin B)^3}+\frac{b+c}{(\sin B+\sin C)^3}+\frac{c+a}{(\sin C+\sin A)^3}\geq 2R$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{(\sin A+\sin B)^3}+\frac{b+c}{(\sin B+\sin C)^3}+\frac{c+a}{(\sin C+\sin A)^3}=$$

$$=\sum_{cyc}\frac{a+b}{(\sin A+\sin B)^3}=\sum_{cyc}\frac{a+b}{\left(\frac{a}{2R}+\frac{b}{2R}\right)^3}=8R^3\sum_{cyc}\frac{a+b}{(a+b)^3}=8R^3\sum_{cyc}\frac{1}{(a+b)^2}\overset{RADON}{\geq}$$

$$\geq 8R^3\cdot\frac{(1+1+1)^3}{(a+b+b+c+c+a)^2}=\frac{8R^3\cdot 27}{4\cdot 4s^2}=\frac{27R^3}{2s^2}\overset{MITRINOVICI}{\geq}\frac{27R^3}{2\cdot\left(\frac{3\sqrt{3}}{2}R\right)^2}=$$

$$=\frac{27R^3}{2\cdot\frac{27R^2}{4}}=2R$$

Equality holds for $a=b=c$.

4067. In ΔABC the following relationship holds:

$$\frac{a}{\sin^3 B}+\frac{b}{\sin^3 C}+\frac{c}{\sin^3 A}\geq 8R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{\sin^3 B}+\frac{b}{\sin^3 C}+\frac{c}{\sin^3 A}=\sum_{cyc}\frac{a}{\sin^3 B}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{abc}{\sin^3 A\sin^3 B\sin^3 C}}=$$

$$=3\sqrt[3]{\frac{abc}{\left(\frac{abc}{2R}\right)^3}}=3\sqrt[3]{\frac{(8R^3)^3}{(abc)^2}}=\frac{24R^3}{\sqrt[3]{(abc)^2}}=\frac{24R^3}{\sqrt[3]{16R^2r^2s^2}}\overset{EULER}{\geq}$$

$$\geq\frac{24R^3}{\sqrt[3]{16R^2\cdot\frac{R^2}{4}s^2}}\overset{MITRINOVIC}{\geq}\frac{24R^3}{\sqrt[3]{4R^4\cdot\frac{27R^2}{4}}}=\frac{24R^3}{3R^2}=8R$$

Equality holds for $a=b=c$.

4068. In ΔABC the following relationship holds:

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}\leq\frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}=\sum_{cyc}\frac{h_a}{b}=\sum_{cyc}\frac{\frac{2F}{a}}{b}=2F\sum_{cyc}\frac{1}{ab}=2F\cdot\frac{a+b+c}{abc}=$$

$$=2F\cdot\frac{2s}{4RF}=\frac{s}{R}\overset{MITRINOVICI}{\leq}\frac{1}{R}\cdot\frac{3\sqrt{3}}{2}R=\frac{3\sqrt{3}}{2}$$

Equality holds for $a=b=c$.

4069. In ΔABC the following relationship holds:

$$\frac{a+b}{\sin A}+\frac{b+c}{\sin B}+\frac{c+a}{\sin C}\geq 12R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{\sin A}+\frac{b+c}{\sin B}+\frac{c+a}{\sin C}=\sum_{cyc}\frac{a+b}{\sin A}=\sum_{cyc}\frac{a+b}{\frac{a}{2R}}=2R\sum_{cyc}\frac{a+b}{a}\overset{AM-GM}{\geq}$$

$$\geq 2R\cdot 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}\overset{CESARO}{\geq}6R\cdot\sqrt[3]{\frac{8abc}{abc}}=6R\sqrt[3]{8}=6R\cdot 2=12R$$

Equality holds for $a=b=c$.

4070. In ΔABC the following relationship holds:

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}\leq\frac{3\sqrt{3}}{4}\cdot\frac{R}{r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}=\sum_{cyc}\frac{h_a}{b}=\sum_{cyc}\frac{\frac{2F}{a}}{b}=2F\sum_{cyc}\frac{1}{ab}=2F\sum_{cyc}\frac{c}{abc}=$$

$$=\frac{2F}{abc}\sum_{cyc}c=\frac{2F}{4RF}(a+b+c)=\frac{1}{2R}\cdot 2s=\frac{s}{r}\overset{EULER}{\leq}\frac{s}{2r}\overset{MITRINOVIC}{\leq}$$

$$\leq\frac{3\sqrt{3}}{2}R\cdot\frac{1}{2r}=\frac{3\sqrt{3}}{4}\cdot\frac{R}{r}$$

Equality holds for $a=b=c$.

4071. In ΔABC the following relationship holds:

$$\frac{r_ar_br_c}{h_ah_bh_c}=\frac{R}{2r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_ar_br_c}{h_ah_bh_c}=\frac{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}=\frac{abc}{8(s-a)(s-b)(s-c)}=$$

$$=\frac{abcs}{8s(s-a)(s-b)(s-c)}=\frac{4RFs}{8F^2}=\frac{4Rr}{8F}=\frac{4Rs}{8rs}=\frac{R}{2r}$$

4072. In any Δ ABC the following relationship holds :

$$\frac{|b^2-c^2|}{a}\le\min\left\{4.\sqrt{R(R-2r)},2\sqrt{s^2-4Rr-19r^2},4.\sqrt{\frac{62}{135}(2s^2-27Rr)}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $x=s-a,y=s-b,z=s-c$ & then : $a=y+z,b=z+x,c=x+y$ and $s=x+y+z$ and furthermore, we denote : $\frac{y+z}{x}=m$ and $\frac{yz}{x^2}=n$ and then, we have the following set "S" of relations : $y^2+z^2=x^2(m^2-2n)$, $y^3+z^3=x^3(m^3-3nn)$, $y^4+z^4=x^4((m^2-2n)^2-2n^2)$, $y^5+z^5=x^5\left(m((m^2-2n)^2+n^2-nm^2)\right)$, $y^6+z^6=x^6((m^3-3nn)^2-2n^3)$,

$$\text{and now, }\frac{|b^2-c^2|}{a}\overset{?}{\le}4.\sqrt{R(R-2r)}$$

$$\Leftrightarrow\frac{(b+c)^2(b-c)^2}{a^2}\overset{?}{\le}16\left(\frac{a^2b^2c^2}{16s(s-a)(s-b)(s-c)}-\frac{abc}{2s}\right)$$

$$\Leftrightarrow\frac{(2x+y+z)^2(y-z)^2}{(y+z)^2}\overset{?}{\le}16.\frac{(y+z)^2(z+x)^2(x+y)^2-8xyz(y+z)(z+x)(x+y)}{16xyz(x+y+z)}$$

$$\Leftrightarrow x^4(y^2+z^2)^2+12x^4.y^2z^2+2x^3(y^5+z^5)-6x^3.yz(y^3+z^3)+$$
$$4x^3.y^2z^2(y+z)+x^2(y^6+z^6)-5x^2.yz(y^4+z^4)-9x^2.y^2z^2(y^2+z^2)-$$
$$6x^2.y^3z^3+x.yz(y^5+z^5)+x.y^2z^2(y^3+z^3)-2x.y^3z^3(y+z)+y^2z^2(y+z)^4$$

