ISSN 2501-0099

ROMANIAN MATHEMATICAL SOCIETY

Mehedinți Branch

R. M. M. - 49

ROMANIAN MATHEMATICAL MAGAZINE

SUMMER EDITION 2026

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Nr.48-SUMMER EDITION 2026

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CONTENT

FURTHER STUDIES ON FERMAT-TORRICELLI AND NAPOLEON POINTS OF A TRIANGLE USING PLAGIOGONAL APPROACH

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Abstract: In this present work, we have attempted to extensively study the geometric characteristics of *Fermat*'s Second point (**X_{14}**) and Second *Napoleon* point (**X_{18}**) using a set of inclined axes, also known as the PLAGIOgonal Axes. As a further continuation of our earlier work similar to the present one, we expanded our discussions on *Fermat*'s First point (**X_{13}**) and First Napoleon point (**X_{17}**) and present some additional relations in this work. In both the cases, we have exemplified our theorems using some challenging geometrical problems available in the social media. We have also tried to make a comparison of the relations between the *Fermat* and *Napoleon* points.

Keywords: *First and Second Fermat – Torricelli points*, **X_{13}**, **X_{14}**, *Napoleon points*, **X_{17}**, **X_{18}**, *PLAGIOGONAL Axes*, *isogonic point, ETC*

1.0 Introduction

In Euclidean Plane Geometry, if three *equilateral* triangles are constructed outwardly and also inwardly on the three sides of a scalene triangle no angles of which are greater than 120^0, and then three lines are drawn from each of the vertices of the triangle to the opposite vertices of the equilateral triangles, then these lines concur at a point which is known as the First *Fermat-Torricelli* point, denoted henceforth as **F_1** and Second *Fermat* point, denoted henceforth as **F_2** and or simply, *Fermat* points of the given triangle. These *Fermat* points are also the first *isogonic* points (***X_{13}*** and **X_{14}** in Kimberling's *Encyclopedia of Triangle Centers*, *ETC*) for the above mentioned type of triangle. Further, if lines from each of the vertices of the triangle are drawn connecting the centroids of the outwardly and inwardly drawn equilateral triangles then these lines also become concurrent the respective points of concurrency of these lines are termed as the First and Second *Napoleon* points **N_1** and **N_2** or according to Clark Kimberling's *ETC* nomenclature, as **X_{17}** and **X_{18}** respectively.

In this present work, we have expanded our analysis carried out in [1] which only had focussed on **F_1** and **N_1** points.

2.0 Second Fermat or Fermat – Torricelli Point (F_2 or X_{14}) and allied Theorems:

According to the theorem, we know that if for any scalene triangle in which the largest angle not exceeding 120^0, if three *equilateral* triangles are developed *inwardly* then the three lines drawn from the opposite vertices of the triangle joining the vertices of the inward equilateral triangles thus formed, are *concurrent* and the point of the concurrency is called Second *Fermat* or *Fermat – Torricelli* point denoted as **F_2** or, as per *Kimberling*'s nomenclature as **X14**. The following figure shows an arbitrary triangle ***ABC*** and three equilateral triangles ***A₁BC***, ***B₁CA*** and ***C₁AB*** drawn inward. As per the above theorem, lines ***AA₁***, ***BB₁*** and ***CC₁*** are *concurrent* at **F₂** (**X₁₄**).

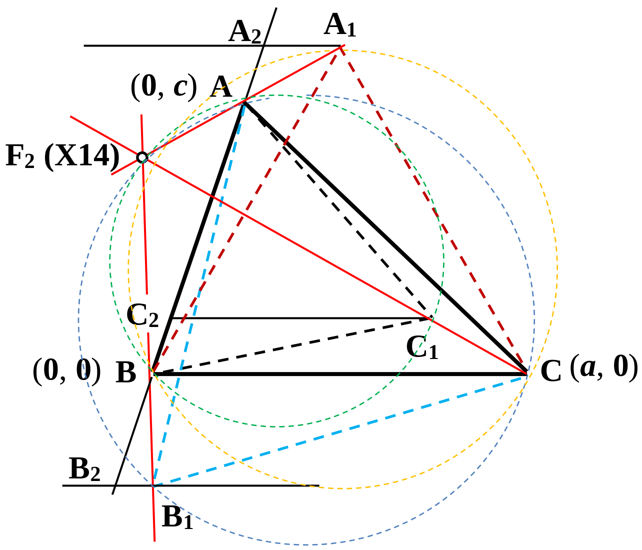


Figure 1: Fermat's Second Point **F₂** (**X₁₄**)

Proof:- Let us choose the PLAGIOgonal Co – ordinate system (Vertex, ***x*** – axis, ***y*** – axis) as (**B**, ***BC***, ***BA***) and it is easy to note the following co-ordinates of the points **A₁**, **B₁** and **C₁** respectively as follows:

$$\left(\frac{a\left(\sin B-\sqrt{3}\cos B\right)}{2\sin B},\frac{\sqrt{3}a}{2\sin B}\right),\left(\frac{a\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}c}{2\sin B},-\frac{\sqrt{3}a-c\left(\sqrt{3}\cos B+\sin B\right)}{2\sin B}\right)$$

and $\left(\frac{\sqrt{3}c}{2\sin B},\frac{c\sin\left(B-60^0\right)}{\sin B}\right)$ $\Rightarrow$ equations of ***AA₁***, ***BB₁*** and ***CC₁*** are thus respectively:

$$y=c+\frac{\sqrt{3}a-2c\sin B}{a\left(\sin B-\sqrt{3}\cos B\right)}x,\ y=-\frac{\sqrt{3}a-c\left(\sqrt{3}\cos B+\sin B\right)}{a\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}c}x,\ y=\frac{c\left(\sin B-\sqrt{3}\cos B\right)}{\sqrt{3}c-2a\sin B}(x-a)$$

Solving these equations in pairs we obtain the following same co-ordinates of point of intersection of the above lines as:

$$\boxed{\left(\frac{ca\left(\sin B-\sqrt{3}\cos B\right)\left[a\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}c\right]}{2\sqrt{3}\sin B\left[ca\left(\sqrt{3}\sin B+\cos B\right)-c^2-a^2\right]},\frac{ca\left(\sin B-\sqrt{3}\cos B\right)\left[c\left(\sin B+\sqrt{3}\cos B\right)-\sqrt{3}a\right]}{2\sqrt{3}\sin B\left[ca\left(\sqrt{3}\sin B+\cos B\right)-c^2-a^2\right]}\right)}$$

or equivalently:

$$\left(\frac{ca\sin\left(B-60^0\right)\left[2a\sin\left(B+60^0\right)-\sqrt{3}c\right]}{\sqrt{3}\sin B\left[2ca\sin\left(B+30^0\right)-c^2-a^2\right]},\frac{ca\sin\left(B-60^0\right)\left[2c\sin\left(B+60^0\right)-\sqrt{3}a\right]}{\sqrt{3}\sin B\left[2ca\sin\left(B+30^0\right)-c^2-a^2\right]}\right) \quad (\mathbf{1a}, \mathbf{1b})$$

and *are considered as the co-ordinates of* **F_2** or **X_{14}** *with respect to the above co-ordinate system*. Alternatively, the above co-ordinates can also be expressed in terms of the sides of the given triangle as follows [eqn (**1c**)]:

$$\left(\frac{a}{8\sqrt{3}\Delta}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]\left[4\Delta-\sqrt{3}\left(c^2+b^2-a^2\right)\right]}{\left[4\sqrt{3}\cdot\Delta-\left(a^2+b^2+c^2\right)\right]},\frac{c}{8\sqrt{3}\Delta}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]\left[4\Delta-\sqrt{3}\left(a^2+b^2-c^2\right)\right]}{\left[4\sqrt{3}\cdot\Delta-\left(a^2+b^2+c^2\right)\right]}\right)$$

Another useful formulation for the co-ordinates of the second *Fermat* point are as follows:

$$\left(\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(A-60^0\right),\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(C-60^0\right)\right) \quad (\mathbf{1d})$$

<u>Theorem_II</u>: For any triangle ***ABC*** whose angles are not more than 120^0, if **F_2** denotes its second *Fermat* point then prove that:

$$AF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta} \quad BF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta} \text{ and}$$

$$CF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(a^2+b^2-c^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$$

<u>Proof</u>:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system (Vertex, ***x*** – axis, ***y*** – axis) as (**B**, ***BC***, ***BA***) and various co-ordinates are marked in the figure itself. For this system, the co-ordinates of **F_2** are shown vide **Theorem I** and are expressed in eqn (1a) – (1d). Considering the co-ordinates as expressed in (**1d**), we can prove by *distance formula*: $BF_2^2 = f_1^2+f_2^2+2f_1\cdot f_2\cos B$, where $f_1=\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(A-60^0\right)$ and $f_2=\frac{2R}{\sqrt{3}}\cdot\frac{4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)}{4\sqrt{3}\Delta-\left(a^2+b^2+c^2\right)}\cdot\sin\left(C-60^0\right)$. With a little trigonometric simplification, we can easily prove that: $\boxed{BF_2^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}}$ [**QED**] and the rest two relations can likewise be derived by choosing respectively the PLAGIOgonal system (**A**, ***AB***, ***AC***) and (**C**, ***CB***, ***CA***) [**QED**]

Note: In both the above theorems, ***Δ*** represents the area of the triangle ***ABC***.

Theorem_III: For any triangle ***ABC*** whose angles are not more than 120^0, if $\mathbf{F_2}$ denotes its second *Fermat* point then prove that: $AF_2^2+BF_2^2+CF_2^2=\frac{a^2+b^2+c^2}{2}+\frac{2\sqrt{3}\Delta}{3}$

Proof:- Using the Theorem II above, we can easily note that:

$$AF_2^2+BF_2^2+CF_2^2=\frac{1}{6}\cdot\frac{\left[\left\{\left[4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)\right]^2+\left[4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right]^2+\left[4\Delta-\sqrt{3}\left(a^2+b^2-c^2\right)\right]^2\right\}\right]}{\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]}$$

which can be simplified to:

$$\Rightarrow AF_2^2+BF_2^2+CF_2^2=\frac{9\left(a^4+b^4+c^4\right)-6\left(a^2b^2+b^2c^2+c^2a^2\right)+48\Delta^2-8\sqrt{3}\Delta\left(a^2+b^2+c^2\right)}{6\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]}$$

$$=\frac{\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]^2+4\left[\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2\right]}{6\left[\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta\right]}$$ and ultimately:

$$AF_2^2+BF_2^2+CF_2^2=\frac{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}{6}+\frac{2}{3}\cdot\frac{\left(a^2-b^2\right)^2+\left(b^2-c^2\right)^2+\left(c^2-a^2\right)^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$$

$$=\frac{\left(a^2+b^2+c^2\right)-8\sqrt{3}\Delta}{6}+\frac{1}{6}\cdot\frac{8\left(a^4+b^4+c^4\right)-8\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\left(a^2+b^2+c^2-8\sqrt{3}\Delta\right)+\frac{8\left(a^4+b^4+c^4\right)-8\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\frac{\left(a^2+b^2+c^2\right)^2-12\sqrt{3}\Delta\left(a^2+b^2+c^2\right)+96\Delta^2+8\left(a^4+b^4+c^4\right)-8\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\frac{\left(a^2+b^2+c^2\right)^2-12\sqrt{3}\Delta\left(a^2+b^2+c^2\right)+2\left(a^4+b^4+c^4\right)+4\left(a^2b^2+b^2c^2+c^2a^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$=\frac{1}{6}\left[\frac{3\left(a^2+b^2+c^2\right)^2-12\sqrt{3}\Delta\left(a^2+b^2+c^2\right)}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}\right]+\frac{2\sqrt{3}\Delta}{3}$$

$$\Rightarrow \boxed{AF_2^2+BF_2^2+CF_2^2=\frac{a^2+b^2+c^2}{2}+\frac{2\sqrt{3}\Delta}{3}}$$ [**QED**]

3.0 Properties of 2nd Fermat Point, F_2 (X_{14}): Having found the basic characters of the 2nd Fermat point, let us observe some of its properties: **Property I:** If ***ABC*** is a triangle with $\angle ABC = 60^0$, then we note from eqn (**1b**) that: F_2 coincides with the vertex **B**. In other words, if any angle of the triangle ***ABC*** becomes 60^0, then *the 2nd Fermat point belongs to the circumcircle of the triangle*.It is also clear that for an *equilateral* triangle, 2nd Fermat point, F_2 or X_{14} becomes undefined. **Property II:** If ***ABC*** is a triangle with $\angle ABC = 120^0$, then we note from eqn (**1b**) that the co-ordinates of F_2 with respect to the PLAGIOgonal system shown above are: $\boxed{\left(\frac{c^2a}{c^2+a^2-ca}, \frac{ca^2}{c^2+a^2-ca}\right)}$

Property III: If ***ABC*** is a triangle with $\angle ABC = 120^0$, then the following is true:

$$\boxed{AF_2 + CF_2 - BF_2 = \sqrt{c^2+a^2-ca} = \sqrt{\frac{a^2+b^2+c^2}{2} - 2\sqrt{3}\Delta}}$$

This can be directly checked if we note that for $\angle ABC = 120^0$ the following results hold good: $AF_2 = \frac{c^2}{\sqrt{c^2+a^2-ca}}, BF_2 = \frac{ca}{\sqrt{c^2+a^2-ca}}, CF_2 = \frac{a^2}{\sqrt{c^2+a^2-ca}}$. So it is obvious that: $AF_2 + CF_2 > BF_2$

Property IV: If ***ABC*** is a triangle with $\angle ABC = 120^0$, then BF_2 is the *Geometric Mean* (**GM**) of AF_2 and CF_2 The fact follows easily from **Property III** mentioned above.Based on the above properties related with F_2 and a special triangle ***ABC*** is a triangle with $\angle ABC = 120^0$, following examples can be discussed in short.

Example # 1: Consider a non-isosceles triangle ***ABC*** with $\angle BAC = 120^0$ and whose *circumcenter*, *orthocentre* and the 2nd *Fermat* point and the middle point of the side ***BC*** be denoted as **O**, **H**, **F** and **M** respectively . Let the '**A**' – angle bisector meets the circumcircle of the triangle at the point **K**. Prove that ***AHOK*** is a *parallelogram* and its area is given as $AM \cdot OF$. [**Problem Courtesy**: *Rachid Iksi*]

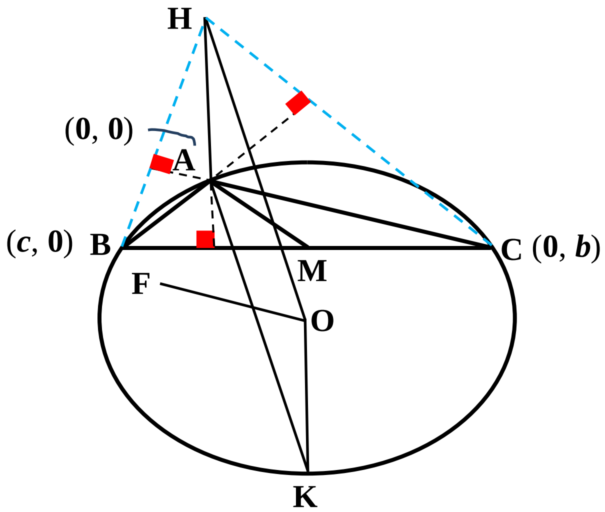


Figure 2: Example #1

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, ***AB***, ***AC***) and various co-ordinates are marked in the figure itself. We note that the co-ordinates of **O**, **H**, **K** and **F** for the given triangle are respectively: $\left(\frac{b+2c}{3},\frac{2b+c}{3}\right),\left(-\frac{2b+c}{3},-\frac{b+2c}{3}\right),(b+c,b+c),\left(\frac{b^2c}{b^2+c^2-bc},\frac{bc^2}{b^2+c^2-bc}\right)\Rightarrow$ now by distance formula we easily obtain:

$$OH=AK=b+c,\ AH=\sqrt{\frac{b^2+c^2+bc}{3}}=\frac{a}{\sqrt{3}},OK=R=\frac{a}{2\sin A}=\frac{a}{\sqrt{3}}\Rightarrow AH=OK$$

(alternatively, we can show that: $AK\parallel OH$ and $OK\parallel HA$) and hence ***AHOK*** *is a parallelogram*.

Further: $[AHOK]=2[\Delta AOK]=\frac{\sqrt{3}}{2}\begin{vmatrix}0 & 0 & 1\\ b+c & b+c & 1\\ \frac{b+2c}{3} & \frac{2b+c}{3} & 1\end{vmatrix}=\frac{1}{2\sqrt{3}}\left|b^2-c^2\right|$ (**1**)

Also we note that: $AM^2=\frac{2(b^2+c^2)-a^2}{4}\Rightarrow AM=\frac{\sqrt{b^2+c^2-bc}}{2}$ since: $a^2=b^2+c^2+bc$

and:

$$OF^2=\left(\frac{b^2c}{b^2+c^2-bc}-\frac{b+2c}{3}\right)^2+\left(\frac{bc^2}{b^2+c^2-bc}-\frac{2b+c}{3}\right)^2-\left(\frac{b^2c}{b^2+c^2-bc}-\frac{b+2c}{3}\right)\left(\frac{bc^2}{b^2+c^2-bc}-\frac{2b+c}{3}\right)$$

$$=\frac{(b^2-c^2)^2}{9(b^2+c^2-bc)^2}\left[(2c-b)^2+(c-2b)^2+(2c-b)(2b-c)\right]=\frac{(b^2-c^2)^2}{3(b^2+c^2-bc)}\Rightarrow OF=\frac{\left|b^2-c^2\right|}{\sqrt{3(b^2+c^2-bc)}}$$

$\Rightarrow AM\cdot OF=\frac{|b^2-c^2|}{2\sqrt{3}}$ (**2**)   From (**1**) and (**2**): $\boxed{[AHOK]=AM\cdot OF}$ [**QED**]

<u>Example</u> # 2: Consider a non-isosceles triangle ***ABC*** with $\angle BAC=120^0$. **F** is the 2nd *Fermat* point, and ***AS*** is the '**A**' – *symmedian* of the given triangle. Prove that: **S** lies on the '**A**' – *symmedian* and further $\frac{1}{AS}\cdot\frac{1}{AF}=\frac{1}{AB^2}+\frac{1}{AC^2}$ [**Problem Courtesy**: *Rachid Iksi*]

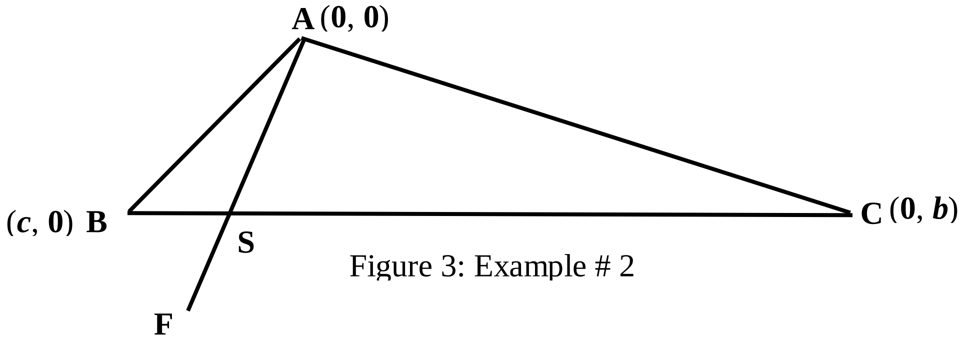


Figure 3: Example # 2

<u>Proof</u>:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, ***AB***, ***AC***) and various co-ordinates are marked in the figure itself. We note that the co-ordinates of **F** for the given triangle are: $\left(\frac{b^2c}{b^2+c^2-bc},\frac{bc^2}{b^2+c^2-bc}\right)\Rightarrow$ equation of ***AF*** is:

$y=\frac{c}{b}x\Rightarrow$ **S** lies on the '**A**' – *symmedian.* Now, we know that by *distance formula*:

$AS=\frac{bc}{b^2+c^2}\sqrt{b^2+c^2-bc}$ and also by **Theorem – II** we know that:

$AF^2=\frac{1}{6}\cdot\frac{\left[4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$ which for $\angle BAC=120^0$ becomes:

$AF=\frac{bc}{\sqrt{b^2+c^2-bc}}\Rightarrow\frac{1}{AS}\cdot\frac{1}{AF}=\frac{b^2+c^2}{b^2c^2}\Rightarrow\boxed{\frac{1}{AS}\cdot\frac{1}{AF}=\frac{1}{AB^2}+\frac{1}{AC^2}}$ [**QED**]

<u>Example</u> # 3: Consider a non-isosceles triangle ***ABC*** with $\angle BAC=120^0$. Prove that its *Lester* Circle (**C_L**), i.e., the circle which passes through the circumcenter **O**, center of *NPC* (**C_N**), **O_9** and *Fermat* points **F_1** and **F_2** and the *NPC* (**C_N**) of the triangle, are congruent. Also prove that (**C_N**) and (**C_L**) intersect at the points **D** and **E** which are the middle points of ***AB*** and ***AC***.

[**Problem Courtesy**: *Rachid Iksi*]

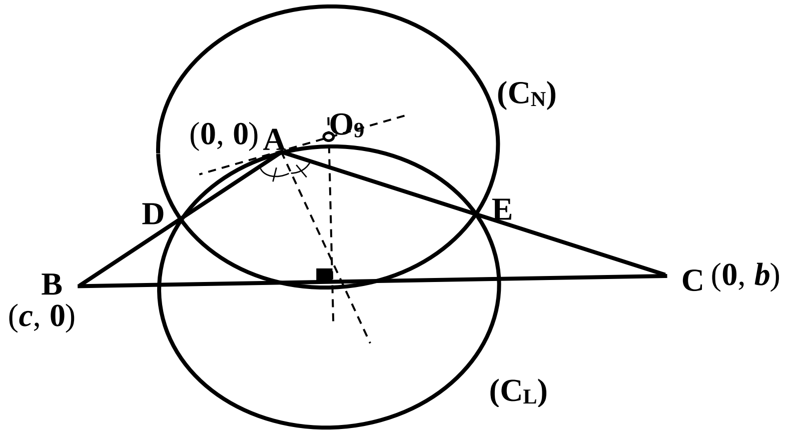


Figure 3: Example #3

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, ***AB***, ***AC***) and various co-ordinates are marked in the figure itself. We note that the co-ordinates of circumcenter **O** and **O₉** are respectively (for $\angle BAC = 120^0$) : $\left(\frac{b+2c}{3}, \frac{2b+c}{3}\right)$ and $\left(-\frac{b-c}{6}, \frac{b-c}{6}\right) \Rightarrow m_{OO_9} = \frac{\frac{4b+2c}{6} - \frac{b-c}{6}}{\frac{2b+4c}{6} + \frac{b-c}{6}} = 1$. Now equation of the perpendicular bisector of the line joining **O** and **O₉** is: $x + y = \frac{b+c}{2}$ (**1**) We now note that the first *Fermat* point, F_1 for this given triangle coincides with the vertex **A** and the co-ordinates of the second *Fermat* point F_2 are: $\left(\frac{b^2c}{c^2+b^2-bc}, \frac{bc^2}{c^2+b^2-bc}\right) \Rightarrow m_{F_1F_2} = \frac{c}{b}$ and the '*m*' value of the line which is perpendicular to the line joining the points F_1 and F_2 is given by: $\left(m_{F_1F_2}\right)_\perp = -\frac{1+m_{F_1F_2}\cos A}{m_{F_1F_2}+\cos A} = \frac{2b-c}{b-2c}$ and hence the equation of the perpendicular bisector of F_1 and F_2 is: $(b-2c)y = -bc + (2b-c)x$. Solving it with (**1**) gives us the co-ordinates of the center O_L of (C_L) as: $\left(\frac{b+2c}{6}, \frac{2b+c}{6}\right)$ and as a result, we get the radius R_L of the *Lester* circle as:

$$R_L^2 = \frac{1}{36}\left[(b+2c)^2 + (2b+c)^2 - (b+2c)(2b+c)\right] = \frac{b^2+c^2+bc}{12} \Rightarrow \boxed{R_L = \frac{R}{2}}$$ (since we have the results: $\angle BAC = 120^0 \Rightarrow a^2 = b^2+c^2+bc,\ R = \frac{a}{\sqrt{3}} \Rightarrow 3R^2 = b^2+c^2+bc$), ***R*** being the circumradius of the given triangle and since this is also the radius of *NPC*, so prove that for the given triangle, *Lester* Circle (C_L) and the *NPC* of the triangle (C_N) are *congruent* [**QED**] Now, equation of (C_L) is: $\left(x - \frac{b+2c}{6}\right)^2 + \left(y - \frac{2b+c}{6}\right)^2 - \left(x - \frac{b+2c}{6}\right)\left(y - \frac{2b+c}{6}\right) = \frac{b^2+c^2+bc}{12}$

$\Rightarrow x^2+y^2-xy-\frac{c}{2}x-\frac{b}{2}y=0 \Rightarrow$ it intersects the sides ***AB*** and ***AC*** at points **D** and **E** which are clearly the middle points of ***AB*** and ***AC*** respectively and thus lies on (C_N) and hence we conclude: (C_N) and (C_L) intersect at the points **D** and **E** which are the middle points of ***AB*** and ***AC*** [**QED**] **Example # 4:** Consider a non-isosceles triangle ***ABC*** with $\angle BAC=120^0$. ***AD*** is the angle – bisector, **Le** is the *Lemonie* point, **O** is the *circumcenter*, **H** is the *orthocentre*, **G** is the *centroid*, O_9 is the center of *NPC* and F_2 is the 2nd *Fermat* point of the triangle ***ABC***. M_a, M_b and M_c are the middle points of the sides ***BC***, ***CA*** and ***AB*** respectively. **N** and **K** are respectively the middle points of ***AD*** and O_9D. **M** is the projection of O_9 on M_bM_c and **L** is the reflection of O_9 with respect to **M**. **E** is defined as the second point of intersection of ***OG*** with C_1. Let us define the following system of circles: C_1: □ (G, GO) C_2: □ (D, DA) C_3: *Lester* Circle (**Example # 3**)C_4: Circle through **A**, O_9, **V** and **D** C_5: *NPC*

C_6: *Circumcircle* of triangle ***ABC*** **Prove that:** 1) **L** belongs to *NPC* (C_5) 2) M_b, M_c, **M** and **N** are *collinear* 3) **O**, **G**, O_9 and **H** are *collinear* 4) **E**, **A**, **Le** and F_2 are *collinear* 5) **P**, **R**, **N**, **K**, **S** and **Q** are *collinear* (**R** and **S** are on ***AB*** and ***AC*** respectively)

6) $LM_cO_9M_b$ and ***DRAS*** are *rhombuses* ($60^0-120^0-60^0-120^0$)

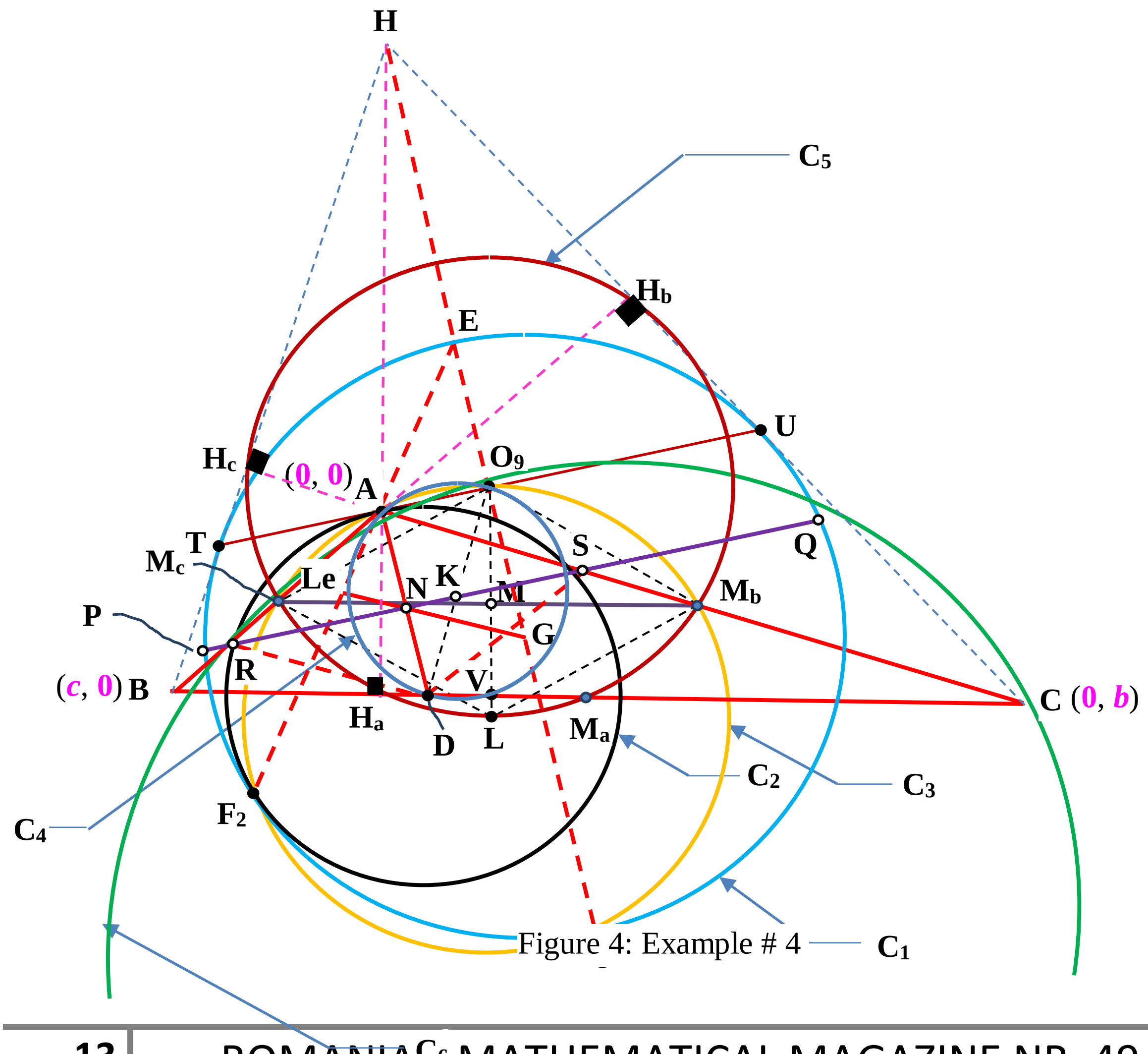


Figure 4: Example # 4

7) ***BH*** and ***CH*** are tangents to C_1 (T and U are touch points)

8) **T**, **A**, O_9 and **U** are *collinear*

[**Problem Courtesy**: *Thanasis Gakopoulos*]

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**A**, ***AB***, ***AC***) and various co-ordinates are marked in the figure itself. We note that the equations of various circles (C_1 through C_6 except C_4) are respectively as follows for the given triangle: $C_1 : x^2+y^2-xy+\left(\frac{b-2c}{3}\right)x-\left(\frac{2b-c}{3}\right)y-\frac{bc}{3}=0$

$C_2 : x^2+y^2-xy-\left(\frac{bc}{b+c}\right)(x+y)=0$

$C_3 : x^2+y^2-xy-\frac{c}{2}x-\frac{b}{2}y=0,\ C_5 : x^2+y^2-xy+(x-y)\left(\frac{b-c}{2}\right)=0$ and finally,

$C_6 : x^2+y^2-xy-cx-by=0$. Now, equation of M_bM_c is: $bx+cy=\frac{bc}{2}\Rightarrow$ co-ordinates of **M** are: $\left(\frac{c}{4},\frac{b}{4}\right)\Rightarrow$ co-ordinates of **L** are: $\left(\frac{b+2c}{6},\frac{2b+c}{6}\right)$

$$\Rightarrow O_9L^2=\frac{1}{36}\left[(2b+c)^2+(b+2c)^2-(2b+c)(b+2c)\right]=\frac{b^2+c^2+bc}{12}$$

$$\Rightarrow O_9L^2=\frac{R^2}{4}=R_9^2\Rightarrow \boxed{L\in C_5}$$ [**QED**]

Clearly, co-ordinates of **N** and **M** are $\left(\frac{bc}{2(b+c)},\frac{bc}{2(b+c)}\right),\left(\frac{c}{4},\frac{b}{4}\right)$ respectively and they satisfy the equation of M_bM_c and hence: M_b, M_c, **M** and **N** are collinear [**QED**]

As the co-ordinates of **O**, **G** and O_9 are: $\left(\frac{b+2c}{3},\frac{2b+c}{3}\right),\left(\frac{c}{3},\frac{b}{3}\right),\left(-\frac{b-c}{6},\frac{b-c}{6}\right)$

$\Rightarrow m_{OG}=1,\ m_{O_9G}=\dfrac{\frac{b-c}{6}-\frac{2b}{6}}{-\frac{b-c}{6}-\frac{2c}{6}}=1\Rightarrow O-G-O_9 \parallel AD\Rightarrow$ equation of this line is: $x-y+\frac{b-c}{3}=0$ and it is clear that the co-ordinates of **H** — the *orthocentre* of the triangle $\left(-\frac{2b+c}{3},-\frac{b+2c}{3}\right)$ satisfy this equation and thus: **O**, **G**, O_9 and **H** are *collinear* [**QED**]

If we now solve the above line with **C₁**, we obtain the co-ordinates of the point **E** as: $\left(-\frac{b}{3},-\frac{c}{3}\right)\Rightarrow E\in y=\frac{c}{b}x\Rightarrow E-Le-A-F_2$ are *collinear* [**QED**]

We now note that the line ***PQ*** is the *common chord* between the circles **C₁** and **C₆** and hence its equation is: $x+y=\frac{bc}{b+c}$ which can be obtained by subtracting the two equations of the circles. Now it intersects the sides ***AB*** and ***AC*** of the triangle at points **R** and **S** respectively having the co-ordinates $\left(\frac{bc}{b+c},0\right)$ and $\left(0,\frac{bc}{b+c}\right)\Rightarrow ARDS$ is a rhombus formed with two equilateral triangles ***ADR*** and ***ADS*** [**QED**]

We note the co-ordinates of the points **N** (middle point of ***AD***) and **K** (middle point of ***O₉D***) as:
$\left(\frac{bc}{2(b+c)},\frac{bc}{2(b+c)}\right),\left(\frac{6bc-b^2+c^2}{12(b+c)},\frac{6bc+b^2-c^2}{12(b+c)}\right)\Rightarrow N,K\in PQ:x+y=\frac{bc}{b+c}\Rightarrow P,R,N,K,S,Q$ are *collinear* [**QED**]

We again note that $O_9L^2=\frac{b^2+c^2+bc}{12},O_9M_c^2=\left(\frac{c}{2}+\frac{b-c}{6}\right)^2+\frac{(b-c)^2}{36}+\left(\frac{c}{2}+\frac{b-c}{6}\right)\frac{b-c}{6}$ $\Rightarrow O_9M_c^2=\frac{1}{36}\left[(b+2c)^2+(b-c)^2+(b+2c)(b-c)\right]=\frac{b^2+c^2+bc}{12}$ Similarly we obtain by *distance formula*:

$LM_c^2=\frac{b^2+c^2+bc}{12}$, $O_9M_b^2=\frac{(b-c)^2}{36}+\left(\frac{b}{2}-\frac{b-c}{6}\right)^2-\frac{b-c}{6}\left(\frac{b}{2}-\frac{b-c}{6}\right)=\frac{b^2+c^2+bc}{12}$ and finally:

$LM_b^2=\left(\frac{b+2c}{6}\right)^2+\left(\frac{2b+c}{6}-\frac{b}{2}\right)^2-\left(\frac{b+2c}{6}\right)\left(\frac{2b+c}{6}-\frac{b}{2}\right)=\frac{b^2+c^2+bc}{12}$ and thus we conclude that ***LMcO₉Mb*** and ***DRAS*** are *rhombuses* (**60⁰ – 120⁰ – 60⁰ – 120⁰**) [**QED**]