$\boxed{\overset{?}{\underset{①}{\geq}}}$ 0 and via set of relations "S", in order to prove ①, it suffices to prove, following simplification : $\overbrace{(m^4-4m^3+20m^2+32m+16)}^{\Omega_1}n^2+\overbrace{(m^5-11m^4-16m^3-4m^2)}^{\Omega_2}n+m^4(m+1)^2\boxed{\overset{?}{\underset{②}{\geq}}}0$ and now, LHS of ② is a quadratic polynomial in "n" with discriminant, $\delta=\Omega_2^2-4\Omega_1.m^4(m+1)^2$

$$= -m^4(m+1)^2(3m+2)(m+6)(m-2)^2 \leq 0 \rightarrow \text{true } (\because m > 0)$$

$$\therefore \boxed{\frac{|b^2-c^2|}{a} \leq 4.\sqrt{R(R-2r)}} \text{ and again, } \frac{|b^2-c^2|}{a} \overset{?}{\leq} 2\sqrt{s^2-4Rr-19r^2}$$

$$\Leftrightarrow \frac{(2x+y+z)^2(y-z)^2}{(y+z)^2} \overset{?}{\leq} \frac{4((x+y+z)^3-(y+z)(z+x)(x+y)-19xyz)}{x+y+z}$$

$$\Leftrightarrow 16x^3.yz + 32x^2.yz(y+z) + 3x(y^4+z^4) - 44x.yz(y^2+z^2) - 94x.y^2z^2 + 3(y+z)^5 \boxed{\overset{?}{\underset{(i)}{\geq}}} 0$$ and via set of relations "S", to prove (i), it suffices to prove, following simplification : $3m^5+3m^4+n(16+32m-56m^2) \boxed{\overset{?}{\underset{(ii)}{\geq}}} 0$ and it's trivially true if : $16+32m-56m^2 \geq 0$ and when : $16+32m-56m^2 < 0$, then, since $n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$, in order to prove (ii), it suffices to prove :

$$3m^5+3m^4+m^2(4+8m-14m^2) \overset{?}{\geq} 0 \Leftrightarrow m^2(3m+1)(m-2)^2 \overset{?}{\geq} 0$$

$$\rightarrow \text{true } \therefore \boxed{\frac{|b^2-c^2|}{a} \leq 2\sqrt{s^2-4Rr-19r^2}} \;\&\; \frac{|b^2-c^2|}{a} \overset{?}{\leq} 4.\sqrt{\frac{62}{135}(2s^2-27Rr)}$$

$$\Leftrightarrow \frac{(2x+y+z)^2(y-z)^2}{(y+z)^2} \overset{?}{\leq} 4.\frac{496(x+y+z)^3-1674(y+z)(z+x)(x+y)}{135(x+y+z)}$$

$$\Leftrightarrow 1444x^3(y+z)^2 + 2160x^3.yz - 1824x^2(y+z)^3 + 4320x^2.yz(y+z) - 1419x(y^2+z^2)^2 - 276x.y^2z^2 - 2976x.yz(y^2+z^2) + 1849\left(y^5+z^5\right) + 3089yz(y^3+z^3) + 22y^2z^2(y+z) \boxed{\overset{?}{\underset{(a)}{\geq}}} 0$$ and via set of relations "S",

in order to prove (a), it suffices to prove, following simplification : $1849m^5 - 1419m^4 - 1824m^3 + 1444m^2 + n(2160+4320m+2700m^2-6156m^3) \boxed{\overset{?}{\underset{(b)}{\geq}}} 0$ and it's trivially true if : $2160+4320m+2700m^2-6156m^3 \geq 0$ and when : $2160+4320m+2700m^2-6156m^3 < 0$, then, since $n \overset{\text{AM-GM}}{\leq} \frac{m^2}{4}$,

$\therefore$ to prove (b), it suffices to prove : $1849m^5 - 1419m^4 - 1824m^3 + 1444m^2 + m^2(540+1080m+675m^2-1539m^3) \overset{?}{\geq} 0 \Leftrightarrow 62m^2(5m+8)(m-2)^2 \overset{?}{\geq} 0$

$$\rightarrow \text{true } (\because m > 0) \therefore \boxed{\frac{|b^2-c^2|}{a} \leq 4.\sqrt{\frac{62}{135}(2s^2-27Rr)}} \text{ and so,}$$

$$\frac{|b^2-c^2|}{a} \leq \min\left\{4.\sqrt{R(R-2r)}, 2\sqrt{s^2-4Rr-19r^2}, 4.\sqrt{\frac{62}{135}(2s^2-27Rr)}\right\}$$

$$\forall\,\Delta\,\mathrm{ABC},\,''=''\text{ for }1^{\mathrm{st}}\text{ inequality iff }\mathrm{b}+\mathrm{c}=2a\text{ and }''=''\text{ for }2^{\mathrm{nd}}\text{ and }3^{\mathrm{rd}}\text{ inequalities iff }\Delta\,\mathrm{ABC}\text{ is equilateral (QED)}$$

4073. In any Δ ABC the following relationship holds :

$$\mathrm{s}(\mathrm{s}-a)+\frac{1}{3}(\mathrm{b}-\mathrm{c})^2\le \mathrm{p}_a\mathrm{m}_a\le \mathrm{s}(\mathrm{s}-a)+\frac{1}{2}(\mathrm{b}-\mathrm{c})^2$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\mathrm{s}(\mathrm{s}-a)+\frac{1}{3}(\mathrm{b}-\mathrm{c})^2\overset{?}{\le}\mathrm{p}_a\mathrm{m}_a$$

$$\Leftrightarrow \mathrm{s}^2(\mathrm{s}-a)^2+\frac{(\mathrm{b}-\mathrm{c})^4}{9}+\frac{2}{3}\mathrm{s}(\mathrm{s}-a)(\mathrm{b}-\mathrm{c})^2\overset{?}{\le}$$

$$\left(\mathrm{s}(\mathrm{s}-a)+\frac{\mathrm{s}(3\mathrm{s}+a)(\mathrm{b}-\mathrm{c})^2}{(2\mathrm{s}+a)^2}\right)\left(\mathrm{s}(\mathrm{s}-a)+\frac{(\mathrm{b}-\mathrm{c})^2}{4}\right)$$

$$\Leftrightarrow \mathrm{s}^2(\mathrm{s}-a)^2+\frac{(\mathrm{b}-\mathrm{c})^4}{9}+\frac{2}{3}\mathrm{s}(\mathrm{s}-a)(\mathrm{b}-\mathrm{c})^2\overset{?}{\le}\mathrm{s}^2(\mathrm{s}-a)^2+\mathrm{s}(\mathrm{s}-a).\frac{(\mathrm{b}-\mathrm{c})^2}{4}+$$

$$\frac{\mathrm{s}(3\mathrm{s}+a)(\mathrm{b}-\mathrm{c})^2}{(2\mathrm{s}+a)^2}.\mathrm{s}(\mathrm{s}-a)+\frac{\mathrm{s}(3\mathrm{s}+a)}{(2\mathrm{s}+a)^2}.\frac{(\mathrm{b}-\mathrm{c})^4}{4}$$

$$\Leftrightarrow\left(\frac{\mathrm{s}(3\mathrm{s}+a)}{4(2\mathrm{s}+a)^2}-\frac{1}{9}\right)(\mathrm{b}-\mathrm{c})^2+$$

$$\mathrm{s}(\mathrm{s}-a).\left(\frac{1}{4}+\frac{\mathrm{s}(3\mathrm{s}+a)}{(2\mathrm{s}+a)^2}-\frac{2}{3}\right)\overset{?}{\ge}0\ (\because(\mathrm{b}-\mathrm{c})^2\ge 0)$$

$$\Leftrightarrow\frac{(\mathrm{s}-a)(11\mathrm{s}+4a)}{36(2\mathrm{s}+a)^2}(\mathrm{b}-\mathrm{c})^2+\mathrm{s}(\mathrm{s}-a)\left(\frac{(\mathrm{s}-a)(16\mathrm{s}+8a)+3a^2}{12(2\mathrm{s}+a)^2}\right)\overset{?}{\ge}0\to\text{true}$$

$$\because \mathrm{s}>a\therefore \mathrm{p}_a\mathrm{m}_a\ge \mathrm{s}(\mathrm{s}-a)+\frac{1}{3}(\mathrm{b}-\mathrm{c})^2$$

$$\text{Again, }\mathrm{p}_a\mathrm{m}_a\overset{?}{\le}\mathrm{s}(\mathrm{s}-a)+\frac{1}{2}(\mathrm{b}-\mathrm{c})^2$$

$$\Leftrightarrow \mathrm{s}^2(\mathrm{s}-a)^2+\mathrm{s}(\mathrm{s}-a).\frac{(\mathrm{b}-\mathrm{c})^2}{4}+\frac{\mathrm{s}(3\mathrm{s}+a)(\mathrm{b}-\mathrm{c})^2}{(2\mathrm{s}+a)^2}.\mathrm{s}(\mathrm{s}-a)+$$

$$\frac{\mathrm{s}(3\mathrm{s}+a)}{(2\mathrm{s}+a)^2}.\frac{(\mathrm{b}-\mathrm{c})^4}{4}\overset{?}{\le}\mathrm{s}^2(\mathrm{s}-a)^2+\frac{(\mathrm{b}-\mathrm{c})^4}{4}+\mathrm{s}(\mathrm{s}-a)(\mathrm{b}-\mathrm{c})^2$$

$$\Leftrightarrow\left(\frac{1}{4}-\frac{\mathrm{s}(3\mathrm{s}+a)}{4(2\mathrm{s}+a)^2}\right).(\mathrm{b}-\mathrm{c})^2+\mathrm{s}(\mathrm{s}-a).\left(\frac{3}{4}-\frac{\mathrm{s}(3\mathrm{s}+a)}{(2\mathrm{s}+a)^2}\right)\overset{?}{\ge}0\ (\because(\mathrm{b}-\mathrm{c})^2\ge 0)$$

$$\Leftrightarrow\frac{\mathrm{s}^2+3\mathrm{s}a+a^2}{4(2\mathrm{s}+a)^2}.(\mathrm{b}-\mathrm{c})^2+\mathrm{s}(\mathrm{s}-a).\frac{8\mathrm{s}a+3a^2}{4(2\mathrm{s}+a)^2}\overset{?}{\ge}0\to\text{true}\because \mathrm{s}>a$$

$$\therefore \mathrm{p}_a\mathrm{m}_a\le \mathrm{s}(\mathrm{s}-a)+\frac{1}{2}(\mathrm{b}-\mathrm{c})^2\text{ and so, }\mathrm{s}(\mathrm{s}-a)+\frac{1}{3}(\mathrm{b}-\mathrm{c})^2\le \mathrm{p}_a\mathrm{m}_a\le$$

$$s(s-a)+\frac{1}{2}(b-c)^2\ \forall\,\Delta\,ABC,''=''\text{ iff } b=c\ (QED)$$

4074. **In any Δ ABC the following relationship holds:**

$$\sum_{cyc}\frac{1}{h_a\cdot\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\geq\sqrt{\frac{3}{2r}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{h_a\cdot\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\overset{?}{\geq}\sqrt{\frac{3}{2r}}\Leftrightarrow\sum_{cyc}\frac{1}{\frac{3r}{x}\cdot\sqrt{\frac{y+z}{3r}}}\overset{?}{\geq}\sqrt{\frac{3}{2r}}\left(x=\frac{3r}{h_a},y=\frac{3r}{h_b},z=\frac{3r}{h_c}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{\sqrt{y+z}}\underset{(*)}{\underbrace{\overset{?}{\geq}}}\frac{3}{\sqrt{2}}$$

$$\text{Indeed, }\sum_{cyc}\frac{x}{\sqrt{y+z}}=\sum_{cyc}\frac{x^2}{x.\sqrt{y+z}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{xy+zx}\right)}\overset{\text{CBS}}{\geq}$$

$$\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)}}\geq\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right).\frac{\left(\sum_{cyc}x\right)^2}{3}}}=\frac{9}{\sqrt{6.\frac{9}{3}}}$$

$$\left(\because\sum_{cyc}x=3r.\sum_{cyc}\frac{1}{h_a}=\frac{3r}{r}=3\right)=\frac{3}{\sqrt{2}}\Rightarrow(*)\text{ is true and so,}$$

$$\sum_{cyc}\frac{1}{h_a\cdot\sqrt{\frac{1}{h_b}+\frac{1}{h_c}}}\geq\sqrt{\frac{3}{2r}}\ \forall\,\Delta\,ABC,''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

4075. **In any Δ ABC the following relationship holds:**

$$\sum_{cyc}\frac{1}{r_a\cdot\sqrt{\frac{1}{r_b}+\frac{1}{r_c}}}\geq\sqrt{\frac{3}{2r}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{r_a\cdot\sqrt{\frac{1}{r_b}+\frac{1}{r_c}}}\overset{?}{\geq}\sqrt{\frac{3}{2r}}\Leftrightarrow\sum_{cyc}\frac{1}{\frac{3r}{x}\cdot\sqrt{\frac{y+z}{3r}}}\overset{?}{\geq}\sqrt{\frac{3}{2r}}\left(x=\frac{3r}{r_a},y=\frac{3r}{r_b},z=\frac{3r}{r_c}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{x}{\sqrt{y+z}}\underset{(*)}{\overset{?}{\geq}}\frac{3}{\sqrt{2}}$$

$$\text{Indeed,}\sum_{cyc}\frac{x}{\sqrt{y+z}}=\sum_{cyc}\frac{x^2}{x.\sqrt{y+z}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{xy+zx}\right)}\overset{\text{CBS}}{\geq}$$

$$\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)}}\geq\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right).\frac{\left(\sum_{cyc}x\right)^2}{3}}}=\frac{9}{\sqrt{6.\frac{9}{3}}}$$

$$\left(\because\sum_{cyc}x=3r.\sum_{cyc}\frac{1}{r_a}=\frac{3r}{r}=3\right)=\frac{3}{\sqrt{2}}\Rightarrow(*)\text{ is true and so,}$$

$$\sum_{cyc}\frac{1}{r_a\cdot\sqrt{\frac{1}{r_b}+\frac{1}{r_c}}}\geq\sqrt{\frac{3}{2r}}\ \forall\,\Delta\,ABC,''=''\ \text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

4076. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\sqrt{\cot A}}{a}\leq\frac{1}{2r^2}\cdot\sqrt{\frac{2R^2+r^2}{3\sqrt{3}}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\sqrt{\cot A}}{a}\overset{\text{CBS}}{\leq}\sqrt{\sum_{cyc}\cot A}\,.\sqrt{\frac{1}{16R^2r^2s^2}\cdot\sum_{cyc}a^2b^2}\overset{\text{Goldstone}}{\leq}$$

$$\sqrt{\frac{s^2-4Rr-r^2}{2rs}}\cdot\sqrt{\frac{4R^2s^2}{16R^2r^2s^2}}\overset{\substack{\text{Mitrinovic}\\ \text{and}\\ \text{Gerretsen}}}{\leq}\sqrt{\frac{4R^2+2r^2}{2r^2.3\sqrt{3}}}\cdot\sqrt{\frac{1}{4r^2}}=\frac{1}{2r^2}\cdot\sqrt{\frac{2R^2+r^2}{3\sqrt{3}}}$$

$$\text{and so,} \sum_{cyc}\frac{\sqrt{\cot A}}{a} \le \frac{1}{2r^2}\cdot\sqrt{\frac{2R^2+r^2}{3\sqrt{3}}}\ \forall\ \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4077. In any Δ ABC the following relationship holds :

$$10\sum_{cyc}\left(\frac{r}{h_a}\right)^3 - 9\sum_{cyc}\left(\frac{r}{h_a}\right)^5 \ge 1$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = \frac{3r}{h_a}, y = \frac{3r}{h_b}, z = \frac{3r}{h_c} \text{ and then : } \sum_{cyc} x = 3r.\sum_{cyc}\frac{1}{h_a} = \frac{3r}{r} = 3 \text{ and so,}$$

$$10\sum_{cyc}\left(\frac{r}{h_a}\right)^3 - 9\sum_{cyc}\left(\frac{r}{h_a}\right)^5 \overset{?}{\ge} 1$$

$$\Leftrightarrow \frac{10}{243}\left(\sum_{cyc} x^3\right)\left(\sum_{cyc} x\right)^2 - \frac{9}{243}\left(\sum_{cyc} x^5\right) \overset{?}{\ge} \frac{1}{243}\left(\sum_{cyc} x\right)^5$$

$$\Leftrightarrow \sum_{cyc} x^4y + \sum_{cyc} xy^4 \overset{?}{\ge} 2xyz\sum_{cyc} xy \rightarrow \text{true} \because \sum_{cyc} x^4y + \sum_{cyc} xy^4 = \sum_{cyc}\left(z(x^4+y^4)\right)$$