Finally we consider that the equation of the tangents that can be drawn from the point **H** $\left(-\frac{2b+c}{3},-\frac{b+2c}{3}\right)$ to the circle **C₁** can be written as: $mx-y+\frac{b(2m-1)+c(m-2)}{3}=0$ and since it is tangent to **C₁** whose center is: $\left(\frac{c}{3},\frac{b}{3}\right)$ and therefore:

$$\frac{\frac{1}{9}\left[mc-b+b(2m-1)+c(m-2)\right]\frac{3}{4}}{m^2-m+1}=\frac{(b+c)^2}{9}\Rightarrow\frac{1}{3}\cdot\frac{(b+c)^2(m-1)^2}{m^2-m+1}=\frac{(b+c)^2}{9}$$

$\Rightarrow m = 2 \wedge \frac{1}{2} \Rightarrow T_1 : \boxed{2x - y + b = 0}$ — which is the equation of ***HC*** for $m = 2$ similarly we can find the equation of the second tangent $\left(m = \frac{1}{2}\right)$ as: $T_2 : \frac{x}{2} - y - \frac{c}{2} = 0 \Rightarrow \boxed{y = \frac{x - c}{2}}$ — – which is the equation of ***HB*** and the co-ordinates of the touch points are easily found to be as **T** and **U** having the co-ordinates: $\left(\frac{c}{3}, -\frac{c}{3}\right)$ and $\left(\frac{b}{3}, -\frac{b}{3}\right)$ satisfying the equation: $x + y = 0$ and thus we note that **T**, **A**, **O₉** and **U** are *collinear* [**QED**]

__Example__ **# 5:** Consider a non-isosceles triangle ***ABC*** with $\angle ABC = 45^0, \angle ACB = 30^0$. **O** and **F₂** are the *circumcenter* and the second *Fermat* point respectively while ***DEF*** is the *orthic* triangle of ***ABC***. Prove that $\Delta^s DEF, ABF_2, AOF_2$ are **30⁰ – 60⁰ – 90⁰** triangles. Also show that: $\frac{[\Delta DEF]}{[\Delta AOF_2]} = \frac{[\Delta DEF]}{[\Delta ABF_2]} = \frac{3}{4}$ [**Problem Courtesy**: *Rachid Iksi*]

__**Proof:**__**-** Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**C**, ***CA***, ***CB***) and various co-ordinates are marked in the figure itself. Now for the given condition $\angle ABC = 45^0, \angle ACB = 30^0$ we note the following:

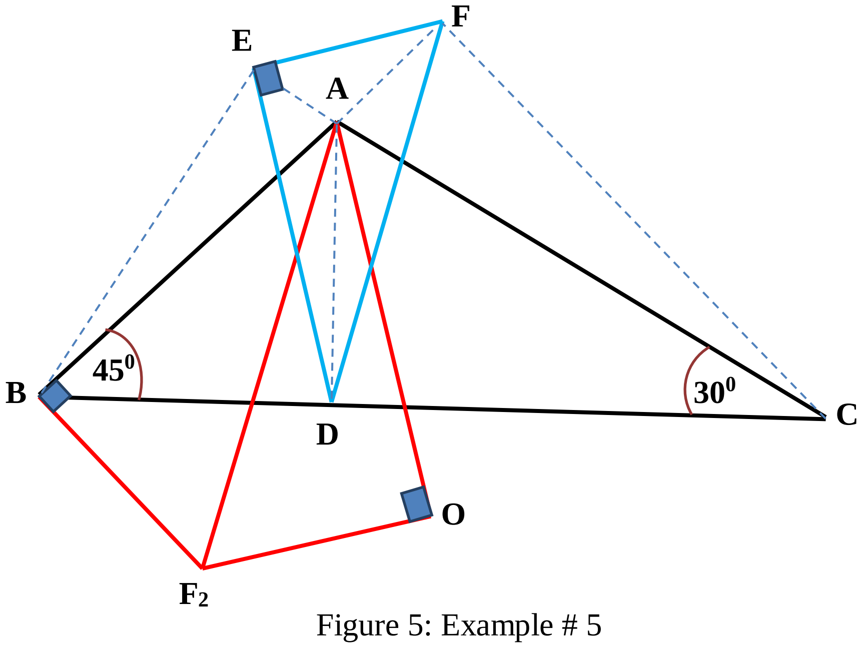


Figure 5: Example # 5

$\frac{a}{\sin 75^0}=\frac{b}{\sin 45^0}=\frac{c}{\sin 30^0}\Rightarrow\frac{a}{\sqrt{3}+1}=\frac{b}{2}=\frac{c}{\sqrt{2}}\Rightarrow a=\left(\sqrt{3}+1\right), b=2, c=\sqrt{2}$ (WLOG). Hence, we note that the co-ordinates of the points **O**, **F_2**, **D**, **E** and **F** are respectively given as follows: $\left(1-\sqrt{3},2\right),\left(-\frac{2}{\sqrt{3}},\frac{2\left(\sqrt{3}+1\right)}{\sqrt{3}}\right),\left(0,\sqrt{3}\right),\left(\frac{\sqrt{3}\left(\sqrt{3}+1\right)}{2},0\right),\left(\sqrt{3}+1,-1\right)\Rightarrow$ we note that:

$m_{CF}=-\frac{\sqrt{3}-1}{2},\quad m_{BF_2}=-\frac{\sqrt{3}-1}{2}\Rightarrow CF\parallel BF_2\Rightarrow BF_2\perp AB$. Also from our known results:

$$AF_2=\frac{1}{\sqrt{6}}\cdot\frac{4\Delta-\sqrt{3}\left(b^2+c^2-a^2\right)}{\sqrt{a^2+b^2+c^2-4\sqrt{3}\Delta}}=\frac{1}{\sqrt{6}}\cdot\frac{2\sqrt{3}+2-2\sqrt{3}\left(1-\sqrt{3}\right)}{2}=\frac{1}{\sqrt{6}}\cdot\frac{8}{2}=\frac{4}{\sqrt{6}}$$ and

$$BF_2=\frac{1}{\sqrt{6}}\cdot\frac{\left|4\Delta-\sqrt{3}\left(c^2+a^2-b^2\right)\right|}{\sqrt{a^2+b^2+c^2-4\sqrt{3}\Delta}}=\frac{1}{\sqrt{6}}\cdot\frac{\left|2\sqrt{3}+2-2\sqrt{3}\left(\sqrt{3}+1\right)\right|}{2}=\frac{2}{\sqrt{6}}$$ Thus:

$\Rightarrow\tan\left(\angle BAF_2\right)=\frac{BF_2}{AB}=\frac{1}{\sqrt{3}}\Rightarrow\angle BAF_2=30^0\Rightarrow$ triangle ABF_2 is a **$30^0-60^0-90^0$** triangle

Again we note that: $m_{AO}=\frac{2a-\sqrt{3}b}{b-\sqrt{3}a}=-\frac{2}{\sqrt{3}+1}=-\left(\sqrt{3}-1\right)$ and also we have:

$$m_{OF_2}=\frac{\frac{2\left(\sqrt{3}+1\right)}{\sqrt{3}}-2}{-\frac{2}{\sqrt{3}}-1+\sqrt{3}}=\frac{\frac{2}{\sqrt{3}}}{\frac{1-\sqrt{3}}{\sqrt{3}}}=-\left(\sqrt{3}+1\right)\Rightarrow 1+m_{AO}\cdot m_{OF_2}+\left(m_{AO}+m_{OF_2}\right)\cos C=1+2-3=0\Rightarrow AO\perp OF_2$$

Thus ***AOF_2*** is right angled triangle and further we note that:

$R=\frac{AB}{2\sin 30^0}=AB\Rightarrow\Delta ABF_2\cong\Delta AOF_2\Rightarrow$ triangle AOF_2 is a **$30^0-60^0-90^0$** triangle.

Finally: $m_{DE}=-\frac{2}{\sqrt{3}+1}=-\left(\sqrt{3}-1\right)\Rightarrow DE\parallel AO$ and also we note that:

$m_{EF}=\frac{2}{\left(\sqrt{3}+1\right)\left(\sqrt{3}-2\right)}=-\frac{2\left(\sqrt{3}+2\right)}{\sqrt{3}+1}=-\left(\sqrt{3}+1\right)\Rightarrow EF\parallel OF_2\Rightarrow EF\perp DE\Rightarrow$ DEF is a right angled triangle and due to the above parallelism we note that ΔAOF_2 is a **$30^0-60^0-90^0$** triangle. [**QED**]

Now we note by *distance formula*: $DF = \sqrt{2\left(\sqrt{3}+1\right)^2 - \sqrt{3}\left(\sqrt{3}+1\right)^2} = \sqrt{2}$ and since

$$\Delta AOF_2 \sim \Delta DEF_2 \Rightarrow \frac{[\Delta DEF_2]}{[\Delta AOF_2]} = \frac{DF_2^2}{AF_2^2} = \frac{6}{4}\cdot\frac{1}{2} = \frac{3}{4} \Rightarrow \boxed{\frac{[\Delta DEF_2]}{[\Delta AOF_2]} = \frac{[\Delta DEF_2]}{[\Delta ABF_2]} = \frac{3}{4}} \quad \text{[QED]}$$

4.0 Comparison between 1st and 2nd Fermat Points, F_1, F_2 (X_{13}, X_{14}):

In our earlier work [1], we had discussed the properties of the First *Fermat* point (also known as the *Fermat – Torricelli* point), F_1 or X_{13}. In this present section, we will discuss the similarity of the two *Fermat* points F_1 and F_2. The following figure shows the F_1 point for any triangle ***ABC***.

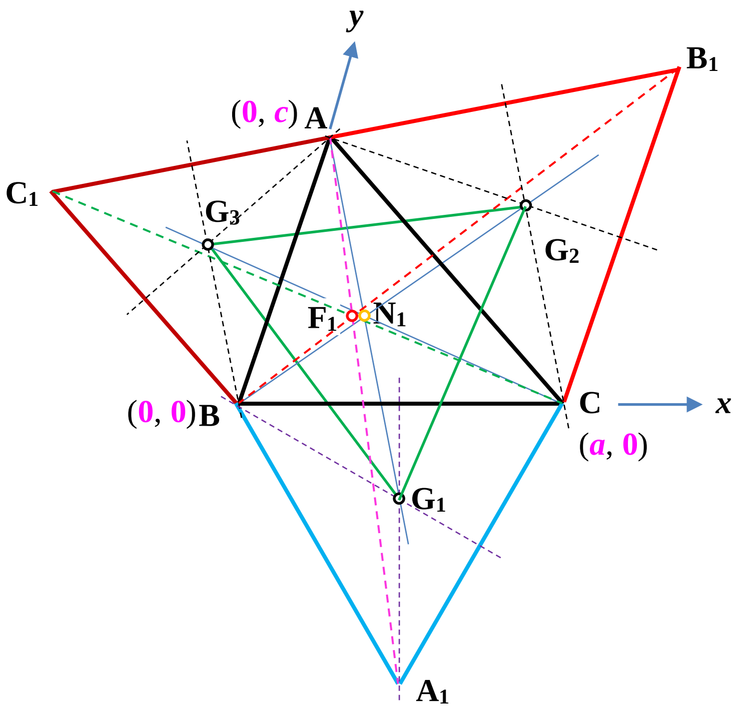


Figure 6: Fermat's First Point (**F1**) and Napoleon's First Point (**N1**)

a) <u>Co-ordinates of **F1** and **F2**:</u>

We list the following co-ordinates with respect to the PLAGIOgonal Co – ordinate system) as (**B**, ***BC***, ***BA***) and various co-ordinates are marked accordingly in the figure itself.

Co-ordinates of **F1**: $\left(\dfrac{ca\sin\left(B+60^0\right)\left[2a\sin\left(60^0-B\right)-\sqrt{3}c\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0-B\right)-a^2-c^2\right]}, \dfrac{ca\sin\left(B+60^0\right)\left[2c\sin\left(60^0-B\right)-\sqrt{3}a\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0-B\right)-a^2-c^2\right]}\right)$

and:

Co-ordinates of **F₂**: $\left(\dfrac{ca\sin\left(B-60^0\right)\left[2a\sin\left(60^0+B\right)-\sqrt{3}c\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0+B\right)-c^2-a^2\right]},\dfrac{ca\sin\left(B-60^0\right)\left[2c\sin\left(60^0+B\right)-\sqrt{3}a\right]}{\sqrt{3}\sin B\left[2ca\sin\left(30^0+B\right)-c^2-a^2\right]}\right)$

Hence from the above it is obvious that if *any one angle of otherwise a scalene triangle is* **120^0***, then* **F_1** *will be on its circumcircle* while, if *any one angle of otherwise a scalene triangle is* **60^0***, then* **F_2** *will be on its circumcircle*.

b) Distances of **F_1** and **F_2** from the Triangle Vertices**:**

We note from the above co-ordinates of **F_1** and **F_2** that:

$$AF_1^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(b^2+c^2-a^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}\quad BF_1^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(c^2+a^2-b^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}\quad CF_1^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(a^2+b^2-c^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}$$

and:

$$AF_2^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(b^2+c^2-a^2\right)-4\Delta\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta},\quad BF_2^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(c^2+a^2-b^2\right)-4\Delta\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta},\quad CF_2^2=\frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(a^2+b^2-c^2\right)-4\Delta\right]^2}{\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}$$

From the previous set of equations we can conclude that:

$$AF_1^2+BF_1^2+CF_1^2=\frac{a^2+b^2+c^2}{2}-\frac{2\sqrt{3}\Delta}{3}$$ for **F_1** and similarly for **F_2**, we have:

$$AF_2^2+BF_2^2+CF_2^2=\frac{a^2+b^2+c^2}{2}+\frac{2\sqrt{3}\Delta}{3}$$ which we have proved in **Theorem III** above.

c) Distances of **F_1** and **F_2** from the *Centroid* of the Triangle**:**

In this case we apply *Leibnitz*' Theorem which states that: $\sum_i MA_i^2=\sum_i GA_i^2+n\cdot MG^2$, where there are ***n*** set of points (**A_1**, **A_2**, **A_3**, ……., **A_n**) whose centroid is **G** and **M** be any arbitrary point. Taking the point **M** as *Fermat* points, we get the following Theorems quite easily:

$$GF_1^2=\frac{a^2+b^2+c^2}{18}-\frac{2\sqrt{3}\Delta}{9}=\frac{a^2+b^2+c^2-4\sqrt{3}\Delta}{18}$$ for **F_1**

and for **F_2** we get: $$GF_2^2=\frac{a^2+b^2+c^2}{18}+\frac{2\sqrt{3}\Delta}{9}=\frac{a^2+b^2+c^2+4\sqrt{3}\Delta}{18}$$

We observe lots of similarities between the characteristics of **F_1** and **F_2** as obvious from the above comparative analysis.

We conclude the session by considering the following examples.

5.0 Examples on 1st and 2nd Fermat Points, F₁, F₂ (X₁₃, X₁₄):

Example # 6: Consider a triangle ***ABC*** of which $\angle ABC = 60^0$. Prove that for such a triangle, $AF_1^2 + BF_1^2 + CF_1^2 = AC^2$ [**Problem Courtesy**: *Rachid Iksi*]

Proof:- We know that: $AF_1^2 + BF_1^2 + CF_1^2 = \frac{a^2+b^2+c^2}{2} - \frac{2\sqrt{3}\Delta}{3}$ and also we note that:

$b^2 = a^2 + c^2 - ca,\ 4\Delta = \sqrt{3}ca \Rightarrow \frac{2\sqrt{3}\Delta}{3} = \frac{ca}{2}$. Now if we substitute these into above relation, we get: $AF_1^2 + BF_1^2 + CF_1^2 = \frac{a^2+b^2+c^2}{2} - \frac{2\sqrt{3}\Delta}{3} = \frac{2(a^2+c^2)-ca}{2} - \frac{ca}{2} = a^2 + c^2 - ca$

$\Rightarrow \boxed{AF_1^2 + BF_1^2 + CF_1^2 = b^2 = AC^2}$ [**QED**]

Example # 7: Consider a triangle ***ABC*** of which $\angle ABC = 60^0$ with ***BS*** and **F₁** as '**B**' – *symmedian* and *Fermat – Torricelli* point respectively of the triangle . It is also given that ***BS = CS***. Show that: $\frac{BF_1}{AF_1} = \frac{CF_1}{BF_1} = \frac{AF_1 + BF_1 + CF_1}{AC} = \varphi = \frac{\sqrt{5}+1}{2}$. Also prove that: $CF_1 = AF_1 + BF_1$ and $\frac{1}{AF_1} = \frac{1}{BF_1} + \frac{1}{CF_1}$ [**Problem Courtesy**: *Rachid Iksi*]

Proof:- Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**B**, ***BC***, ***BA***) and various co-ordinates are marked in the figure itself. Since $\angle ABC = 60^0$, we note: $\Rightarrow b^2 = a^2 + c^2 - ca,\ 4\Delta = \sqrt{3}ca$. Also: We note that the co-ordinates of **S** are: $\left(\frac{c^2a}{a^2+c^2}, \frac{ca^2}{a^2+c^2}\right) \Rightarrow BS = \frac{ca}{a^2+c^2}\sqrt{a^2+c^2+ca}$ and also: $CS = \frac{ba^2}{a^2+c^2}$ Thus $BS = CS \Rightarrow \frac{ba^2}{a^2+c^2} = \frac{ca}{a^2+c^2}\sqrt{a^2+c^2+ca} \Rightarrow a^2 - c^2 = ca \Rightarrow \frac{a}{c} = \frac{\sqrt{5}+1}{2} = \varphi$ Also we know that: $\frac{b^2}{c^2} = 1 + \frac{a^2}{c^2} - \frac{a}{c} = 2 \Rightarrow \frac{b}{c} = \sqrt{2}$. Now from our foregoing analysis we know that: $\frac{BF_1}{AF_1} = \frac{\sqrt{3}(c^2+a^2-b^2)+4\Delta}{\sqrt{3}(b^2+c^2-a^2)+4\Delta} = \frac{c^2+a^2-b^2+ca}{b^2+c^2-a^2+ca} = \frac{a}{c} = \varphi$, $\frac{CF_1}{BF_1} = \frac{a^2+b^2-c^2+ca}{c^2+a^2-b^2+ca} = \frac{a}{c} = \varphi$.