$$\overset{AM-GM}{\ge} 2\sum_{cyc}(zx^2y^2) = 2xyz\sum_{cyc} xy \therefore 10\sum_{cyc}\left(\frac{r}{h_a}\right)^3 - 9\sum_{cyc}\left(\frac{r}{h_a}\right)^5 \ge 1\ \forall\ \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4078. In any Δ ABC the following relationship holds :

$$10\sum_{cyc}\left(\frac{r}{r_a}\right)^3 - 9\sum_{cyc}\left(\frac{r}{r_a}\right)^5 \ge 1$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = \frac{3r}{r_a}, y = \frac{3r}{r_b}, z = \frac{3r}{r_c} \text{ and then : } \sum_{cyc} x = 3r.\sum_{cyc}\frac{1}{r_a} = \frac{3r}{r} = 3 \text{ and so,}$$

$$10\sum_{cyc}\left(\frac{r}{r_a}\right)^3 - 9\sum_{cyc}\left(\frac{r}{r_a}\right)^5 \overset{?}{\ge} 1$$

$$\Leftrightarrow \frac{10}{243}\left(\sum_{cyc} x^3\right)\left(\sum_{cyc} x\right)^2 - \frac{9}{243}\left(\sum_{cyc} x^5\right) \overset{?}{\geq} \frac{1}{243}\left(\sum_{cyc} x\right)^5$$

$$\Leftrightarrow \sum_{cyc} x^4y + \sum_{cyc} xy^4 \overset{?}{\geq} 2xyz\sum_{cyc} xy \to \text{true} \because \sum_{cyc} x^4y + \sum_{cyc} xy^4 = \sum_{cyc}\left(z(x^4+y^4)\right)$$

$$\overset{AM-GM}{\geq} 2\sum_{cyc}(zx^2y^2) = 2xyz\sum_{cyc} xy \therefore 10\sum_{cyc}\left(\frac{r}{r_a}\right)^3 - 9\sum_{cyc}\left(\frac{r}{r_a}\right)^5 \geq 1 \ \forall \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4079. In ΔABC the following relationship holds:

$$r_ar_b + r_br_c + r_cr_a \geq h_ah_b + h_bh_c + h_ch_a$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc} r_ar_b \geq \sum_{cyc} h_ah_b, \quad \sum_{cyc} \frac{F}{s-a}\cdot\frac{F}{s-b} \geq \sum_{cyc} \frac{2F}{a}\cdot\frac{2F}{b}$$

$$\sum_{cyc} \frac{1}{(s-a)(s-b)} \geq 4\sum_{cyc}\frac{1}{ab}, \quad \frac{s-c+s-b+s-a}{(s-a)(s-b)(s-c)} \geq 4\cdot\frac{a+b+c}{abc}$$

$$\frac{s}{(s-a)(s-b)(s-c)} \geq 4\cdot\frac{2s}{4RF}, \quad \frac{s^2}{s(s-a)(s-b)(s-c)} \geq \frac{2s}{R\cdot rs}$$

$$\frac{s^2}{F^2} \geq \frac{2}{Rr}, \quad \frac{s^2}{r^2s^2} \geq \frac{2}{Rr}$$

$$\frac{1}{r^2} \geq \frac{2}{Rr} \Leftrightarrow R \geq 2r \quad \text{(Euler)}$$

Equality holds for $a = b = c$.

4080. In ΔABC the following relationship holds:

$$216r^3 \leq (h_a + h_b)(h_b + h_c)(h_c + h_a) \leq 27R^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(h_a + h_b)(h_b + h_c)(h_c + h_a) = \prod_{cyc}(h_a + h_b) =$$

$$=\prod_{cyc}\left(\frac{2F}{a}+\frac{2F}{b}\right)=8F^3\prod_{cyc}\left(\frac{1}{a}+\frac{1}{b}\right)=$$

$$=8F^3\prod_{cyc}\left(\frac{a+b}{ab}\right)=8F^3\cdot\frac{(a+b)(b+c)(c+a)}{abc\cdot abc}\overset{CESARO}{\geq}$$

$$\geq 8F^3\cdot\frac{8abc}{abc\cdot abc}=\frac{64F^3}{abc}=\frac{64F^3}{4RF}=\frac{16F^2}{R}=\frac{16r^2s^2}{R}$$

Remains to prove:

$$216r^3\leq\frac{16r^2s^2}{R}\leq 27R^3$$

$$\frac{16r^2s^2}{R}\overset{EULER}{\leq}\frac{16\cdot\left(\frac{R}{2}\right)^2\cdot s^2}{R}=4Rs^2\overset{MITRINOVIC}{\leq}4R\cdot\left(\frac{3\sqrt{3}}{2}R\right)^2=4R\cdot\frac{27R^2}{4}=27R^3$$

$$\frac{16r^2s^2}{R}\geq 216r^3\Leftrightarrow\frac{2r^2s^2}{R}\geq 27r^3\Leftrightarrow\frac{2s^2}{R}\geq 27r\Leftrightarrow 2s^2\geq 27Rr\quad\text{(to prove)}$$

$$2s^2\overset{GERRESTEN}{\geq}2(16Rr-5r^2)=32Rr-10r^2\geq 27Rr\Leftrightarrow$$

$$\Leftrightarrow 5Rr\geq 10r^2\Leftrightarrow R\geq 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.

4081. In ΔABC the following relationship holds:

$$216r^3\leq(r_a+r_b)(r_b+r_c)(r_c+r_a)\leq 27R^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(r_a+r_b)(r_b+r_c)(r_c+r_a)=\prod_{cyc}(r_a+r_b)=$$

$$=\prod_{cyc}\left(\frac{F}{s-a}+\frac{F}{s-b}\right)=F^3\prod_{cyc}\left(\frac{1}{s-a}+\frac{1}{s-b}\right)=$$

$$=F^3\prod_{cyc}\frac{s-b+s-a}{(s-a)(s-b)}=F^3\prod_{cyc}\frac{2s-a-b}{(s-a)(s-b)}=$$

$$=F^3\prod_{cyc}\frac{a+b+c-a-b}{(s-a)(s-b)}=F^3\prod_{cyc}\frac{c}{(s-a)(s-b)}=$$

$$=F^3\cdot\frac{abc}{(s-a)^2(s-b)^2(s-c)^2}=\frac{F^3\cdot abc\cdot s^2}{s^2(s-a)^2(s-b)^2(s-c)^2}=\frac{F^3\cdot 4RF\cdot s^2}{F^4}=4Rs^2$$

Remains to prove:

$$216r^3\le 4Rs^2\le 27R^3$$

$$4Rs^2\overset{MITRINOVIC}{\le}4R\cdot\left(\frac{3\sqrt{3}}{2}R\right)^2=4R\cdot\frac{27R^2}{4}=27R^3$$

$$4Rs^2\overset{MITRINOVIC}{\ge}4R\cdot\left(3\sqrt{3}r\right)^2=4R\cdot 27r^2\overset{EULER}{\ge}4\cdot 2r\cdot 27r^2=216r^3$$

Equality holds for $a=b=c$.

4082. In ΔABC the following relationship holds:

$$(h_a+h_b)^{\cos\frac{C}{2}}\cdot(h_b+h_c)^{\cos\frac{A}{2}}\cdot(h_c+h_a)^{\cos\frac{B}{2}}\le(3R)^{\frac{3\sqrt{3}}{2}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\text{WLOG: } a\le b\le c\Rightarrow\frac{1}{a}\ge\frac{1}{b}\ge\frac{1}{c}\Rightarrow\frac{2F}{a}\ge\frac{2F}{b}\ge\frac{2F}{c}$$

$$\Rightarrow h_a\ge h_b\ge h_c\Rightarrow\begin{cases}h_a+h_c\ge h_b+h_c\\h_b+h_a\ge h_c+h_a\end{cases}\Rightarrow h_a+h_b\ge h_a+h_c\ge h_b+h_c\qquad(1)$$

$$a\le b\le c\Rightarrow\cos\frac{A}{2}\ge\cos\frac{B}{2}\cos\frac{C}{2}\Rightarrow\cos\frac{C}{2}\le\cos\frac{B}{2}\le\cos\frac{A}{2}\qquad(2)$$

By (1); (2) and Cebyshev's inequality:

$$(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{A}{2}\le$$

$$\le\frac{1}{3}(h_a+h_b+h_b+h_c+h_c+h_a)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)=$$

$$=\frac{2}{3}(h_a+h_b+h_c)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\qquad(3)$$

$$\prod_{cyc}(h_a+h_b)\cos\frac{C}{2}\overset{WEIGHTED\ AM-GM}{\le}$$

$$\le\left(\frac{(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{(3)}{\le}\left(\frac{\frac{2}{3}(h_a+h_b+h_c)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{Jensen}{\le}\left(\frac{2}{3}\left(\frac{2F}{a}+\frac{2F}{b}+\frac{2F}{c}\right)\right)^{3\cos\frac{A+B+C}{6}}=\left(\frac{4F}{3}\cdot\frac{ab+bc+ca}{abc}\right)^{3\cos\frac{\pi}{6}}=$$

$$=\left(\frac{4F}{3}\cdot\frac{s^2+r^2+4Rr}{4RF}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{s^2+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\le$$

$$\overset{MITRINOVIC}{\le}\left(\frac{\frac{27R^2}{4}+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\overset{EULER}{\le}$$

$$\le\left(\frac{\frac{27R^2}{4}+\frac{R^2}{4}+4R\cdot\frac{R}{2}}{3R}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{7R^2+2R^2}{3R}\right)^{\frac{3\sqrt{3}}{2}}=(3R)^{\frac{3\sqrt{3}}{2}}$$

Equality holds for $a=b=c$.

4083. In ΔABC the following relationship holds:

$$(ab)^{\sin C}\cdot(bc)^{\sin A}\cdot(ca)^{\sin B}\le(3R^2)^{\frac{3\sqrt{3}}{2}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(ab)^{\sin C}\cdot(bc)^{\sin A}\cdot(ca)^{\sin B}\overset{WEIGHTED\ AM-GM}{\le}$$

$$\le\left(\frac{ab\sin C+bc\sin A+ca\sin B}{\sin A+\sin B+\sin C}\right)^{\sin A+\sin B+\sin C}=\left(\frac{2F+2F+2F}{\frac{s}{R}}\right)^{\frac{s}{R}}\overset{MITRINOVIC}{\le}$$

$$\le\left(\frac{6F}{\frac{s}{R}}\right)^{\frac{1}{R}\cdot\frac{3\sqrt{3}}{2}R}=\left(\frac{6FR}{s}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{6rsR}{s}\right)^{\frac{3\sqrt{3}}{2}}=(6rR)^{\frac{3\sqrt{3}}{2}}\le\left(6\cdot\frac{R}{2}\cdot R\right)^{\frac{3\sqrt{3}}{2}}=(3R^2)^{\frac{3\sqrt{3}}{2}}$$

Equality holds for $a=b=c$.

4084. In ΔABC the following relationship holds:

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}\ge 9$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}=\sum_{cyc}\frac{r_a+2r_b}{h_c}=\sum_{cyc}\frac{r_a}{h_c}+2\sum_{cyc}\frac{r_b}{h_c}\overset{AM-GM}{\ge}$$

$$\ge 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}+2\cdot 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}=9\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\ge 9\Leftrightarrow\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\ge 1\Leftrightarrow r_ar_br_c\ge h_ah_bh_c$$

$$\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\ge\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}$$

$$abc\ge 8(s-a)(s-b)(s-c)$$

$$abc\cdot s\ge 8s(s-a)(s-b)(s-c)$$

$$4RF\cdot s\ge 8F^2,\qquad Rs\ge 2F,\qquad Rs\ge 2rs$$

$$R\ge 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.

4085. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a+b}{\sqrt{r_c}}\le 2\sum_{cyc}\frac{a}{\sqrt{h_a}}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let $\sqrt{s-a}=x,\sqrt{s-b}=y,\sqrt{s-c}=z\ (\because (s-a),(s-b),(s-c)>0)$

and then : $s-a=x^2,s-b=y^2,s-c=z^2\Rightarrow 3s-2s=\sum_{cyc}x^2$ and so,

$$a=y^2+z^2,b=z^2+x^2,c=x^2+y^2\ \therefore\sum_{cyc}\frac{b+c}{\sqrt{r_a}}\overset{?}{\le}2\sum_{cyc}\frac{a}{\sqrt{h_a}}\Leftrightarrow$$

$$\sum_{cyc}\frac{x(2x^2+y^2+z^2)}{\sqrt{rs}}\overset{?}{\le}\sum_{cyc}\frac{(y^2+z^2).\sqrt{2(y^2+z^2)}}{\sqrt{rs}}$$

$$\Leftrightarrow \sum_{cyc}\left((y^2+z^2).\sqrt{2(y^2+z^2)}\right) \boxed{\overset{?}{\underset{(*)}{\geq}}} \sum_{cyc}\left(x(2x^2+y^2+z^2)\right)$$

$$\text{Now, } \sum_{cyc}\left((y^2+z^2).\sqrt{2(y^2+z^2)}\right) \geq$$

$$\sum_{cyc}\left((y^2+z^2)(y+z)\right) \overset{?}{\geq} \sum_{cyc}\left(x(2x^2+y^2+z^2)\right)$$

$$\Leftrightarrow \sum_{cyc}\left((y^2+z^2)\left(\sum_{cyc} x - x\right)\right) \overset{?}{\geq} \left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) + \sum_{cyc} x^3$$

$$\Leftrightarrow 2\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right) - P \overset{?}{\geq} 2\sum_{cyc} x^3 + P \quad \left(P = \sum_{cyc} x^2 y + \sum_{cyc} xy^2\right)$$

$$\Leftrightarrow 2\sum_{cyc} x^3 + 2P - P \overset{?}{\geq} 2\sum_{cyc} x^3 + P \to \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{a+b}{\sqrt{r_c}} \leq 2\sum_{cyc}\frac{a}{\sqrt{h_a}} \ \forall \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

4086. In any acute triangle ABC the following relationship holds :

$$r_a\sqrt{\cos A} + r_b\sqrt{\text{cst}\, B} + r_c\sqrt{\cos C} \leq \frac{9\sqrt{2}}{4}R$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}(r_a.\sqrt{\cos A}) = s\sum_{cyc}\left(\sqrt{\tan\frac{A}{2}}.\sqrt{\tan\frac{A}{2}.\cos A}\right) \overset{CBS}{\leq}$$

$$s.\sqrt{\sum_{cyc}\tan\frac{A}{2}}.\sqrt{\sum_{cyc}\left(\tan\frac{A}{2}.\left(2\cos^2\frac{A}{2}-1\right)\right)} = s.\sqrt{\frac{4R+r}{s}}.\sqrt{\sum_{cyc}\sin A - \sum_{cyc}\tan\frac{A}{2}}$$

$$= s.\sqrt{\frac{4R+r}{s}}.\sqrt{\frac{s}{R} - \frac{4R+r}{s}} = s.\sqrt{\frac{4R+r}{s}}.\sqrt{\frac{s^2-4R^2-Rr}{Rs}} \overset{\text{Gerretsen}}{\leq}$$

$$\sqrt{\frac{4R+r}{R}}.\sqrt{3Rr+3r^2} = \sqrt{\frac{3r(4R+r)(R+r)}{R}} \overset{\text{Euler}}{\leq} \sqrt{\frac{3.\frac{R}{2}.\frac{9R}{2}.\frac{3R}{2}}{R}} = 9R.\sqrt{\frac{2}{16}}$$

$$=\frac{9\sqrt{2}}{4}\mathrm{R}\ \textit{and so}, \mathrm{r}_a.\sqrt{\cos \mathrm{A}}+\mathrm{r_b}.\sqrt{\mathrm{cst\,B}}+\mathrm{r_c}.\sqrt{\cos \mathrm{C}} \leq \frac{9\sqrt{2}}{4}\mathrm{R}\ \forall\ \textit{acute}\ \Delta\ \mathrm{ABC},$$