Further from [1], it is a known result that: $AF_1 + BF_1 + CF_1 = \sqrt{\frac{a^2+b^2+c^2+4\sqrt{3}\Delta}{2}} \Rightarrow$ for the present triangle, we have:

$$AF_1+BF_1+CF_1=\sqrt{\frac{a^2+b^2+c^2+3ca}{2}}=\sqrt{a^2+c^2+ca}\Rightarrow\frac{AF_1+BF_1+CF}{AC}=\sqrt{\frac{a^2+c^2+ca}{b^2}}=\sqrt{\frac{1+\frac{a}{c}+\frac{a^2}{c^2}}{\frac{b^2}{c^2}}}$$

$$\Rightarrow\frac{AF_1+BF_1+CF}{AC}=\sqrt{\frac{1+\varphi+\varphi^2}{2}}=\sqrt{\varphi+1}\Rightarrow\frac{AF_1+BF_1+CF}{AC}=\varphi$$

$\Rightarrow\boxed{\frac{BF_1}{AF_1}=\frac{CF_1}{BF_1}=\frac{AF_1+BF_1+CF}{AC}=\varphi}$ Now from the above ratios we get:

$CF_1=\varphi BF_1,\ AF_1=\frac{1}{\varphi}\cdot BF_1\Rightarrow CF_1-AF_1=\left(\varphi-\frac{1}{\varphi}\right)BF_1\ \Rightarrow\boxed{CF_1=AF_1+BF_1}$ [**QED**]

Further, we note that: $\frac{1}{AF_1}=\varphi\cdot\frac{1}{BF_1},\ \frac{1}{CF_1}=\frac{1}{\varphi}\cdot\frac{1}{BF_1}$

$\Rightarrow\frac{1}{AF_1}-\frac{1}{CF_1}=\left(\varphi-\frac{1}{\varphi}\right)\frac{1}{BF_1}\Rightarrow\boxed{\frac{1}{AF_1}=\frac{1}{BF_1}+\frac{1}{CF_1}}$ [**QED**]

<u>Example</u> # 8: Consider a triangle ***ABC*** of which $\angle ABC=60^0,\angle ACB=45^0$ with ***BM*** and **F₁** as '**B**' –*median* and *Fermat – Torricelli* point respectively of the triangle. If ***R*** denotes the circumradius of the triangle, then prove that: $R^2=AF_1\cdot BM$ [**Problem Courtesy**: *Rachid Iksi*] **<u>Proof</u>:-** Let us refer to the above figure and choose the PLAGIOgonal Co – ordinate system) as (**B**, ***BC***, ***BA***) and various co-ordinates are marked in the figure itself. Since $\angle ABC=60^0,\angle ACB=45^0\Rightarrow$ we note: $\frac{a}{\sin 75^0}=\frac{b}{\sin 60^0}=\frac{c}{\sin 45^0}\Rightarrow\frac{a}{\sqrt{3}+1}=\frac{b}{\sqrt{6}}=\frac{c}{2}$. Thus

WLOG we assume: $a=\sqrt{3}+1,\ b=\sqrt{6},\ c=2\Rightarrow 2R=\frac{c}{\sin 45^0}=2\sqrt{2}\Rightarrow R=\sqrt{2}$ Also we

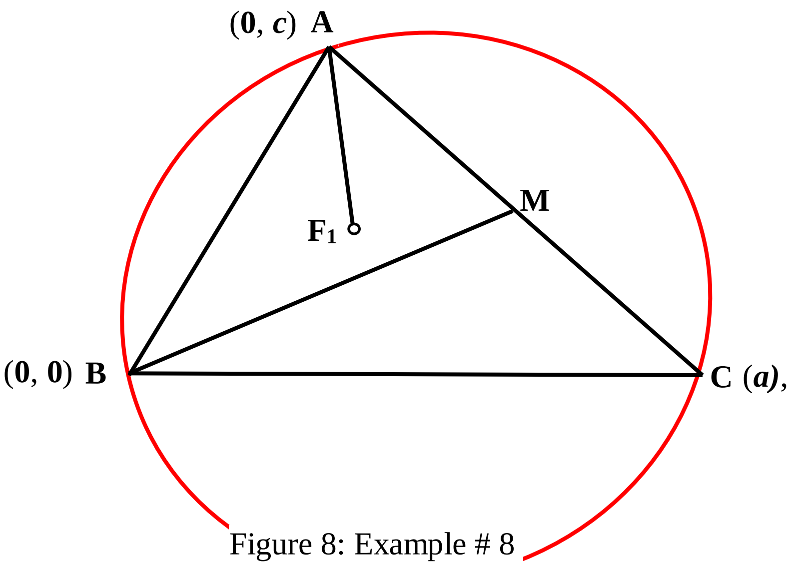


Figure 8: Example # 8

note that: $BM^2 = \frac{2(a^2+c^2)-b^2}{4} \Rightarrow BM = \frac{\sqrt{10+4\sqrt{3}}}{2}$ and also we find from our previous work that:

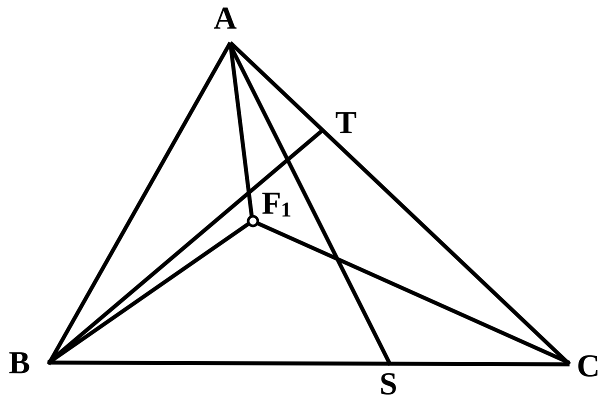
$$AF_1 = \frac{1}{\sqrt{6}} \cdot \frac{\sqrt{3}(b^2+c^2-a^2)+4\Delta}{\sqrt{4\sqrt{3}\Delta + a^2+b^2+c^2}} = \frac{1}{\sqrt{2}} \cdot \frac{b^2+c^2-a^2+ca}{\sqrt{3ca+a^2+b^2+c^2}} = \frac{c^2}{\sqrt{a^2+c^2+ca}} \Rightarrow AF_1 = \frac{4}{\sqrt{10+4\sqrt{3}}}$$

$$\Rightarrow AF_1 \cdot BM = \frac{4}{\sqrt{10+4\sqrt{3}}} \cdot \frac{\sqrt{10+4\sqrt{3}}}{2} = \left(\sqrt{2}\right)^2 \Rightarrow \boxed{R^2 = AF_1 \cdot BM}$$ [**QED**]

Example # 9: Consider a triangle ***ABC*** of which ***AS*** and ***BT*** are two *symmedian*s such that ***AS = BS*** and ***BT = CT***. **F_1** is the *Fermat – Torricelli* point of the triangle. Prove that:

$$\frac{AC}{AF_1+BF_1+CF_1} = \sqrt{1-\frac{\sqrt{3}}{\sqrt{7}}}$$

[**Problem Courtesy**: *Rachid Iksi*]



Proof:- From our earlier exercise (Figure 9: Example # 9 we know that $AS = BS \Rightarrow a = \sqrt{2}b$ and similarly, $BT = CT \Rightarrow b = \sqrt{2}c$. Thus we must have: $a = \sqrt{2}b = 2c \Rightarrow \frac{a}{2} = \frac{b}{\sqrt{2}} = \frac{c}{1}$. Thus, WLOG we assume: $a = 2, b = \sqrt{2}, c = 1$. Hence $4\Delta = \sqrt{7}$. Now we note from our analysis [1]:

$$\frac{AC}{AF_1+BF_1+CF_1} = \frac{\sqrt{2}b}{\sqrt{a^2+b^2+c^2+4\sqrt{3}\Delta}} = \frac{2}{\sqrt{7+\sqrt{21}}} = \frac{\sqrt{7-\sqrt{21}}}{\sqrt{7}}$$

$$\Rightarrow \boxed{\frac{AC}{AF_1+BF_1+CF_1} = \sqrt{1-\frac{\sqrt{3}}{\sqrt{7}}}}$$ [**QED**]

Example # 10: Consider a triangle ***ABC*** of which the largest angle is not more than 120^0 and **G** and $\mathbf{F_1}$ be its *centroid* and the first *Fermat* point respectively. Prove that: $CF_1 \perp GF_1 \Leftrightarrow c^2 = \frac{a^2+b^2}{2}$ [**Problem Courtesy**: *Rachid Iksi*]

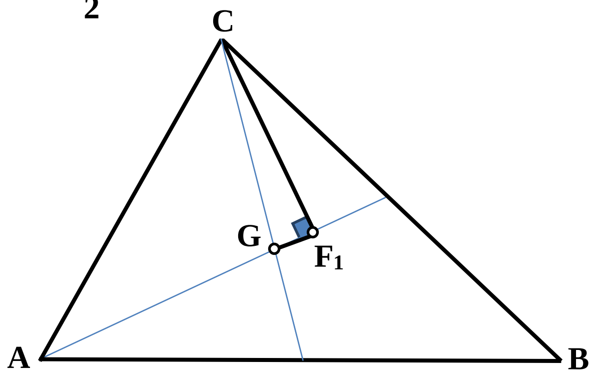


Figure 10: Example # 10

Proof:- From our earlier theorems ([1] and art 4.0 above) we know that:

$$CF_1^2 = \frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(a^2+b^2-c^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta}, CG^2 = \frac{2\left(a^2+b^2\right)-c^2}{9}, GF_1^2 = \frac{a^2+b^2+c^2-4\sqrt{3}\Delta}{18}$$

Let us first assume that: $CF_1 \perp GF_1 \Rightarrow CF_1^2 + GF_1^2 = CG^2$

$$\Rightarrow \frac{1}{6}\cdot\frac{\left[\sqrt{3}\left(a^2+b^2-c^2\right)+4\Delta\right]^2}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta} + \frac{a^2+b^2+c^2-4\sqrt{3}\Delta}{18} = \frac{2\left(a^2+b^2\right)-c^2}{9}$$

$$\Rightarrow 5\left(a^4+b^4+c^4\right)+10a^2b^2-8b^2c^2-8a^2c^2+12\sqrt{3}\Delta\left(a^2+b^2-c^2\right)$$
$$= 2a^4+2b^4-c^4+4a^2b^2+b^2c^2+c^2a^2+4\sqrt{3}\Delta\left(2a^2+2b^2-c^2\right)$$

$$\Rightarrow 3\left(a^4+b^4+2c^4+2a^2b^2-3b^2c^2-3a^2c^2\right)+4\sqrt{3}\Delta\left(a^2+b^2-2c^2\right)=0$$

$$\Rightarrow \sqrt{3}\left(a^2+b^2-c^2\right)\left(a^2+b^2-2c^2\right)+4\Delta\left(a^2+b^2-2c^2\right)=0$$

$$\Rightarrow a^2+b^2-2c^2=0 \Rightarrow \boxed{CF_1 \perp GF_1 \Rightarrow c^2 = \frac{a^2+b^2}{2}}$$

Conversely we now assume that $c^2 = \frac{a^2+b^2}{2}$ and we now consider the expression:

$$BF_1^2 + GF_1^2 - CG^2 = \frac{1}{18}\cdot\frac{9\left(a^2+b^2-c^2\right)^2+\left(a^2+b^2+c^2\right)^2+24\sqrt{3}\Delta\left(a^2+b^2-c^2\right)}{\left(a^2+b^2+c^2\right)+4\sqrt{3}\Delta} - \frac{4\left(a^2+b^2\right)-2c^2}{18}$$

$$\Rightarrow BF_1^2+GF_1^2-CG^2=\frac{c^2}{3}\left[\frac{3c^2+4\sqrt{3}\Delta}{3c^2+4\sqrt{3}\Delta}\right]-\frac{c^2}{3}=0\Rightarrow \boxed{BF_1^2+GF_1^2=CG^2}$$

$$\therefore\ c^2=\frac{a^2+b^2}{2}\Rightarrow BF_1^2+GF_1^2=CG^2\Rightarrow CF_1\perp GF_1\ \therefore\ \boxed{CF_1\perp GF_1\Leftrightarrow c^2=\frac{a^2+b^2}{2}}$$ [QED]

6.0 2nd Napoleon Point, N_2 (X_{18}):

<u>Theorem</u>: Figure 11 shows the second *Napoleon* Point N_2 (X_{18}). In this figure again arbitrary triangle ***ABC*** is shown and the second *Napoleon* Point is defined as the point of concurrency of the three lines: AG_1', BG_2', CG_3' - where G_1', G_2', G_3' represent the *centroid*s of the inner *equilateral* triangles A_1CB, B_1AC and C_1BA respectively. With the PLAGIOgonal system chosen as shown and from our foregoing analysis we can say that the co-ordinates of the above *centroid*s are given as:

$$\left(\frac{\sqrt{3}a\left(\sqrt{3}\sin B-\cos B\right)}{6\sin B},\frac{\sqrt{3}a}{6\sin B}\right),$$

$$\left(\frac{\sqrt{3}\left\{a\left(\sqrt{3}\sin B+\cos B\right)-c\right\}}{6\sin B},\frac{\sqrt{3}\left\{c\left(\sqrt{3}\cos B+\sin B\right)-a\right\}}{6\sin B}\right),\left(\frac{\sqrt{3}c}{6\sin B},\frac{\sqrt{3}c\left(\sqrt{3}\sin B-\cos B\right)}{6\sin B}\right)$$

And consequently, the equations of the lines AG_1', BG_2', CG_3' are:

$$y=c+\frac{\left(a-2\sqrt{3}c\sin B\right)}{a\left(\sqrt{3}\sin B-\cos B\right)}x,\ y=\frac{c\left(\sqrt{3}\sin B+\cos B\right)-a}{a\left(\sqrt{3}\sin B+\cos B\right)-c}x,\ y=\frac{c\left(\sqrt{3}\sin B-\cos B\right)}{\left(c-2\sqrt{3}a\sin B\right)}x-\frac{ca\left(\sqrt{3}\sin B-\cos B\right)}{\left(c-2\sqrt{3}a\sin B\right)}$$

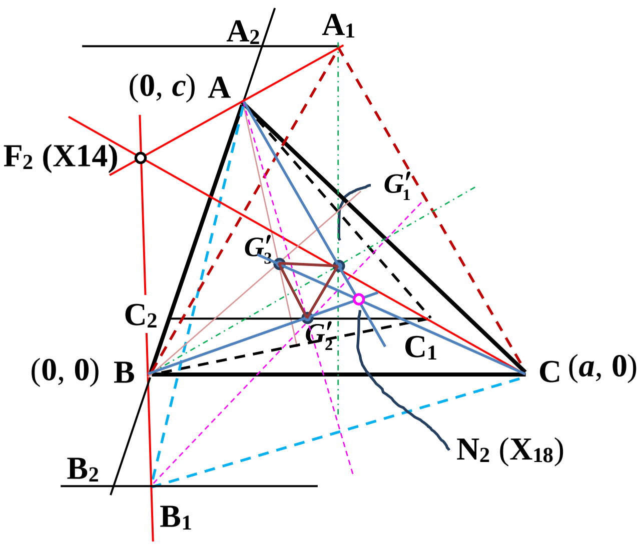


Figure 11: Second Napoleon's point (N_2 or X_{18})

Now solving the first and the last of the above three we get the point of intersection of the lines AG_1' and BG_2' as:

$$\left(\frac{2ca\sin\left(30^0-B\right)\left[2a\sin\left(B+30^0\right)-c\right]}{ca\left[4\sin^2\left(B-30^0\right)-1-12\sin^2 B\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B},\frac{2ca\sin\left(30^0-B\right)\left[2c\sin\left(B+30^0\right)-a\right]}{ca\left[4\sin^2\left(30^0-B\right)-12\sin^2 B-1\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B}\right)$$

and it is easy to see that the above point of intersection lies on the line CG_3' and hence we conclude: AG_1', BG_2', CG_3' are concurrent and thus the required co-ordinates of **N_2** or **X_{18}** are given by:

$$\left(\frac{2ca\sin\left(30^0-B\right)\left[2a\sin\left(B+30^0\right)-c\right]}{ca\left[4\sin^2\left(B-30^0\right)-1-12\sin^2 B\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B},\frac{2ca\sin\left(30^0-B\right)\left[2c\sin\left(B+30^0\right)-a\right]}{ca\left[4\sin^2\left(30^0-B\right)-12\sin^2 B-1\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B}\right)$$

and for reference, we note the co-ordinates of **N_1** or **X_{17}** are [1]:

$$\left(\frac{2ca\sin\left(30^0+B\right)\left[2a\sin\left(30^0-B\right)-c\right]}{ca\left[4\sin^2\left(B-30^0\right)-12\sin^2 B-1\right]-2\sqrt{3}\left(a^2+c^2\right)\sin B},\frac{2ca\sin\left(30^0+B\right)\left[2c\sin\left(30^0-B\right)-a\right]}{ca\left[4\sin^2\left(B-30^0\right)-12\sin^2 B-1\right]-2\sqrt{3}\left(a^2+c^2\right)\sin B}\right)$$

It is easy to observe that: co-ordinates of the *centroid* of $\Delta G_1'G_2'G_3'$ are: $\left(\frac{a}{3},\frac{c}{3}\right)\equiv G$ —— centroidal co-ordinates of the triangle ***ABC***. In other words, *centroid* of $\Delta G_1'G_2'G_3'$ coincide with the centroid of the triangle ***ABC*** —— the property shared by the first Napoleon point also i.e., *centroid*s of ***$\Delta G_1G_2G_3$*** and $\Delta G_1'G_2'G_3'$ are coincident.