$$'' =''\ \text{iff}\ \Delta\ \mathrm{ABC}\ \text{is equilateral (QED)}$$

4087. In ΔABC the following relationship holds:

$$\frac{w_a}{h_a}+\frac{h_a}{w_a} \leq \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \leq min\left\{\frac{w_b}{w_c}+\frac{w_c}{w_b}, \frac{2w_a}{h_a}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Jenish Rijal-Nepal

$$Let\ u=\cos\left(\frac{B-C}{2}\right)\ and\ v=\cos\left(\frac{B+C}{2}\right)=\sin\frac{A}{2}.$$

$$Also, assume\ M\ to\ be\ the\ middle\ term.$$

$$We\ know\ \frac{|B-C|}{2}<\frac{B+C}{2}<\frac{\pi}{2} \Rightarrow 0<v<u\leq 1.\quad Now, let's\ prove\ the\ LHS:$$

$$* \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}+\frac{\cos\frac{C}{2}}{\cos\frac{B}{2}}=\frac{1+\cos\left(\frac{B+C}{2}\right)\cos\left(\frac{B-C}{2}\right)}{\frac{1}{2}\left[\cos\left(\frac{B+C}{2}\right)+\cos\left(\frac{B-C}{2}\right)\right]}=\frac{2+2vu}{v+u}.$$

$$The\ inequality\ \frac{w_a}{h_a}+\frac{h_a}{w_a}\leq M\ is\ equivalent\ to:$$

$$\frac{w_a}{h_a}+\frac{h_a}{w_a}=\frac{u^2+1}{u}\leq M=\frac{2+2vu}{v+u} \Leftrightarrow (u^2-1)(u-v)\leq 0$$

$$which\ is\ true\ \because v<u\leq 1.$$

$$Now, let\ us\ prove\ the\ RHS\ \ i.e.\ M\leq\frac{2w_a}{h_a}=\frac{2}{u}:$$

$$M=\frac{2+2vu}{v+u}\leq\frac{2}{u}\Leftrightarrow u(1+vu)\leq v+u\Leftrightarrow u^2\leq 1\quad which\ is\ true\ \because u\leq 1.$$

$$At\ last, WLOG\ assume: b\geq c.$$

$$Let\ t=\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}\leq 1. Now, \frac{w_b}{w_c}=\left[\frac{c(a+b)}{b(a+c)}\right]\cdot\frac{\cos\frac{B}{2}}{\cos\frac{C}{2}}=k\cdot t\ \ ;\frac{c(a+b)}{b(a+c)}=k$$

$$As\ b \geq c \Rightarrow k = \frac{ac+bc}{ab+bc} \leq 1. Thus, 0 < kt \leq t \leq 1.$$

We can observe that the function $f(x) = x + \frac{1}{x}$ *is strictly decreasing on* $(0,1]$.

And the middle term is $f(t)$, *i.e.*, $f(t) = M$.

$$0 < kt \leq t \leq 1 \implies f(kt) \overbrace{\geq}^{Monotonicity} f(t) \implies \frac{w_b}{w_c} + \frac{w_c}{w_b} \geq M.$$

$$\therefore \frac{w_a}{h_a} + \frac{h_a}{w_a} \leq \frac{\cos\frac{B}{2}}{\cos\frac{C}{2}} + \frac{\cos\frac{C}{2}}{\cos\frac{B}{2}} \leq min\left\{\frac{w_b}{w_c} + \frac{w_c}{w_b}, \frac{2w_a}{h_a}\right\}$$

Equality holds if and only if the triangle is isosceles $(b = c)$.

4088. In ΔABC the following relationship holds:

$$m_a^{m_a} \cdot m_b^{m_b} \cdot m_c^{m_c} \geq (3r)^{m_a+m_b+m_c}$$

Proposed by Daniel Sitaru – Romania

Solution by Khaled Abd Imouti, Kasem Abotrabi – Syria

Let be $f: (0,\infty) \to \mathbb{R}; f(x) = x\ln x$

$$f'(x) = \ln x + x \cdot \frac{1}{x} = \ln x + 1; f''(x) = \frac{1}{x} > 0 \Rightarrow f \text{ convex}$$

By Jensen's inequality:

$$f(m_a) + f(m_b) + f(m_c) \geq 3f\left(\frac{m_a+m_b+m_c}{3}\right)$$

$$m_a \ln m_a + m_b \ln m_b + m_c \ln m_c \geq 3\frac{m_a+m_b+m_c}{3} \cdot \ln\left(\frac{m_a+m_b+m_c}{3}\right)$$

$$\ln\left(m_a^{m_a}\right) + \ln\left(m_b^{m_b}\right) + \ln\left(m_c^{m_c}\right) \geq (m_a+m_b+m_c)\ln\left(\frac{m_a+m_b+m_c}{3}\right) \quad (1)$$

$$h_a \leq m_a; h_b \leq m_b; h_c \leq m_c \Rightarrow$$

$$\frac{1}{h_a} \geq \frac{1}{m_a}; \frac{1}{h_b} \geq \frac{1}{m_c}; \frac{1}{h_c} \geq \frac{1}{m_c}$$

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \geq \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}$$

$$\frac{1}{r} \geq \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \overset{BERGSTROM}{\geq} \frac{(1+1+1)^2}{m_a+m_b+m_c} =$$

$$=\frac{9}{m_a+m_b+m_c}\Rightarrow m_a+m_b+m_c\ge 9r\Rightarrow\frac{m_a+m_b+m_c}{3}\ge 3r\quad \textbf{(2)}$$

$$\ln\left(\frac{m_a+m_b+m_c}{3}\right)\ge\ln(3r)$$

By (1):

$$\ln\left(m_a^{m_a}\cdot m_b^{m_b}\cdot m_c^{m_c}\right)\ge(m_a+m_b+m_c)\ln\left(\frac{m_a+m_b+m_c}{3}\right)\ge$$

$$\overset{(2)}{\ge}(m_a+m_b+m_c)\ln(3r)=\ln(3r)^{m_a+m_b+m_c}$$

$$\ln\left(m_a^{m_a}\cdot m_b^{m_b}\cdot m_c^{m_c}\right)\ge\ln(3r)^{m_a+m_b+m_c}$$

$$m_a^{m_a}\cdot m_b^{m_b}\cdot m_c^{m_c}\ge(3r)^{m_a+m_b+m_c}$$

Equality holds for: $a=b=c$.

4089. In ΔABC the following relationship holds:

$$\frac{(a+b)^2}{h_c}+\frac{(b+c)^2}{h_a}+\frac{(c+a)^2}{h_b}\ge 48r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(a+b)^2}{h_c}+\frac{(b+c)^2}{h_a}+\frac{(c+a)^2}{h_b}=\sum_{cyc}\frac{(a+b)^2}{h_c}\ge$$

$$\overset{AM-GM}{\ge}3\sqrt[3]{\frac{\left((a+b)(b+c)(c+a)\right)^2}{h_ah_bh_c}}\overset{CESARO}{\ge}3\sqrt[3]{\frac{(8abc)^2}{h_ah_bh_c}}=$$

$$=3\sqrt[3]{\frac{64\cdot(4RF)^2}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=\frac{3}{2F}\cdot 4\sqrt[3]{(4RF)^2\cdot abc}=$$

$$=\frac{6}{F}\sqrt[3]{(4RF)^3}=\frac{6}{F}\cdot 4RF=24R\overset{EULER}{\ge}24\cdot 2r=47r$$

Equality holds for $a=b=c$.

4090. **In ΔABC the following relationship holds:**

$$\frac{(r_a+r_b)^2}{h_c}+\frac{(r_b+r_c)^2}{h_a}+\frac{(r_c+r_a)^2}{h_b}\ge 36r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(r_a+r_b)^2}{h_c}+\frac{(r_b+r_c)^2}{h_a}+\frac{(r_c+r_a)^2}{h_b}=\sum_{cyc}\frac{(r_a+r_b)^2}{h_c}\ge$$

$$\overset{AM-GM}{\ge}3\sqrt[3]{\frac{\left((r_a+r_b)(r_b+r_c)(r_c+r_a)\right)^2}{h_ah_bh_c}}\overset{CESARO}{\ge}3\sqrt[3]{\frac{(8r_ar_br_c)^2}{h_ah_bh_c}}=$$

$$=3\sqrt[3]{\frac{64\cdot\left(\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\right)^2}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=12\sqrt[3]{\frac{F^6}{\left((s-a)(s-b)(s-c)\right)^2}\cdot\frac{abc}{8F^3}}=$$

$$=\frac{12}{2}\sqrt[3]{\frac{F^3\cdot 4RF\cdot s^2}{\left(s(s-a)(s-b)(s-c)\right)^2}}=6\sqrt[3]{\frac{4Rs^2F^4}{F^4}}=$$

$$=6\sqrt[3]{4Rs^2}\overset{MITRINOVIC}{\ge}6\sqrt[3]{4R\cdot 27r^2}\overset{EULER}{\ge}6\cdot 3\sqrt[3]{8r\cdot r^2}=36r$$

Equality holds for $a=b=c$.

4091. **In ΔABC the following relationship holds:**

$$\frac{(a+b)^2}{r_c}+\frac{(b+c)^2}{r_a}+\frac{(c+a)^2}{r_b}\ge 48r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(a+b)^2}{r_c}+\frac{(b+c)^2}{r_a}+\frac{(c+a)^2}{r_b}=\sum_{cyc}\frac{(a+b)^2}{r_c}\ge$$

$$\overset{AM-GM}{\ge}3\sqrt[3]{\frac{\left((a+b)(b+c)(c+a)\right)^2}{r_ar_br_c}}\overset{CESARO}{\ge}3\sqrt[3]{\frac{(8abc)^2}{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}}=3\sqrt[3]{\frac{64(abc)^2}{\frac{F^3s}{F^2}}}=$$

$$=12\sqrt[3]{\frac{(4RF)^2}{Fs}}=12\sqrt[3]{\frac{16R^2F^2}{Fs}}=12\sqrt[3]{\frac{16R^2F}{s}}=12\sqrt[3]{\frac{16R^2rs}{s}}=$$

$$=12\sqrt[3]{16R^2r}\overset{EULER}{\geq}12\sqrt[3]{16\cdot 4r^2\cdot r}=12\sqrt[3]{64r^3}=12\cdot 4r=48r$$

Equality holds for $a=b=c$.

4092. In ΔABC the following relationship holds:

$$\frac{r_a+r_b}{h_c}+\frac{r_b+r_c}{h_a}+\frac{r_c+r_a}{h_b}\leq\frac{3R}{r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{r_a+r_b}{h_c}=\sum_{cyc}\frac{\frac{F}{s-a}+\frac{F}{s-b}}{\frac{2F}{c}}=\sum_{cyc}\frac{\frac{1}{s-a}+\frac{1}{s-b}}{\frac{2}{c}}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{c(s-a+s-b)}{(s-a)(s-b)}=\frac{1}{2}\sum_{cyc}\frac{c(a+b+c-a-b)}{(s-a)(s-b)}=$$

$$=\frac{1}{2}\sum_{cyc}\frac{c^2}{(s-a)(s-b)}=\frac{1}{2(s-a)(s-b)(s-c)}\sum_{cyc}c^2(s-c)=$$

$$=\frac{s}{2s(s-a)(s-b)(s-c)}\cdot 4rs(R+r)=$$

$$=\frac{4rs^2(R+r)}{2F^2}=\frac{2rs^2(R+r)}{r^2s^2}=\frac{2(R+r)}{r}\leq\frac{2\left(R+\frac{R}{2}\right)}{r}=\frac{2\cdot\frac{3R}{2}}{r}=\frac{3R}{r}$$

Equality holds for $a=b=c$.

4093. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}\geq 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}=$$

$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c}=$$

$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c}+\sum_{cyc}c\ge$$

$$\overset{BERGSTROM}{\ge}\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+2s=$$

$$=\frac{a+b+c}{4}+\frac{a+b+c}{4}+2s=\frac{2s}{4}+\frac{2s}{4}+2s=$$

$$=\frac{4s}{4}+2s=3s\overset{MITRINOVIC}{\ge}3\cdot3\sqrt{3}r=9\sqrt{3}r$$

Equality holds for $a=b=c$.

4094. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}\ge\frac{9}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}=$$

$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c^2}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c^2}=$$

$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c^2}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c^2}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c^2}+\sum_{cyc}\frac{4c^2}{4c^2}\ge$$

$$\overset{AM-GM}{\ge}\frac{1}{4}\cdot3\sqrt[3]{\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}\cdot\frac{a^2}{b^2}}+\frac{1}{4}\cdot3\sqrt[3]{\frac{a^2}{c^2}\cdot\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}}+3=$$

$$=\frac{3}{4}+\frac{3}{4}+3=\frac{6}{4}+3=\frac{3}{2}+3=\frac{9}{2}$$

Equality holds for $a = b = c$.

4095. In ΔABC the following relationship holds:

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}\ge 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}=\sum_{cyc}\frac{r_a^2+r_b^2}{c}=\sum_{cyc}\frac{r_a^2}{c}+\sum_{cyc}\frac{r_b^2}{c}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(r_a+r_b+r_c)^2}{a+b+c}+\frac{(r_a+r_b+r_c)^2}{a+b+c}=\frac{2(r_a+r_b+r_c)^2}{a+b+c}=$$

$$=\frac{2(4R+r)^2}{2s}=\frac{(4R+r)^2}{s}\overset{DOUCET}{\ge}\frac{(4R+r)^2}{\frac{4R+r}{\sqrt{3}}}=\sqrt{3}(4R+r)\overset{EULER}{\ge}$$

$$\ge\sqrt{3}(4\cdot 2r+r)=9\sqrt{3}r$$

Equality holds for $a = b = c$.

4096. In ΔABC the following relationship holds:

$$r_a^2r_b^2+r_b^2r_c^2+r_c^2r_a^2+n_a^2n_b^2+n_b^2n_c^2+n_c^2n_a^2\ge 2h_ah_bh_c(h_a+h_b+h_c)$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Jenish Rijal-Nepal

$$h_c=\frac{2r_ar_b}{r_a+r_b}\overset{HM-GM}{\overbrace{\le}}\sqrt{r_ar_b}\Leftrightarrow r_ar_b\ge h_c^2\Leftrightarrow$$

$$r_ar_b\cdot n_an_b\ge h_c^2\cdot h_ah_b\ [\because n_a\ge h_a\ and\ n_b\ge h_b]$$

$$\therefore\ r_ar_bn_an_b\overset{①}{\overbrace{\ge}}h_ah_bh_c^2$$

$Now, via\ AM-GM\ Inequality$:

$$r_a^2r_b^2+n_a^2n_b^2\ge 2\sqrt{r_a^2r_b^2n_a^2n_b^2}=r_ar_bn_an_b\overset{\text{Via }①}{\overbrace{\ge}}2h_ah_bh_c^2\Rightarrow$$

$$\sum_{cyc}(r_a^2r_b^2+n_a^2n_b^2)\ge 2\sum_{cyc}(h_ah_bh_c^2)$$

$$\Leftrightarrow\ r_a^2r_b^2+r_b^2r_c^2+r_c^2r_a^2+n_a^2n_b^2+n_b^2n_c^2+n_c^2n_a^2\ge 2h_ah_bh_c(h_a+h_b+h_c).$$

$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$

4097. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^2\frac{B}{2}\sqrt{\sin\frac{A}{2}}+\sin^2\frac{C}{2}\sqrt{\sin\frac{C}{2}}\right)}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}\geq 6\sqrt{2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Jenish Rijal-Nepal

$$Let\ (x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right).\quad \Rightarrow x,y,z\in(0,1)$$

Lemma 1: $xyz\leq\frac{1}{8}.$

Proof: $We\ know\ that{:}\ r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\Rightarrow xyz=\frac{r}{4R}\overset{Euler}{\leq}\frac{1}{8}.$