Now we note from the triangle $\Delta BG_1'G_3'$ that:

$$\angle G_1'BG_3' = B-60^0 \Rightarrow \left(G_1'G_3'\right)^2 = \frac{1}{3}\left[a^2+c^2-2ca\cos\left(B-60^0\right)\right]$$

$\Rightarrow \left(G_1'G_3'\right)^2 = \frac{1}{3}\left[c^2+a^2-\frac{c^2+a^2-b^2}{2}-2\sqrt{3}\Delta\right] = \frac{1}{6}\left(a^2+b^2+c^2-4\sqrt{3}\Delta\right) =$ constant and is always positive (by *Weitzenbock*'s inequality in triangles) and thus we conclude from the above that: $\Delta G_1'G_2'G_3'$ is an *equilateral* triangle – a similar property of **N_1** (since ***$\Delta G_1G_2G_3$*** is *equilateral* [1]). Further we note that:

$$BN_2^2 = \frac{4c^2a^2\sin^2\left(30^0-B\right)}{\left\{ca\left[4\sin^2\left(30^0-B\right)-12\sin^2 B-1\right]+2\sqrt{3}\left(a^2+c^2\right)\sin B\right\}^2}\left[\begin{array}{l}\left\{2a\sin\left(B+30^0\right)-c\right\}^2+\left\{2c\sin\left(B+30^0\right)-a\right\}^2\\ +2\cos B\left\{2a\sin\left(B+30^0\right)-c\right\}\left\{2c\sin\left(B+30^0\right)-a\right\}\end{array}\right]$$

$$= \frac{4c^2a^2\sin^2\left(30^0-B\right)}{\left[\sqrt{3}\left(a^2+c^2\right)\sin B-2\Delta\left(5\sin B+\sqrt{3}\cos B\right)\right]^2}\left[4\left(c^2+a^2\right)\sin^2 B+4ca\sin^2 B\cos B-4\sqrt{3}ca\sin^3 B\right]$$

$$\Rightarrow BN_2^2 = \frac{16\Delta^2\sin^2\left(30^0-B\right)}{\left[\sqrt{3}\left(a^2+c^2\right)\sin B-2\Delta\left(5\sin B+\sqrt{3}\cos B\right)\right]^2}\left[\frac{3\left(a^2+b^2+c^2\right)-4\sqrt{3}\Delta}{2}-2b^2\right]$$

$\Rightarrow BN_2^2 = \dfrac{4c^2a^2\sin^2\left(30^0 - B\right)}{\left[\sqrt{3}\left(a^2+c^2\right) - ca\left(5\sin B + \sqrt{3}\cos B\right)\right]^2}\left[\dfrac{3\left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta}{2} - 2b^2\right]$ and similarly we can write that:

$$AN_2^2 = \frac{8b^2c^2\sin^2\left(30^0 - A\right)\cdot\left[2\Delta\sin 2A + \left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta\right]}{\left\{bc\left[4\sin^2\left(30^0 - A\right) - 12\sin^2 A - 1\right] + 2\sqrt{3}\left(b^2+c^2\right)\right\}^2} = \frac{4b^2c^2\sin^2\left(30^0 - A\right)}{\left[\sqrt{3}\left(b^2+c^2\right) - bc\left(5\sin A + \sqrt{3}\cos A\right)\right]^2}\left[\frac{3\left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta}{2} - 2a^2\right]$$

and:

$$CN_2^2 = \frac{8a^2b^2\sin^2\left(30^0 - C\right)\cdot\left[2\Delta\sin 2C + \left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta\right]}{\left\{ab\left[4\sin^2\left(30^0 - C\right) - 12\sin^2 C - 1\right] + 2\sqrt{3}\left(a^2+b^2\right)\right\}^2} = \frac{4a^2b^2\sin^2\left(30^0 - C\right)}{\left[\sqrt{3}\left(a^2+b^2\right) - ab\left(5\sin C + \sqrt{3}\cos C\right)\right]^2}\left[\frac{3\left(a^2+b^2+c^2\right) - 4\sqrt{3}\Delta}{2} - 2c^2\right]$$

In the above expressions we note:

$$2bc\sin\left(30^0 - A\right) = bc\left(\cos A - \sqrt{3}\sin A\right) = \frac{b^2+c^2-a^2}{2} - 2\sqrt{3}\Delta = \frac{a^2+b^2+c^2-4\sqrt{3}\Delta}{2} - a^2$$

$$AN_2^2 = \frac{4b^2c^2\sin^2\left(30^0 - A\right)}{\left[\sqrt{3}\left(b^2+c^2\right) - bc\left(5\sin A + \sqrt{3}\cos A\right)\right]^2}\left[b^2+c^2+2bc\sin\left(30^0 - A\right)\right]$$

$$BN_2^2 = \frac{4c^2a^2\sin^2\left(30^0 - B\right)}{\left[\sqrt{3}\left(a^2+c^2\right) - ca\left(5\sin B + \sqrt{3}\cos B\right)\right]^2}\left[c^2+a^2+2ca\sin\left(30^0 - B\right)\right]$$

$$CN_2^2 = \frac{4a^2b^2\sin^2\left(30^0 - C\right)}{\left[\sqrt{3}\left(a^2+b^2\right) - ab\left(5\sin C + \sqrt{3}\cos C\right)\right]^2}\left[a^2+b^2+2ab\sin\left(30^0 - C\right)\right]$$

And for reference we note [1]:

$$AN_1{}^2 = \frac{4b^2c^2\sin^2\left(30^0 + A\right)}{\left[\sqrt{3}\left(b^2+c^2\right) + bc\left(5\sin A - \sqrt{3}\cos A\right)\right]}\left[b^2+c^2+2bc\sin\left(30^0 + A\right)\right]$$

$$BN_1{}^2 = \frac{4c^2a^2\sin^2\left(30^0 + B\right)}{\left[\sqrt{3}\left(c^2+a^2\right) + ca\left(5\sin B - \sqrt{3}\cos B\right)\right]}\left[c^2+a^2+2ca\sin\left(30^0 + B\right)\right]$$

$$CN_1{}^2 = \frac{4a^2b^2\sin^2\left(30^0 + C\right)}{\left[\sqrt{3}\left(a^2+b^2\right) + ab\left(5\sin C - \sqrt{3}\cos C\right)\right]}\left[a^2+b^2+2ab\sin\left(30^0 + C\right)\right]$$

Finally, we note from the above co-ordinates of $\mathbf{N_2}$ that for a triangle ***ABC*** with $\angle ABC = 120^0 \Rightarrow$ the co-ordinates of $\mathbf{N_2}$ or $\mathbf{X_{18}}$ are: $\left(\dfrac{2ca}{3(a-c)}, -\dfrac{2ca}{3(a-c)}\right) \Rightarrow$ $\mathbf{N_2}$ lies on the *external angle bisector* of angle **B**.

7.0 Conclusion:

In the present work an extensive discussion has been attempted focussing on the second *Fermat* point and the second *Napoleon* point of a triangle using the PLAGIOgonal system developed by the first author [2], [3]. This work is considered as the extension of the earlier research work [1] focussing on the First *Fermat* and the First *Napoleon* point only. It is observed that there are lots of similarities between the various theorems and the properties of these pairs of isogonic points $\mathbf{X_{13}}$, $\mathbf{X_{14}}$ and $\mathbf{X_{17}}$ and $\mathbf{X_{18}}$. The discussion is further exemplified by solving various challenging problems available in the social media.

8.0 Acknowledgement:

Both the authors are deeply indebted to Sir Rachid Iksi [4] for using freely his challenging geometry problems which he authored and posted in the social media for the benefit of the mathematical community.

9.0 References:

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[3] PLANE ANALYTICAL GEOMETRY WITH PLAGIOGONAL (OBLIQUE) AXES - Thanasis Gakopoulos and Debabrata Nag (in Press)

[4] Various Facebook posts of Rachid Iksi

A FEW NON-ELEMENTARY GEOMETRICAL PROOFS

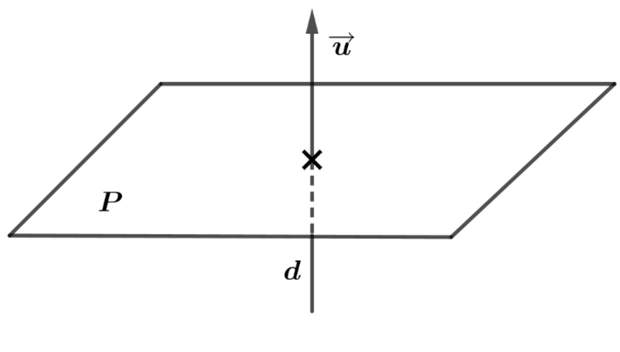
By Daniel Sitaru-Romania

PERPENDICULAR LINE ON THE REAL PLAN

Let be $A(1,1,1)$ and the real plan:

$$P: x + 2y + 3z - 4 = 0$$

Find the equations of the perpendicular line from A to the real plan P.



$\vec{u}(1,2,3), \qquad \vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$

$$d:\frac{x-x_A}{1}=\frac{y-y_A}{2}=\frac{z-z_A}{3},\qquad d:x-1=\frac{y-1}{2}=\frac{z-1}{3}$$

Let's find also the parametrical equations of d:

$$x-1=t;\frac{y-1}{2}=t;\frac{z-1}{3}=t,\qquad d:\begin{cases}x=1+t\\ y=1+2t\,;t\in\mathbb{R}\\ z=1+3t\end{cases}$$

THE DISTANCE FROM A POINT TO A LINE

Let be $A(2,-1,1)$; $B(0,1,3)$; $C(-1,2,2)$. Find the distance from the point A to the line BC.

$$\overrightarrow{AB}=(x_B-x_A)\vec{\imath}+(y_b-y_A)\vec{\jmath}+(z_B-z_A)\vec{k}=$$

$$=(0-2)\vec{\imath}+(1+1)\vec{\jmath}+(3-1)\vec{k}=-2\vec{\imath}+2\vec{\jmath}+2\vec{k}$$

$$\overrightarrow{AC}=(x_C-x_A)\vec{\imath}+(y_C-y_A)\vec{\jmath}+(z_C-z_A)\vec{k}$$

$$=(-1-2)\vec{\imath}+(2+1)\vec{\jmath}+(2-1)\vec{k}=-3\vec{\imath}+3\vec{\jmath}+\vec{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{\imath}&\vec{\jmath}&\vec{k}\\-2&2&2\\-3&3&1\end{vmatrix}=2\vec{\imath}-6\vec{\jmath}-6\vec{k}+6\vec{k}-6\vec{\imath}+2\vec{\jmath}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=-4\vec{\imath}-4\vec{\jmath},\qquad A[ABC]=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$A[ABC]=\frac{1}{2}\sqrt{(-4)^2+(-4)^2}=\frac{\sqrt{32}}{2}=2\sqrt{2}$$

$$BC=\sqrt{(x_C-x_B)^2+(y_c-y_B)^2+(z_C-z_B)^2}=\sqrt{(-1-0)^2+(2-1)^2+(2-3)^2}=\sqrt{3}$$

$$d(A,BC)=\frac{2A[ABC]}{BC}=\frac{2\cdot 2\sqrt{2}}{\sqrt{3}}=\frac{4\sqrt{2}}{\sqrt{3}}=\frac{4\sqrt{6}}{3}$$

THE DISTANCE FROM A POINT TO A REAL PLAN

1. Let be $A(2,1,3)$ and the real plan: $P: 3x+4y+5z-10=0$. Find the distance from A to P.

$$d(A,P)=\frac{|3x_A+4y_A+5z_A-10|}{\sqrt{3^2+4^2+5^2}}$$

$$d(A,P)=\frac{|3\cdot 2+4\cdot 1+5\cdot 3-10|}{\sqrt{9+16+25}}=\frac{|6+4+15-10|}{\sqrt{50}},\qquad d(A,P)=\frac{15}{5\sqrt{2}}=\frac{3}{\sqrt{2}}=\frac{3\sqrt{2}}{2}$$

2. Let be $A(1,1,0); B(0,1,0); C(0,1,1); D(8,8,8)$.

Find the distance from the point D to the real plan (ABC).

$$(ABC):\begin{vmatrix} x & y & z & 1\\ 1 & 1 & 0 & 1\\ 0 & 1 & 0 & 1\\ 0 & 1 & 1 & 1\end{vmatrix}=0,\qquad (ABC):\begin{vmatrix} x & y-1 & z-1 & 0\\ 1 & 0 & -1 & 0\\ 0 & 0 & -1 & 0\\ 0 & 1 & 1 & 1\end{vmatrix}=0$$

$$(ABC):\begin{vmatrix} x & y-1 & z-1\\ 1 & 0 & -1\\ 0 & 0 & -1\end{vmatrix}=0,\qquad (ABC): y-1=0$$

$$d\big(D,(ABC)\big)=\frac{|8-1|}{\sqrt{1^2+0^2+0^2}}=7$$

THE MEDIATOR PLAN OF A SEGMENT

Let be $A(1,5,1); B(2,3,4)$. Find the mediator plan of AB.

Let M be the middle of AB.

$$M\left(\frac{1+2}{2};\frac{5+3}{2};\frac{1+4}{2}\right)\Rightarrow M\left(\frac{3}{2},4,\frac{5}{2}\right),\qquad \overrightarrow{AB}=(x_B-x_A)\vec{\imath}+(y_B-y_A)\vec{\jmath}+(z_B-z_A)\vec{k}$$

$$\overrightarrow{AB}=(2-1)\vec{\imath}+(3-5)\vec{\jmath}+(4-1)\vec{\jmath},\qquad \overrightarrow{AB}=\vec{\imath}-2\vec{\jmath}+3\vec{\jmath}$$

Let P be the mediator plan of AB.

$$P:(x-x_M)\cdot 1+(y-y_M)\cdot(-2)+(z-z_M)\cdot 3=0$$

$$P:\left(x-\frac{3}{2}\right)+(y-4)\cdot(-2)+\left(z-\frac{5}{2}\right)\cdot 3=0$$

$$P:x-\frac{3}{2}-2y+8+3z-\frac{15}{2}=0,\qquad P:x-2y+3z-1=0$$

AREA OF THE TRIANGLE

Let be $A(1,1,0); B(0,1,1); C(2,2,2)$. Find the area of ΔABC.

$$\overrightarrow{AB}=(x_B-x_A)\vec{\imath}+(y_B-y_A)\vec{\jmath}+(z_B-z_A)\vec{k}$$

$$\overrightarrow{AB} = (0-1)\vec{i} + (1-1)\vec{j} + (1-0)\vec{k} = -\vec{i} + \vec{k}$$

$$\overrightarrow{AC} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (2-1)\vec{i} + (2-1)\vec{j} + (2-0)\vec{k} = \vec{i} + \vec{j} + 2\vec{k}$$

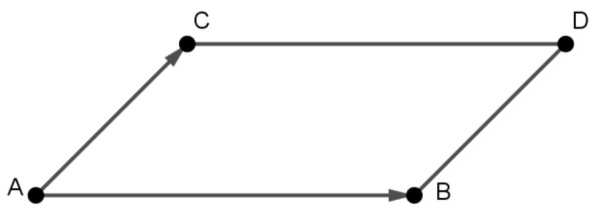
$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{vmatrix} = \vec{j} - \vec{k} - \vec{i} + 2\vec{j}, \qquad \overrightarrow{AB} \times \overrightarrow{AC} = -\vec{i} + 3\vec{j} - \vec{k}$$

$$\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{(-1)^2 + 3^2 + (-1)^2} = \sqrt{11}$$

$$Area\ (\Delta ABC) = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{\sqrt{11}}{2}$$

THE AREA OF THE PARALLELOGRAM

Let be $A(2,0,0)$; $B(0,1,0)$; $C(1,2,2)$. Find the area of the parallelogram builded on $\overrightarrow{AB}$ and $\overrightarrow{AC}$.



$$\overrightarrow{AB} = (x_B - x_A)\vec{j} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}$$

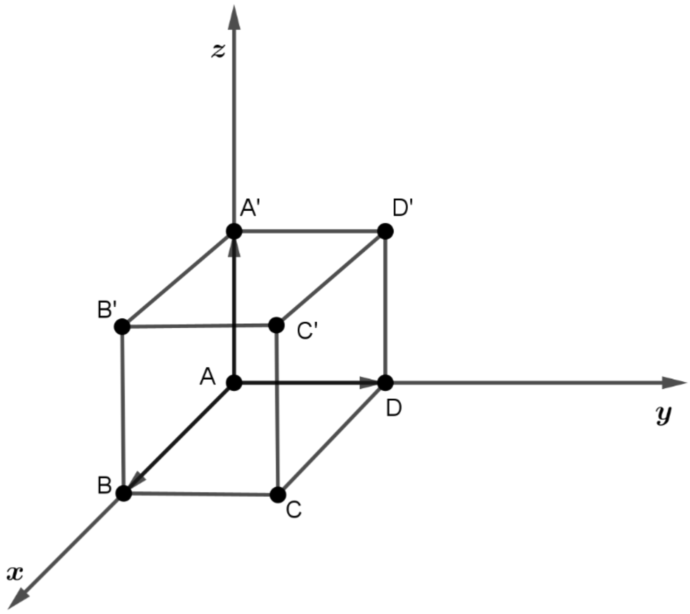
$$\overrightarrow{AB} = (0-2)\vec{i} + (1-0)\vec{j} + (0-0)\vec{k} = -2\vec{i} + \vec{j}, \qquad \overrightarrow{AC} = (x_C - x_A)\vec{i} + (y_C - y_A)\vec{j} + (z_C - z_A)\vec{k}$$

$$\overrightarrow{AC} = (1-2)\vec{i} + (2-0)\vec{j} + (2-0)\vec{k} = -\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 1 & 0 \\ -1 & 2 & 2 \end{vmatrix} = 2\vec{i} - 4\vec{k} + \vec{k} + 4\vec{j} = 2\vec{i} + 4\vec{j} - 3\vec{k}$$

$$\text{Area } (ABCD) = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \sqrt{2^2 + 4^2 + (-3)^2} = \sqrt{29}$$

VOLUME OF THE CUBE



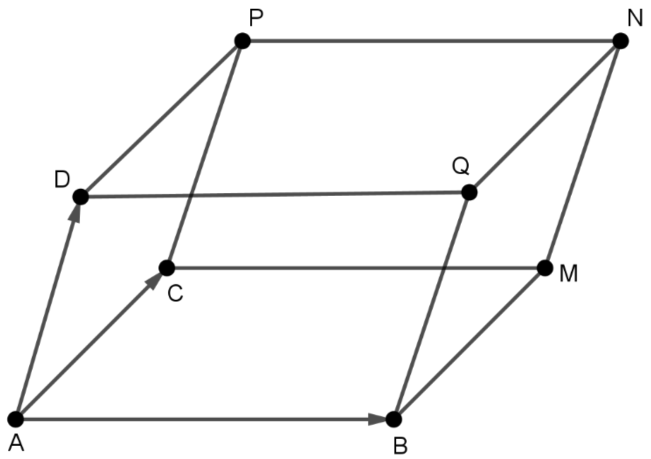
$$B(l,0,0); D(0,l,0); A'(0,0,l), \qquad \overrightarrow{AB}=l\vec{i};\ \overrightarrow{AD}=l\vec{j}; \overrightarrow{AA'}=l\vec{k}$$

$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ l & 0 & 0\\ 0 & l & 0\end{vmatrix}=l^2\vec{k}, \qquad \overrightarrow{AA'}=(0-l)\vec{i}+0\vec{j}+l\vec{k}$$

$$\overrightarrow{AA'}\cdot\left(\overrightarrow{AB}+\overrightarrow{AD}\right)=\left(0\vec{i}+0\vec{j}+l^2\vec{k}\right)\cdot\left(-l\vec{i}+0\vec{j}+l\vec{k}\right)=0\cdot(-l)+0\cdot 0+l^2\cdot l=l^3$$

$$V[ABCDA'B'C'D']=l^3$$

VOLUME OF THE PARALLELIPIPED



Let be $A(1,1,1)$; $B(1,2,3)$; $C(2,3,1)$; $D(4,4,4)$. Find the volume of the parallelepiped builded on the vectors $\overrightarrow{AB}$; $\overrightarrow{AC}$; $\overrightarrow{AD}$.

$$\overrightarrow{AB}=(x_B-x_A)\vec{\imath}+(y_B-y_A)\vec{\jmath}+(z_B-z_A)\vec{k}=(1-1)\vec{\imath}+(2-1)\vec{\jmath}+(3-1)\vec{k}=\vec{\jmath}+2\vec{k}$$

$$\overrightarrow{AC}=(x_C-x_A)\vec{\imath}+(y_C-y_A)\vec{\jmath}+(3-1)\vec{k}=\vec{\jmath}+2\vec{k}$$

$$\overrightarrow{AC}=(x_C-x_A)\vec{\imath}+(y_C-y_A)\vec{\jmath}+(z_C-z_A)\vec{k}=$$

$$=(2-1)\vec{\imath}+(3-1)\vec{\jmath}+(1-1)\vec{k}=\vec{\imath}+2\vec{\jmath}$$

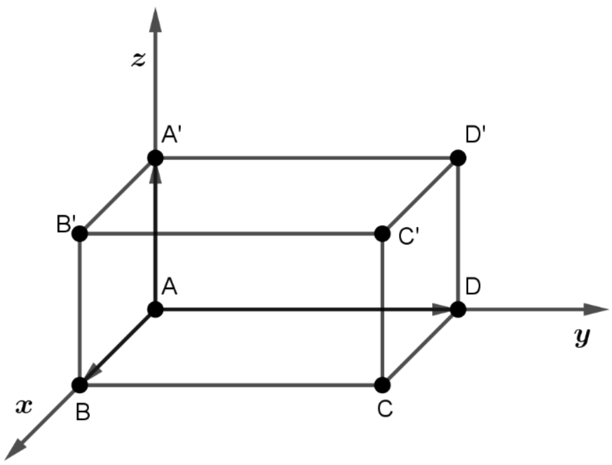
$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ 0 & 1 & 2\\ 1 & 2 & 0\end{vmatrix}=2\vec{\jmath}-\vec{k}-4\vec{\imath}=-4\vec{\imath}+2\vec{\jmath}-\vec{k}$$

$$\overrightarrow{AD}=(x_D-x_A)\vec{\imath}+(y_D-y_A)\vec{\jmath}+(z_D-z_A)\vec{k}=(4-1)\vec{\imath}+(4-1)\vec{\jmath}+(4-1)\vec{k}=3\vec{\imath}+3\vec{\jmath}+3\vec{k}$$

$$\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=-4\cdot 3+2\cdot 3-1\cdot 3=-12+6-3=-9$$

$$V[ABMCDQNP]=\left|\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|=9$$

VOLUME OF THE RIGHT PARALLELIPIPED



$$B(L,0,0);D(0,l,0);A'(0,0,h),\qquad \overrightarrow{AB}=L\vec{\imath};\overrightarrow{AD}=l\vec{\jmath};\overrightarrow{AA'}=h\vec{k}$$

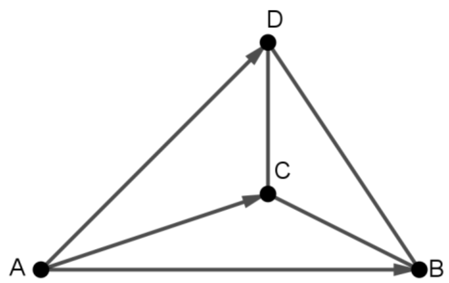
$$\overrightarrow{AB}\times\overrightarrow{AD}=\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ L & 0 & 0\\ 0 & l & 0\end{vmatrix}=Ll\vec{k}$$

$$\overrightarrow{AA'}=(0-L)\vec{\imath}+(0-0)\vec{\jmath}+(h-0)\vec{k}=-L\vec{\imath}+h\vec{k}$$

$$\overrightarrow{AA'}\cdot\left(\overrightarrow{AB}+\overrightarrow{AD}\right)=\left(-L\vec{\imath}+h\vec{k}\right)\left(0\cdot\vec{\imath}+0\cdot\vec{\jmath}+Ll\vec{k}\right)=-L\cdot 0+0\cdot 0+h\cdot L\cdot l=Llh$$

$$V[ABCDA'B'C'D']=Llh$$

VOLUME OF THE TETRAHEDRON



Let be $A(1,1,1); B(1,2,3); C(2,3,1); D(4,4,4)$. Find the volume of the tetrahedron $ABCD$.

$$\overrightarrow{AB}=(x_B-x_A)\vec{\imath}+(y_B-y_A)\vec{\jmath}+(z_B-z_A)\vec{k}=(1-1)\vec{\imath}+(2-1)\vec{\jmath}+(3-1)\vec{k}=\vec{\jmath}+2\vec{k}$$

$$\overrightarrow{AC}=(x_C-x_A)\vec{\imath}+(y_C-y_A)\vec{\jmath}+(z_C-z_A)\vec{k}=(2-1)\vec{\imath}+(3-1)\vec{\jmath}+(1-1)\vec{k}=\vec{\imath}+2\vec{\jmath}$$

$$\overrightarrow{AB}\times\overrightarrow{AC}=\begin{vmatrix}\vec{\imath}&\vec{\jmath}&\vec{k}\\0&1&2\\1&2&0\end{vmatrix}=2\vec{\jmath}-\vec{k}-4\vec{\imath}=-4\vec{\imath}+2\vec{\jmath}-\vec{k}$$

$$\overrightarrow{AD}=(x_D-x_A)\vec{\imath}+(y_D-y_A)\vec{\jmath}+(z_D-z_A)\vec{k}=$$

$$=(4-1)\vec{\imath}+(4-1)\vec{\jmath}+(4-1)\vec{k}=3\vec{\imath}+3\vec{\jmath}+3\vec{k}$$

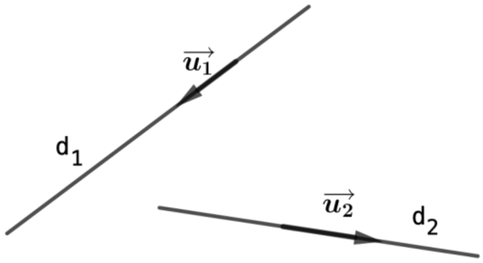
$$\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)=-4\cdot 3+2\cdot 3-1\cdot 3=-12+6-3=-9$$

$$V[ABCD]=\frac{1}{6}\left|\overrightarrow{AD}\cdot\left(\overrightarrow{AB}\times\overrightarrow{AC}\right)\right|=\frac{1}{6}|-9|=\frac{3}{2}$$

THE ANGLE BETWEEN TWO LINES

Let be the lines:

$$\boldsymbol{d_1}:\begin{cases}\boldsymbol{x=2+t}\\\boldsymbol{y=3+2t}\\\boldsymbol{z=1+3t}\end{cases};\ \boldsymbol{d_2}:\begin{cases}\boldsymbol{x=3+t}\\\boldsymbol{y=1+3t}\\\boldsymbol{z=4+2t}\end{cases};\ \boldsymbol{t}\in\mathbb{R}$$



$$d_1: \begin{cases} x-2=t \\ \dfrac{y-3}{2}=t \\ \dfrac{z-1}{3}=t \end{cases}; \quad d_2: \begin{cases} x-3=t \\ \dfrac{y-1}{3}=t \\ \dfrac{z-4}{2}=t \end{cases}; \quad t \in \mathbb{R}$$

$$d_1: \frac{x-2}{1}=\frac{y-3}{2}=\frac{z-1}{3}; d_2: \frac{x-3}{1}=\frac{y-1}{3}=\frac{z-4}{2}$$

$$\overrightarrow{u_1}(1,2,3) = \vec{i} + \overrightarrow{2j} + 3\vec{k}$$

$$\overrightarrow{u_2}(1,3,2) = \vec{i} + 3\vec{j} + 2\vec{k}$$

$$\overrightarrow{u_1} \cdot \overrightarrow{u_2} = 1 \cdot 1 + 2 \cdot 3 + 3 \cdot 2 = 13$$

$$|u_1| = \sqrt{1^2+2^2+3^2} = \sqrt{14}, \qquad |\overrightarrow{u_2}| = \sqrt{1^2+3^2+2^2} = \sqrt{14}$$

$$\cos\left(\sphericalangle(\overrightarrow{u_1}, \overrightarrow{u_2})\right) = \frac{\overrightarrow{u_1} \cdot \overrightarrow{u_2}}{|\overrightarrow{u_1}| \cdot |\overrightarrow{u_2}|} = \frac{13}{\sqrt{14} \cdot \sqrt{14}} = \frac{13}{14}$$

$$\mu\left(\sphericalangle(\overrightarrow{u_1}, \overrightarrow{u_2})\right) = \arccos\left(\frac{13}{14}\right)$$

THE ANGLE BETWEEN TWO REAL PLANS

Let be the real plans:

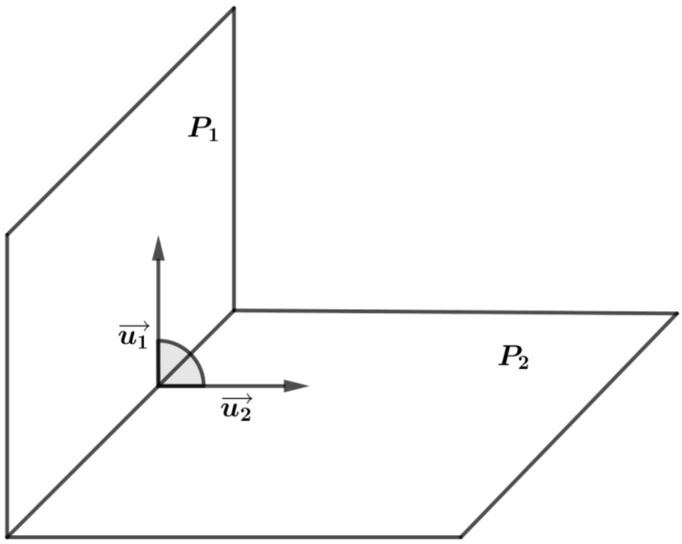
$$\boldsymbol{P_1: x + 2y - 3z + 1 = 0}$$

$$\boldsymbol{P_2: 5x - 3y + 4z - 2 = 0}$$

The normal vectors of P_1, P_2 are:

$$\overrightarrow{u_1}(1,2,-3) = \vec{i} + 2\vec{j} - 3\vec{k}$$

$$\overrightarrow{u_2}(5,-3,4) = 5\vec{i} - 3\vec{j} + 4\vec{k}$$



$$\overrightarrow{u_1}\cdot\overrightarrow{u_2}=1\cdot 5+2\cdot(-3)-3\cdot 4=5-6-12=-13$$

$$|\overrightarrow{u_1}|=\sqrt{1^2+2^2+(-3)^2}=\sqrt{1+4+9}=\sqrt{14}$$

$$|\overrightarrow{u_2}|=\sqrt{5^2+(-3)^2+4^2}=\sqrt{25+9+16}=\sqrt{50}=5\sqrt{2}$$

$$\cos\big(\sphericalangle(P_1,P_2)\big)=\cos\big(\sphericalangle(\overrightarrow{u_1},\overrightarrow{u_2})\big)=\frac{\overrightarrow{u_1}\cdot\overrightarrow{u_2}}{|\overrightarrow{u_1}|\cdot|\overrightarrow{u_2}|}=\frac{-13}{\sqrt{14}\cdot 5\sqrt{2}}=\frac{-13}{10\sqrt{7}}=\frac{-13\sqrt{7}}{70}$$

THE ANGLE BETWEEN A LINE AND A REAL PLAN

Let be the line:

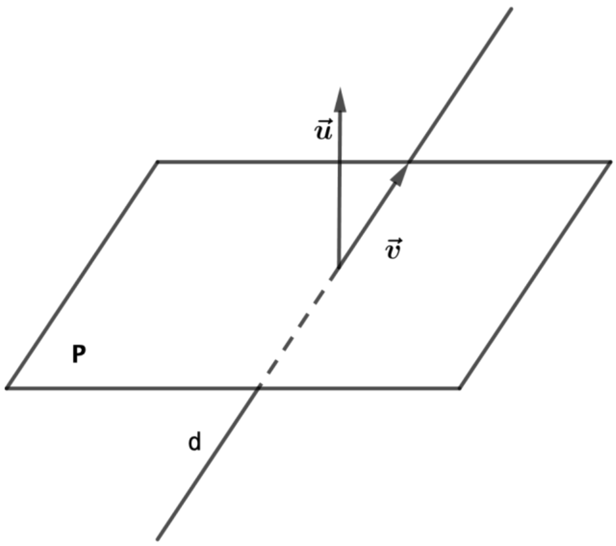
$$\boldsymbol{d}:\begin{cases}\boldsymbol{x=1+t}\\ \boldsymbol{y=2+3t}\\ \boldsymbol{z=-2+5t}\end{cases};\boldsymbol{t}\in\mathbb{R}$$

Let be the real plan: $\boldsymbol{P:2x+3y+z-4=0}$. Find the angle between $\boldsymbol{d}$ and $\boldsymbol{P}$.

$$d:\begin{cases}x-1=t\\ \dfrac{y-2}{3}=t\\ \dfrac{z+2}{5}=t\end{cases};d:\frac{x-1}{1}=\frac{y-2}{3}=\frac{z+2}{5}$$

The line d has the directory vector: $\vec{v}(1,3,5)=\vec{\imath}+3\vec{\jmath}+5\vec{k}$

The normal vector of the real plan P is: $\vec{u}(2,3,1)=2\vec{\imath}+3\vec{\jmath}+\vec{k}$



$$\vec{u}\cdot\vec{v}=1\cdot 2+3\cdot 3+5\cdot 1=16$$

$$|\vec{u}|=\sqrt{2^2+3^2+1^2}=\sqrt{14},\ \ |\vec{v}|=\sqrt{1^2+3^2+5^2}=\sqrt{35}$$

$$\cos(\sphericalangle(\vec{u},\vec{v})\,)=\frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\cdot|\vec{v}|}=\frac{16}{\sqrt{14}\cdot\sqrt{15}}=\frac{16}{\sqrt{210}}=\frac{8\sqrt{210}}{105}$$

SUPRAARITHMETIC MEANS, SUBHARMONIC MEANS

By Dorin Mărghidanu-Romania

In this article, supraarithmetic and subharmonic means are introduced and studied.

Some examples of such means are analyzed. The correlation between these types of means and the mean’s conjugation operator are also highlighted, as well as the refinement relation that is established using these means.

Keywords: classical means, supraarithmetic means, subharmonic means, conjugate means, refinement, Hermite-Hadamard inequality, Jensen’s inequality

Mathematics Subject Classification: 26D15

We recall that a (n-ary) mean in a set $S\subset\mathbb{R}$ is a function $M: S^n\to S$, with the property of internality:

$$\min\{a_1,a_2,\dots,a_n\}\le M(a_1,a_2,\dots,a_n)\le \max\{a_1,a_2,\dots,a_n\}\quad (1)$$

$$(\forall)a_1,a_2,\dots,a_n\in S$$

In general, for the numbers $a_1,a_2,\dots,a_n>0, n\in\mathbb{N}^*$ we will denote define their classical means, as follows, [1]:

$$A_n(a_1, a_2, \dots, a_n) \coloneqq \frac{a_1+a_2+\dots+a_n}{n}, \text{ (arithmetic mean)} \quad (2)$$

$$G_n(a_1, a_2, \dots, a_n) \coloneqq \sqrt[k]{a_1 \cdot a_2 \cdot \dots \cdot a_n}, \text{ (geometric mean)} \quad (3)$$

$$H_n(a_1, a_2, \dots, a_n) \coloneqq \frac{n}{\frac{1}{a_1}+\frac{1}{a_2}+\dots+\frac{1}{a_n}}, \text{ (harmonic mean)} \quad (4)$$

Obviously, there are numerous other means, theoretically – infinitely many. Among them, we will be interested here (and we will introduce them in the following) in two categories of means, namely supraarithmetic and subharmonic means:

1. Definition

Let the numbers $a_1, a_2, \dots, a_n \in S \subset \mathbb{R}, n \in \mathbb{N}^*$.

A mean $\mathcal{M}(a_1, a_2, \dots, a_n)$ is called supraarithmetic mean if,

$$\mathcal{M}(a_1, a_2, \dots, a_n) \geq A_n(a_1, a_2, \dots, a_n). \quad (5)$$

A mean $\mathcal{N}(a_1, a_2, \dots, a_n)$ it is called subharmonic mean if,

$$\mathcal{N}(a_1, a_2, \dots, a_n) \leq H_n(a_1, a_2, \dots, a_n). \quad (6)$$

2. Remark.

When there is no possibility of confusion regarding the numbers $a_1, a_2, \dots, a$ then relations (5) and (6) are written in simplified form: $\mathcal{M} \geq A_n, \mathcal{N} \leq H_n$.

Furthermore, if we also note: $m \coloneqq \min\{a_1, a_2, \dots, a_n\}, M \coloneqq \max\{a_1, a_2, \dots, a_n\}$, then we can express the relations from the previous definition in the language of membership to intervals:

$\mathcal{M}$ is the supraarithmetic mean if, $\mathcal{M} \in [A_n, M]$ (7)

$\mathcal{N}$ is the subharmonic mean if, $\mathcal{N} \in [m, H_n]$ (8)

Characteristic of many types of means are the properties (see e.g:[1],[4],[5]):

- of symmetry: if $M(a,b) = M(b,a), (\forall)a, b \in S$;

- of homogeneity: if $M(ka,kb) = kM(b,a), (\forall)a, b \in S, k \in R_{>0}$.

3. Remember.

If M and N there are two binary means, we say that M and N are inverse (with respect to G) if $M \cdot N = G^2$, and we will say that N is the inverse of M and we will write $N = {}^iM$.

Of course, as well, M is the inverse of N, too an ${}^i({}^iM) = M$.

The notion of inverse mean was introduced by Corrado Gini, in [1]. See also [4], [5], [8].

For example – for classical means, the arithmetic mean and the harmonic mean are inverse means: ${}^iA = H, {}^iH = A$, as can be easily observed. The geometric mean is its own inverse, ${}^iG = G$. In [7], the notation of conjugation was also introduced.

4. Definition.

For the mean $\mathcal{M}(a_1, a_2, \dots, a_n)$ of numbers $a_1, a_2, \dots., a_n \in \mathbb{R}_{>0}$ we will call its conjugate, the expression denoted and defined as follows:

$$^{c}\mathcal{M}(a_1, a_2, \dots, a_n) \coloneqq 1/M\,[1/a_1\,, 1/a_2\,, \dots, 1/a_n].$$

The expressions $\mathcal{M}(a_1, a_2, \dots, a_n)$ and $^{c}\mathcal{M}(a_1, a_2, \dots, a_n)$ will be called conjugate expressions.

Also, the conjugation operator is involutive: $^{c}(^{c}M) = M$, as can be easily observed.

As well as inverse means, the arithmetic mean and the harmonic mean are conjugate means: $^{c}A = H, {}^{c}H = A$, as can be easily observed. The geometric mean is its own.

There are situations when the inversion and conjugation operators are different, see for example in [4].

For what follows, the following is very important.

5. Lemma.

For the numbers $a_1, a_2, \dots, a_n \in \mathbb{R}_{>0}$, the following identities holds:

(a) $\frac{1}{\max\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)} = \min(a_1, a_2, \dots, a_n)\,;$ (9)

(b) $\frac{1}{\min\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)} = \max(a_1, a_2, \dots, a_n)$ (10)

Proof.

(a) We will prove by induction. For $n = 2$, we have to prove that:

$$\frac{1}{\max\left(\frac{1}{a_1}, \frac{1}{a_2}\right)} = \min(a_1, a_2). \qquad (9')$$

Considering $a_1 \le a_2$, the relation (9') becomes, $\frac{1}{\frac{1}{a_1}} = a_1$, obviously true.

We assume that $\frac{1}{\max\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_{n-1}}\right)} = \min(a_1, a_2, \dots, a_{n-1})$; and we will demonstrate that

$\frac{1}{\max\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)} = \min(a_1, a_2, \dots, a_n)$. Indeed, we have:

$$\frac{1}{\max\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right)} = \frac{1}{\max\left(\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_{n-1}}\right), \frac{1}{a_n}\right)} =$$

$$= \min\big((a_1, a_2, \dots, a_{n-1}), a_n\big) = \min(a_1, a_2, \dots, a_n).$$

(b) In a similar way

With the following result, we have a characterization of supraarithmetic and subharmonic means.

6. Theorem.

If $a_1, a_2, \dots, a_n \in \mathbb{R}_{>0}$, then occurs the statement:

$\mathcal{M}(a_1, a_2, \dots, a_n)$ is a supraarithmetic mean if and only if ${}^c\mathcal{M}(a_1, a_2, \dots, a_n)$is the subharmonic mean.

Proof.

$\mathcal{M}(a_1, a_2, \dots, a_n)$ is a supraarithmetic mean if and only if

$\mathcal{M}(a_1, a_2, \dots, a_n) \geq A_n(a_1, a_2, \dots, a_n), (\forall)a_1, a_2, \dots, a_n \in \mathbb{R}_{>0}$. In particular, making the substitutions: $a_1 \to \frac{1}{a_1}, a_2 \to \frac{1}{a_2}, \dots, a_n \to \frac{1}{a_n}$, we obtain:

$$M\left[\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right] \geq A_n\left(\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}\right) \Leftrightarrow$$

$$\Leftrightarrow 1/M\,[1/a_1\,, 1/a_2\,, \dots, 1/a_n] \leq 1/A_n\,(1/a_1, 1/a_2\,, \dots, 1/a_n) \Leftrightarrow$$

$$\Leftrightarrow {}^cM[a_1, a_2, \dots, a_n] \leq H_n(a_1, a_2, \dots, a_n).$$

Definition 3 was used, as well as the well – known identity:

$$1/A_n\,(1/a_1\,, 1/a_2\,, \dots, 1/a_n) = H_n(a_1, a_2, \dots, a_n).$$

Several supraarithmetic and subharmonic means have been proposed in [3].

7. Proposition, [3]

For any real numbers $a, b > 0$, the following inequalities hold:

$$\frac{a+b}{2} \overset{(1)}{\leq} \sqrt{\frac{a^2+ab+b^2}{3}} \overset{(2)}{\leq} \sqrt{\frac{a^2+b^2}{2}} \overset{(3)}{\leq}$$

(a) $\overset{(3)}{\leq} \frac{a^2+b^2}{a+b} \overset{(4)}{\leq} \sqrt{a^2-ab+b^2} \overset{(5)}{\leq} \max\{a, b\}$ (11)

(inequalities for superarithmetic means)

$$\frac{2}{\frac{1}{a}+\frac{1}{b}} \overset{(1)}{\geq} \sqrt{\frac{3}{\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}}} \overset{(2)}{\geq} \sqrt{\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}} \overset{(3)}{\geq}$$

(b) $\overset{(3)}{\geq} \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}} \overset{(4)}{\geq} \frac{1}{\sqrt{\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}}} \overset{(5)}{\geq} \min\{a, b\}$

(inequalities for subharmonic means)

Proof.

(a) After routine calculations, the inequalities (1), (2), (3), (4) reduce to the obvious inequality, $a^2 + b^2 \geq 2ab$.

For inequality (5) (due to symmetry in a, b) we assume $a \leq b$ and then the inequality becomes,

$$\sqrt{a^2-ab+b^2} \leq b \Leftrightarrow a^2-ab+b^2 \leq b^2 \Leftrightarrow a^2 \leq ab \Leftrightarrow a \leq b.$$

In all inequalities, the equality relation occurs when $a = b$.

(b) The inequalities in (b) are obtained from (a), based on Theorem 6.

8. Remark.

In Proposition 7, besides the classical binary means:

$$A_2(a,b) := \frac{a+b}{2}, \quad \text{(arithmetic mean);} \quad (13)$$

$$H_2(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad \text{(harmonic mean);} \quad (14)$$

$$Q_2(a,b) := \sqrt{\frac{a^2+b^2}{2}}, \quad \text{(quadratic mean),} \quad (15)$$

other means also appear, which we will name and note in a sui=generis way:

$${}^cQ_2(a,b) := \sqrt{\frac{2}{\frac{1}{a^2}+\frac{1}{b^2}}}, \quad \text{(c-quadratic mean),} \quad (16)$$

$$C_2(a,b) := \frac{a^2+b^2}{a+b}, \quad \text{(contraharmonic mean),} \quad (17)$$

$${}^cC_2(a,b) := \frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}+\frac{1}{b^2}}, \quad \text{(c-contraharmonic mean),} \quad (18)$$

$$P_2(a,b) := \sqrt{\frac{a^2+ab+b^2}{3}}, \quad \text{(extended power mean),} \quad (19)$$

$${}^cP_2(a,b) := \sqrt{\frac{3}{\frac{1}{a^2}+\frac{1}{ab}+\frac{1}{b^2}}}, \quad \text{(c-extended power mean)} \quad (20)$$

$$R_2(a,b) := \sqrt{a^2-ab+b^2}, \quad \text{(radical mean),} \quad (21)$$

$${}^cR_2(a,b) := \frac{1}{\sqrt{\frac{1}{a^2}-\frac{1}{ab}+\frac{1}{b^2}}}, \quad \text{(c-radical mean)} \quad (22)$$

All the above means are symmetric and homogeneous. In addition, in each of the pairs:

$$\big(A_2(a,b), H_2(a,b)\big), (Q_2(a,b), {}^cQ_2(a,b)), \big(C_2(a,b), {}^cC_2(a,b)\big),$$

$(P_2(a,b), {}^cP_2(a,b)), (R_2(a,b), {}^cR_2(a,b)), (\max(a,b), \min(a,b))$, the first component is the supraarithmetic mean, and the second component is the subharmonic mean.

In [6], the following multiple inequality was proposed:

If $a, b > 0$, then:

$$\frac{a+b}{2} \le \ln\frac{e^a-e^b}{b-a} \le \ln\frac{e^a+e^b}{2} \le \frac{e^a+e^b-2}{2} \quad (23)$$

As we have, $\frac{e^a+e^b-2}{2} \ge \max\{a,b\}$, (24)

we will retain from (23) only the first two inequalities:

9. Proposition.

If $a, b > 0$, then we have the following inequalities:

(a) $\frac{a+b}{2} \overset{(1)}{\leq} \ln\frac{e^b-e^a}{b-a} \overset{(2)}{\leq} \ln\frac{e^a+e^b}{2} \overset{(3)}{\leq} \max\{a,b\}$; (25)

(inequality for supraarithmetic means)

(b) $\frac{2}{\frac{1}{a}+\frac{1}{b}} \overset{(1)}{\geq} \frac{1}{\ln\frac{ab\left(e^{\frac{1}{a}}-e^{\frac{1}{b}}\right)}{b-a}} \overset{(2)}{\geq} \frac{1}{\ln\frac{e^{\frac{1}{a}}+e^{\frac{1}{b}}}{2}} \overset{(3)}{\geq} \min\{a,b\}$. (26)

(inequalities for subharmonic means)

Proof.

(a) Applying Hermite – Hadamard inequality for convex functions,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\cdot\int_a^b f(x)\cdot dx \leq \frac{f(a)+f(b)}{2} \qquad (27)$$

for the function $f(x) = e^x$, we get:

$$e^{\frac{a+b}{2}} \leq \frac{e^b-e^a}{b-a} \leq \frac{e^a+e^b}{2}.$$

By applying the logarithm, the inequalities (1), (2) from the statement are obtained. For inequality (3), for $a \leq b$, we have:

$$\ln\frac{e^a+e^b}{2} \leq \ln\frac{2e^b}{2} \leq \ln e^b = b = \max\{a,b\}.$$

In all inequalities, the equality relation occurs when $a = b$.

(b) The inequalities in (b) are obtained from (a), based on Theorem 6.

10. Remark.

And here let us notice that in Proposition 9, besides the classical binary means: the arithmetic mean and the harmonic mean, new binary means also appear, which we will name and note as follows:

$$Lde_2(a,b) := \ln\frac{e^b-e^a}{b-a}, \text{ (log.diff.exp. mean)}, \quad (28)$$

$$^cLde_2(a,b) := \frac{1}{\ln\frac{ab\left(e^{\frac{1}{a}}-e^{\frac{1}{b}}\right)}{b-a}}, \text{ (c – log.diff.exp.mean)}, \quad (29)$$

$$Lse_2(a,b) := \ln\frac{e^a+e^b}{2}, \text{ (log.sum exp. mean)}, \quad (30)$$

$$^cLse_2(a,b) := \frac{1}{\ln\frac{e^{\frac{1}{a}}+e^{\frac{1}{b}}}{2}}, \text{ (c-log.sum exp. mean)}. \quad (31)$$

The means (28) – (31) are symmetric and homogeneous. In addition, in each of the pairs:

$\left(A_2(a,b), H_2(a,b)\right), \left(Lde_2(a,b), {}^cLde_2(a,b)\right), \left(Lse_2(a,b), {}^cLse_2(a,b)\right),$

$(\max(a,b), \min(a,b))$, the first component is the supraarithmetic mean, and the second component is the subharmonic mean.

In [5], [6], [7] – among others the following two inequalities were presented:

$$\min(a,b) \le \left(a^b \cdot b^a\right)^{\frac{1}{a+b}} \le \frac{a+b}{2} \le \left(a^a \cdot b^b\right)^{\frac{1}{a+b}} \le \max(a,b). \quad (32)$$

$$\min(a,b,c) \le \left(a^{bc} \cdot b^{ca} \cdot c^{ab}\right)^{\frac{1}{ab+bc+ca}} \le \frac{a+b+c}{3} \le \left(a^a \cdot b^b \cdot c^c\right)^{\frac{1}{a+b+c}} \le \max(a,b,c) \quad (33)$$

In these inequalities we observe binary and trinary means which are supraarithmetical means. We will try to generalize these means to the n – ary case.

11. Definition.

For the numbers $a_1, a_2, \dots, a_n \in \mathbb{R}_{>0}$, we will consider the following two expressions:

$$E_n(a_1, a_2, \dots, a_n) := \left(\prod_{k=1}^n a_k^{a_k}\right)^{\frac{1}{\sum_{k=1}^n a_k}} \quad (34)$$

$$F_n(a_1, a_2, \dots, a_n) := \left(\prod_{k=1}^n a_k^{\prod_{i=1, i\neq k}^n a_k}\right)^{\frac{1}{\sum_{k=1}^n \left(\prod_{i=1, i\neq k}^n a_i\right)}} \quad (35)$$

12. Proposition.

For any numbers $a_1, a_2, \dots, a_n \in \mathbb{R}_{>0}$, the statement hold:

(a) The expression $E_n(a_{1,2}, \dots, a_n)$ is supraarithmetic mean.

(b) The expression $F_n(a_1, a_2, \dots, a_n)$ is subharmonic mean.

Proof.

The expression $E_n(a_1, a_2, \dots, a_n)$ is supraarithmetic mean of numbers $a_1, a_2, \dots, a_n$ if we have, according to Remark 2:

$$A_n(a_1, a_2, \dots, a_n) \overset{(1)}{\le} E_n(a_1, a_2, \dots, a_n) \overset{(2)}{\le} \max\{a_1, a_2, \dots, a_n\}. \quad (36)$$

For the inequality (1), we successively have

$$A_n(a_1, a_2, \dots, a_n) \le E_n(a_1, a_2, \dots, a_n) \Leftrightarrow \ln A_n(a_1, a_2, \dots, a_n) \le \ln E_n(a_1, a_2, \dots, a_n) \Leftrightarrow$$

$$\Leftrightarrow \ln A_n(a_1, a_2, \dots, a_n) \le \frac{1}{\sum_{k=1}^n a_k} \cdot \sum_{k=1}^n a_k \cdot \ln a_k \Leftrightarrow \left(\sum_{k=1}^n a_k\right) \cdot \ln A_n(a_1, a_2, \dots, a_n) \le$$

$$\le \sum_{k=1}^n a_k \cdot \ln a_k \Leftrightarrow$$

$$\Leftrightarrow n \cdot A_n(a_1, a_2, \dots, a_n) \cdot \ln A_n(a_1, a_2, \dots, a_n) \le \sum_{k=1}^n a_k \cdot \ln a_k \Leftrightarrow$$

$$\Leftrightarrow A_n(a_1,a_2,\dots,a_n)\cdot\ln A_n(a_1,a_2,\dots,a_n)\le\frac{1}{n}\cdot\sum_{k=1}^{n}a_k\cdot\ln a_k \quad (*)$$

Using Jensen's inequality,

$$f\left(\frac{a_1+a_2+\cdots+a_n}{n}\right)\le\frac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n}\Leftrightarrow$$

$$\Leftrightarrow f\big(A_n(a_1,a_2,\dots,a_n)\big)\le\frac{f(a_1)+f(a_2)+\cdots+f(a_n)}{n},\quad (37)$$

for the convex function $f:(0,\infty)\to\mathbb{R}, f(x)=x\cdot\ln x$, we obtain:

$$A_n(a_1,a_2,\dots,a_n)\cdot\ln\big(A_n(a_1,a_2,\dots,a_n)\big)\le\frac{a_1\cdot\ln a_1+a_2\cdot\ln a_2+\cdots+a_n\cdot\ln a_n}{n}.$$

what is the inequality (*) above.

For the inequality (2), WLOG $a_1\le a_2\le\cdots\le a_n$ and we will have:

$$E_n(a_1,a_2,\dots,a_n):=\left(\prod_{k=1}^{n}a_k^{a_k}\right)^{\frac{1}{\sum_{k=1}^{n}a_k}}\le\left(\prod_{k=1}^{n}a_n^{a_k}\right)^{\frac{1}{\sum_{k=1}^{n}a_k}}=$$

$$=\left(a_n^{\sum_{k=1}^{n}a_k}\right)^{\frac{1}{\sum_{k=1}^{n}a_k}}=a_n=\max\{a_1,a_2,\dots,a_n\}.$$

(b) We will demonstrate that $F_n(a_1,a_2,\dots,a_n)$ is the conjugate of the mean $E_n(a_1,a_2,\dots,a_n)$. Indeed, by Definition 4, we have:

$${}^{c}E_n(a_1,a_2,\dots,a_n)\overset{Def.4}{=}\frac{1}{E_n\left(\frac{1}{a_1},\frac{1}{a_2},\dots,\frac{1}{a_n}\right)}=\frac{1}{\left(\prod_{k=1}^{n}\left(\frac{1}{a_k}\right)^{\frac{1}{a_k}}\right)^{\frac{1}{\sum_{k=1}^{n}\frac{1}{a_k}}}}=$$

$$=\frac{1}{\frac{1}{\left(\prod_{k=1}^{n}a_k^{\frac{1}{a_k}}\right)^{\frac{\prod_{k=1}^{n}a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\ne k}^{n}a_i\right)}}}}=\left(\prod_{k}^{n}a_k^{\frac{1}{a_k}}\right)^{\frac{\prod_{k=1}^{n}a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\ne k}^{n}a_i\right)}}=\prod_{k}^{n}a_k^{\frac{1}{ak}\frac{\prod_{k=1}^{n}a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\ne k}^{n}a_i\right)}}=$$

$$=\prod_{k=1}^{n}a_k^{\frac{\prod_{i=1,i\ne k}^{n}a_k}{\sum_{k=1}^{n}\left(\prod_{i=1,i\ne k}^{n}a_i\right)}}=\left(\prod_{k=1}^{n}a_k^{\prod_{i=1,i\ne k}^{n}a_i}\right)^{\frac{1}{\sum_{k=1}^{n}\left(\prod_{i=1,i\ne k}^{n}a_i\right)}}=F_n(a_1,a_2,\dots,a_n).$$

The rest follows from Theorem 6.

Regarding the supraarithmetic and subharmonic means presented in this material it would be interesting to compare them and then establish an ordering of them. We will deal with this in a future paper.

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SELECTED PROBLEMS IN GEOMETRY AND INEQUALITIES

By Andrei Ștefan Mihalcea – Romania

Abstract:

This note presents a short collection of original problems in geometry and inequalities. The solutions use elementary but effective tools such as orthogonal coordinates, barycentric representations, the triangle inequality, Schur's inequality, Cauchy's inequality, and Jensen's inequality. The aim is to give clean statements and rigorous proofs suitable for olympiad-style mathematics.

1. A Geometry problem in an Orthodiagonal Quadrilateral

Problem 1: Let $ABCD$ be an othodiagonal quadrilateral and let

$$AC \cap BD = \{O\}$$

Assume that

$$M \in Int(\Delta AOB), \qquad N \in Int(\Delta BOC), \qquad P \in Int(\Delta COD), \qquad Q \in Int(\Delta DOA).$$

Let

$$MM_1 \perp AO, \quad MM_2 \perp BO, NN_1 \perp BO, \quad NN_2 \perp CO, \quad PP_1 \perp CO, \quad PP_2 \perp DO,$$
$$QQ_1 \perp DO, \quad QQ_2 \perp AO.$$

Prove that: $\left|\overrightarrow{MN} + \overrightarrow{PQ}\right| \le M_1Q_2 + N_1M_2 + P_1N_2 + Q_1P_2$

Proof:

Since $ABCD$ is orthodiagonal, the diagonals AC and BD are perpendicular. We take them as coordinate axes, with origin O. Denote by $\vec{r}_X$ the position vector of a point X.

For a point lying inside one of the four triangles determined by the two axes, its position vector can be expressed in terms of its perpendicular distances to the two axes. Thus, for $M \in \Delta AOB$, we have

$$\vec{r}_M = \frac{MM_2}{OA}\vec{r}_A + \frac{MM_1}{OB}\vec{r}_B.$$

Similarly,

$$\vec{r}_N = \frac{NN_2}{OB}\vec{r}_B + \frac{NN_1}{OC}\vec{r}_C, \vec{r}_P = \frac{PP_2}{OC}\vec{r}_C + \frac{PP_1}{OD}\vec{r}_D,$$

and

$$\vec{r}_Q = \frac{QQ_1}{OA}\vec{r}_A + \frac{QQ_2}{OD}\vec{r}_D.$$

Therefore

$$\overrightarrow{MN} + \overrightarrow{PQ} = \vec{r}_N - \vec{r}_M + \vec{r}_Q - \vec{r}_P.$$

Using the expressions above, we get

$$\overrightarrow{MN} + \overrightarrow{PQ} = \frac{QQ_1 - MM_2}{OA}\vec{r}_A + \frac{NN_2 - MM_1}{OB}\vec{r}_B + \frac{NN_1 - PP_2}{OC}\vec{r}_C + \frac{QQ_2 - PP_1}{OD}\vec{r}_D.$$

Since

$$\left|\frac{\vec{r}_A}{OA}\right| = \left|\frac{\vec{r}_B}{OB}\right| = \left|\frac{\vec{r}_C}{OC}\right| = \left|\frac{\vec{r}_D}{OD}\right| = 1,$$

the tringle inequality gives

$$\left|\overrightarrow{MN} + \overrightarrow{PQ}\right| \le |QQ_1 - MM_2| + |NN_2 - MM_1| + |NN_1 - PP_2| + |QQ_2 - PP_1|.$$

The four absolute values are precisely the distances between the corresponding orthogonal projections:

$$|QQ_1 - MM_2| = M_1Q_2, \quad |NN_2 - MM_1| = N_1M_2, \quad |NN_1 - PP_2| = P_1N_2,$$

and

$$|QQ_2 - PP_1| = Q_1P_2.$$

Hence

$$\left|\overrightarrow{MN}+\overrightarrow{PQ}\right| \le M_1Q_2+N_1M_2+P_1N_2+Q_1P_2.$$

2. A Cyclic Inequality

Problem 2: Let $a, b, c > 0$ with

$$a+b+c=3.$$

Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge \frac{5b+2c-a}{9-3a}+\frac{5c+2a-b}{9-3b}+\frac{5a+2b-c}{9-3c}.$$

Proof: Since $a+b+c=3$, we have

$$9-3a=3(b+c), \qquad 9-3b=3(c+a), \qquad 9-3c=3(a+b).$$

Thus the right – hand side becomes

$$\sum_{cyc}\frac{5b+2c-a}{3(b+c)}.$$

A direct simplification gives

$$\sum_{cyc}\frac{5b+2c-a}{3(b+c)}=2\sum_{cyc}\frac{1}{a+b}+\sum_{cyc}\frac{b-a}{b+c}.$$

By the AM-HM inequality,

$$\frac{1}{a}+\frac{1}{b} \ge \frac{4}{a+b}.$$

Adding the three analogous inequalities, we obtain

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \ge 2\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right).$$

Therefore it remains to prove that

$$\sum_{cyc}\frac{b-a}{b+c} \le 0.$$

Multiplying by the positive number $(a+b)(b+c)(c+a)$, this is equivalent to

$$a^3+b^3+c^3 \ge a^2b+b^2c+c^2a.$$

This follows from Schur's inequality:

$$a^3+b^3+c^3+3abc \ge a^2b+a^2c+b^2a+b^2c+c^2a+c^2b$$

Hence: $a^3+b^3+c^3-(a^2b+b^2c+c^2a) \ge a^2c+b^2a+c^2b-3abc$.

By AM-GM: $a^2c+b^2a+c^2b \ge 3abc$. Thus: $a^3+b^3+c^3 \ge a^2b+b^2c+c^2a$.

Consequently,

$$\sum_{cyc}\frac{b-a}{b+c}\leq 0,$$

and the desired inequality follows. Equality holds for $a=b=c=1$.

3. A Bound Involving a Parameter

Problem 3: Let $t>1$ and let $x_1,\dots,x_n$ be real numbers such that $\frac{1}{t}\leq x_1\leq x_2\leq\cdots\leq x_n\leq t$

and

$$\sum_{k=1}^{n}x_k=1.$$

Prove that: $t\geq n$ and

$$t+\frac{1}{t}\geq n+\sum_{k=1}^{n}x_k^2.$$

Proof: Since $x_k\geq\frac{1}{t}$ for every k, summing over $k=1,\dots,n$ gives $1=\sum_{k=1}^{n}x_k\geq\frac{n}{t}$.

Therefore $t\geq n$. Now, for each $k\in\{1,\dots,n\}$, we have $\frac{1}{t}\leq x_k\leq t$.

Hence: $(t-x_k)\left(x_k-\frac{1}{t}\right)\geq 0$. Expanding, we get $tx_k-x_k^2-1+\frac{x_k}{t}\geq 0$,

or equivalently: $x_k\left(t+\frac{1}{t}\right)\geq x_k^2+1$. Summing from $k=1$ to n, we obtain

$$\left(t+\frac{1}{t}\right)\sum_{k=1}^{n}x_k\geq\sum_{k=1}^{n}x_k^2+n.$$

Since

$$\sum_{k=1}^{n}x_k=1,$$

it follows that

$$t+\frac{1}{t}\geq n+\sum_{k=1}^{n}x_k^2.$$

4. A Four – Variable Quadratic Inequality

Problem 4: Let $a,b,c,d\in\mathbb{R}$ with

$$a+b+c+d=0.$$

Prove that: $5(a^2+b^2+c^2+d^2)\geq 4(a+c)(b+d)+4(bd-ac)$.

Proof: Since $a+b+c+d=0$, we may write $d=-a-b-c$.

The desired inequality is equivalent to

$$5(a^2+b^2+c^2+d^2)-4(a+c)(b+d)-4(bd-ac)\geq 0.$$

Substituting $d=-a-b-c$, the left-hand side becomes

$$14a^2+14b^2+14c^2+14ab+14bc+22ac.$$

Let $u=a+c,\ v=a-c$. Then $a=\frac{u+v}{2}$ and $c=\frac{u-v}{2}$. Therefore

$$14a^2+14b^2+14c^2+14ab+14bc+22a=\frac{1}{2}(28b^2+28bu+25u^2+3v^2).$$

We rewrite this as

$$\frac{1}{2}\left(28\left(b+\frac{u}{2}\right)^2+18u^2+3v^2\right).$$

This quantity is nonnegative. Hence

$$5(a^2+b^2+c^2+d^2)\geq 4(a+c)(b+d)+4(bd-ac),$$

as required.

5. A Jensen – Type Inequality

Problem 5: Let $f:I\to\mathbb{R}$ be a convex function, and let $0\leq a,b,c\in I,\ a+b+c=1$.

Assume also that $ab+bc+ca\in I$ and $\frac{1-ab-bc-ca}{2}\in I$. Prove that:

$$f(a)+f(b)+f(c)\geq 2f\left(\frac{1-ab-bc-ca}{2}\right)+f(ab+bc+ca).$$

Proof: Since $a,b,c\geq 0$ and $a+b+c=1$, the numbers $1+a,\ 1+b,\ 1+c$

are positive and have sum 4. By Jensen's inequality,

$$(1+a)f(a)+(1+b)f(b)+(1+c)f(c)\geq 4f\left(\frac{(1+a)a+(1+b)b+(1+c)c}{4}\right).$$

Now

$$(1+a)a+(1+b)b+(1+c)c=a+b+c+a^2+b^2+c^2=1+a^2+b^2+c^2.$$

Since

$$(a+b+c)^2=1,$$

we have: $a^2+b^2+c^2+2(ab+bc+ca)=1$.

Thus

$$\frac{1+a^2+b^2+c^2}{4}=\frac{1-ab-bc-ca}{2}.$$

Therefore

$$(1+a)f(a)+(1+b)f(b)+(1+c)f(c)\geq 4f\left(\frac{1-ab-bc-ca}{2}\right).$$

Similarly, the numbers

$$1-a, \qquad 1-b, \qquad 1-c$$

are nonnegative and have sum 2. Applying Jensen's inequality again, we get

$$(1-a)f(a)+(1-b)f(b)+(1-c)f(c)\geq 2f\left(\frac{(1-a)a+(1-b)b+(1-c)c}{2}\right).$$

But

$$(1-a)a+(1-b)b+(1-c)c=a+b+c-(a^2+b^2+c^2)=1-(a^2+b^2+c^2)=$$

$$=2(ab+bc+ca).$$

Hence: $(1-a)f(a)+(1-b)f(b)+(1-c)f(c)\geq 2f(ab+bc+ca)$.

Adding the two inequalities gives

$$2\big(f(a)+f(b)+f(c)\big)\geq 4f\left(\frac{1-ab-bc-ca}{2}\right)+2f(ab+bc+ca).$$

Dividing by 2, we conclude that

$$f(a)+f(b)+f(c)\geq 2f\left(\frac{1-ab-bc-ca}{2}\right)+f(ab+bc+ca).$$

A FEW LIMITS OF INTEGRALS

By Neculai Stanciu-Romania

Abstract: In the next paper we will prove a theorem and we will solve three problems from the well-known math magazine Crux Mathematicorum, Vol. 52, No. 2, February, 2026.

Theorem: Let $I=[a,b]\subset R$, and the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, and $(c_n)_{n\geq 1}$:

(i) $a_n\leq b_n$, $c_n>0$, for any $n\in N^*$; (ii) $a_n,b_n,c_nx\in I, n\in N^*$ for any $x\in I, n\in N^*$;

(iii) $(a_nc_n)_{n\geq 1}$ and $(b_nc_n)_{n\geq 1}$ have finite limits and $\lim_{n\to\infty} a_nc_n=\lim_{n\to\infty} b_nc_n=l\in I$;

(iv) there exists $\alpha\in R$ such that $\lim_{n\to\infty} n^{\alpha}(b_n-a_n)=L\in R$.

If the function $f:I\to R$ is continuous on I, then:

$$\lim_{n\to\infty} n^{\alpha}\int_{a_n}^{b_n} f(c_n x)dx = L\cdot f(l).$$

Proof: For a $n\in N^*$, let the function $g_n:I\to R$ defined by $g_n(x)=f(c_n x)$, $x\in I$. Since g_n it is a continuous function on I, according to the mean theorem (MVT) of integral calculus we have that there exists $\xi_n\in[a_n,b_n]$ such that

$$\int_{a_n}^{b_n} g_n(x)dx=(b_n-a_n)g_n(\xi_n),\ (*),$$

i.e. there exists $a_n\le \xi_n\le b_n$ such that

$$\int_{a_n}^{b_n} f(c_n x)dx=(b_n-a_n)f(c_n\xi_n),\ n\in N^*,\ (**).$$

Since $c_n>0$, from (*) we have $a_n c_n\le c_n\xi_n\le b_n c_n$. Taking into account (iii), according to the squeeze theorem it follows that

$\lim_{n\to\infty} f_n(c_n\xi_n)=f(l)$, (***).

Taking into account (**) and (***), we get the conclusion of the proposition.

The idea of the proof in the proposition above can be used in solving problems with limits of definite integrals.

Problem 1: Find the limit $\lim_{n\to\infty}\sqrt[n]{n!}\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx$, where $(s_n)_{n\ge1}, s_n=\sum_{k=1}^{n}\frac{1}{k^2}$ and $f:(0,\infty)\to(0,\infty)$ is a continuous function on $(0,\infty)$.

Solution: Using $\lim_{n\to\infty} s_n=\frac{\pi^2}{6}$, by (MVT) there exists $\xi_n\in\left(s_n,\frac{\pi^2}{6}\right)$ such that

$$\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx=\left(\tfrac{\pi^2}{6}-s_n\right)f(\xi_n),\ \forall n\in N^*,\ (1).\ \text{So,}$$

$$\sqrt[n]{n!}\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx=\frac{\sqrt[n]{n!}}{n}\cdot f(\xi_n)\cdot n\left(\frac{\pi^2}{6}-s_n\right)=\frac{\sqrt[n]{n!}}{n}\cdot f(\xi_n)\cdot\frac{\frac{\pi^2}{6}-s_n}{\frac{1}{n}}$$

and since

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e},\ \lim_{n\to\infty}\sqrt[n]{n!}\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx=\frac{1}{e}\cdot f\left(\lim_{n\to\infty}\xi_n\right)\cdot\lim_{n\to\infty}\frac{\frac{\pi^2}{6}-s_n}{\frac{1}{n}}\underset{\left(\frac{0}{0}\right)}{\overset{Stolz}{=}}$$

$$=\frac{1}{e}f\left(\frac{\pi^2}{6}\right)\lim_{n\to\infty}\frac{-s_{n+1}+s_n}{\frac{1}{n+1}-\frac{1}{n}}=\frac{f\left(\frac{\pi^2}{6}\right)}{e}\lim_{n\to\infty}\frac{s_{n+1}-s_n}{\frac{1}{n(n+1)}}=$$

$$=\frac{f\left(\frac{\pi^2}{6}\right)}{e}\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n(n+1)}}=\frac{f\left(\frac{\pi^2}{6}\right)}{e}\lim_{n\to\infty}\frac{n}{n+1}=\frac{f\left(\frac{\pi^2}{6}\right)}{e}.$$

Problem 2: Find the limit $\lim_{n\to\infty}\sqrt[n]{n!}\int_{\gamma}^{\gamma_n} f(x)dx$, where $(\gamma_n)_{n\ge1}$, $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$, $\lim_{n\to\infty}\gamma_n=\gamma$ and $f:(0,\infty)\to(0,\infty)$ is a continuous function on $(0,\infty)$.

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Solution: Using $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$, (1), by (MVT) there exists $\xi_n\in(\gamma,\gamma_n)$ such that

$$\int_{\gamma}^{\gamma_n} f(x)dx=(\gamma_n-\gamma)f(\xi_n),\ \forall n\in N^*, \text{ so}$$

$$\sqrt[n]{n!}\int_{\gamma}^{\gamma_n} f(x)dx=\sqrt[n]{n!}(\gamma_n-\gamma)f(\xi_n)=\frac{\sqrt[n]{n!}}{n}\cdot n(\gamma_n-\gamma)f(\xi_n),\ (2).$$

From (1) and (2)

$$\lim_{n\to\infty}\sqrt[n]{n!}\int_{\gamma}^{\gamma_n} f(x)dx = \frac{1}{e} f\left(\lim_{n\to\infty}\xi_n\right)\lim_{n\to\infty} n(\gamma_n-\gamma) =$$

$$= \frac{1}{e} f(\gamma)\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}} \underset{\left(\frac{0}{0}\right)}{\overset{Cesaoro-Stols}{=}} \frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}} = \frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{-\gamma_{n+1}+\gamma_n}{\frac{1}{n(n+1)}} =$$

$$= \frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}-\frac{1}{n+1}}{\frac{1}{n^2}} \overset{\left(\frac{1}{n}=x\right)}{=} \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{\ln(1+x)-\frac{x}{1+x}}{x^2} =$$

$$= \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{(x+1)\ln(1+x)-x}{x^2(1+x)} = \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{(x+1)\ln(1+x)-x}{x^2} =$$

$$= \frac{f(\gamma)}{e}\lim_{\substack{x\to 0\\ x>0}}\frac{\ln(1+x)+1-1}{2x} = \frac{f(\gamma)}{2e}\lim_{\substack{x\to 0\\ x>0}}\ln(1+x)^{\frac{1}{x}} = \frac{f(\gamma)}{2e}\ln e = \frac{f(\gamma)}{2e}.$$

Problem 3: Find the limit $\lim_{n\to\infty}\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}}\int_{\gamma}^{\gamma_n} f(x)dx$, where $(\gamma_n)_{n\geq 1}$, $\gamma_n = -\ln n + \sum_{k=1}^{n}\frac{1