Lemma 2: $x+y+z\leq\frac{3}{2}$

Proof: *Since the sine function is concave on* $(0,\pi)$, *by Jensen's Inequality*,

$$x+y+z=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq 3\cdot\sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=3\cdot\sin\left(\frac{\pi}{6}\right)=\frac{3}{2}$$

Claim 1: $\prod(x+y)\leq 1$

Proof: $\prod(x+y)\overset{AM-GM}{\leq}\left(\frac{x+y+y+z+z+x}{3}\right)^3=\left(\frac{2(x+y+z)}{3}\right)^3\overset{Lemma\ 2}{\leq}1$

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^2\frac{B}{2}\sqrt{\sin\frac{A}{2}}+\sin^2\frac{C}{2}\sqrt{\sin\frac{C}{2}}\right)}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}=\sum_{cyc}\frac{x\left(y^2\sqrt{x}+z^2\sqrt{z}\right)}{y^2z^2(y+z)}=$$

$$=\sum_{cyc}\left(\frac{x\sqrt{x}}{z^2(y+z)}+\frac{x\sqrt{z}}{y^2(y+z)}\right)$$

By AM − GM Inequality,

$$\sum_{cyc}\left(\frac{x\sqrt{x}}{z^2(y+z)}+\frac{x\sqrt{z}}{y^2(y+z)}\right)\geq 6\cdot\sqrt[6]{\prod\left(\frac{x\sqrt{x}}{z^2(y+z)}\cdot\frac{x\sqrt{z}}{y^2(y+z)}\right)}=$$

$$=6\cdot\sqrt[6]{\frac{x^3y^3z^3}{x^4y^4z^4(x+y)^2(y+z)^2(x+z)^2}}=6\cdot\sqrt[6]{\frac{1}{xyz\cdot[\ \prod(x+y)\ ]^2}}$$

From Lemma 1 and Claim 1 it follows that:

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^2\frac{B}{2}\sqrt{\sin\frac{A}{2}}+\sin^2\frac{C}{2}\sqrt{\sin\frac{C}{2}}\right)}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}=\sum_{cyc}\left(\frac{x\sqrt{x}}{z^2(y+z)}+\frac{x\sqrt{z}}{y^2(y+z)}\right)\ge$$

$$\ge 6\cdot\sqrt[6]{\frac{1}{\frac{1}{8}\cdot(1)^2}}=6\sqrt{2}$$

Equality holds if and only if the triangle is equilateral.

4098. In ΔABC the following relationship holds:

$$1+\sum_{cyc}\frac{r^2}{{h_a}^2}\ge 4\sum_{cyc}\frac{r^2}{h_ah_b}$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

$$\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=1\quad\left[\because\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}=r\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)=r\cdot\frac{1}{r}=1\right]$$

$$\Rightarrow\left(\frac{r}{r_a}+\frac{r}{r_b}+\frac{r}{r_c}\right)^2=1^2=1\Rightarrow\sum_{cyc}\frac{r^2}{{r_a}^2}+2\cdot\sum_{cyc}\frac{r^2}{r_ar_b}=1$$

$$\Rightarrow 4\sum_{cyc}\frac{r^2}{r_ar_b}=2-2\sum_{cyc}\frac{r^2}{{r_a}^2}$$

Substituting this into our original inequality:

$$1+\sum_{cyc}\frac{r^2}{{r_a}^2}\ge 2-2\sum_{cyc}\frac{r^2}{{r_a}^2}\Leftrightarrow 3\sum_{cyc}\frac{r^2}{{r_a}^2}\ge 1\Leftrightarrow\sum_{cyc}\frac{r^2}{{r_a}^2}\ge\frac{1}{3}$$

Thus it suffices to show that:

$$\sum_{cyc}\frac{r^2}{{r_a}^2}\ge\frac{1}{3}$$

$$\sum_{cyc}\frac{r^2}{{r_a}^2}=\sum_{cyc}\frac{\left(\frac{r}{r_a}\right)^2}{1}\overbrace{\ge}^{BERGSTROM}\frac{\left(\sum_{cyc}\frac{r}{r_a}\right)^2}{3}=\frac{1^2}{3}=\frac{1}{3}$$

$$\therefore 1+\sum_{cyc}\frac{r^2}{{r_a}^2}\ge 4\sum_{cyc}\frac{r^2}{r_ar_b}$$

Equality holds if and only if the triangle is equilateral.

4099. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{r_b\sqrt{r_c}}{{r_a}^2}\ge\sqrt{\frac{3}{r}}$$

Proposed by Marin Chirciu-Romania

Solution by Jenish Rijal-Nepal

$$\text{Let } r_a=\frac{1}{x}, r_b=\frac{1}{y}, \text{ and } r_c=\frac{1}{z}.$$

$$\text{Using the identity }\sum_{cyc}\frac{1}{r_a}=\frac{1}{r}, \text{ we have } x+y+z=\frac{1}{r}.$$

$$\text{Substituting these into the LHS: } LHS=\sum_{cyc}\frac{r_b\sqrt{r_c}}{{r_a}^2}=\sum_{cyc}\frac{\frac{1}{y}\sqrt{\frac{1}{z}}}{\frac{1}{x^2}}=\sum_{cyc}\frac{x^2}{y\sqrt{z}}$$

$$\text{Now, }\quad\sum_{cyc}\frac{x^2}{y\sqrt{z}}\overset{BERGSTROM}{\overbrace{\ge}}\frac{(x+y+z)^2}{\sum_{cyc}y\sqrt{z}}\mapsto ①$$

Again by Cauchy's Inequality,

$$\left(\sum_{cyc}\sqrt{y}\cdot\sqrt{yz}\right)^2=\left(\sum_{cyc}y\sqrt{z}\right)^2\le(x+y+z)(xy+yz+zx)\Rightarrow$$

$$\sum_{cyc}y\sqrt{z}\le\sqrt{(x+y+z)(xy+yz+zx)}$$

Substituting this bound back into ①:

$$\sum_{cyc}\frac{x^2}{y\sqrt{z}}\ge\frac{(x+y+z)^2}{\sqrt{(x+y+z)(xy+yz+zx)}}=\sqrt{\frac{(x+y+z)^3}{(xy+yz+zx)}}$$

Since $(a+b+c)^2\ge 3(ab+bc+ac)$, *it follows that*:

$$\sum_{cyc}\frac{h_b\sqrt{h_c}}{{h_a}^2}=\sum_{cyc}\frac{x^2}{y\sqrt{z}}\ge\sqrt{\frac{(x+y+z)^3}{(xy+yz+zx)}}\ge\sqrt{3(x+y+z)}=\sqrt{3\cdot\frac{1}{r}}$$

$$\therefore\sum_{cyc}\frac{r_b\sqrt{r_c}}{{r_a}^2}\ge\sqrt{\frac{3}{r}}$$

Equality holds if and only if the triangle is equilateral.

4100. In ΔABC the following relationship holds:

$$\frac{a^2+b^2}{r_c}+\frac{b^2+c^2}{r_a}+\frac{c^2+a^2}{r_b}\geq 24r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a^2+b^2}{r_c}+\frac{b^2+c^2}{r_a}+\frac{c^2+a^2}{r_b}=\sum_{cyc}\frac{a^2+b^2}{r_c}\overset{AM-GM}{\geq}\sum_{cyc}\frac{2ab}{r_c}=2\sum_{cyc}\frac{ab}{r_c}\overset{AM-GM}{\geq}$$

$$=2\cdot 3\sqrt[3]{\frac{(abc)^2}{r_ar_br_c}}=6\sqrt[3]{\frac{16r^2F^2}{Fs}}=6\cdot 2\sqrt[3]{\frac{2R^2F}{s}}=12\sqrt[3]{\frac{2R^2rs}{s}}=12\sqrt[3]{2R^2r}\geq$$

$$\geq 12\sqrt[3]{2\cdot 4r^2\cdot r}=12\cdot 2r=24r$$

Equality holds for $a=b=c$.

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru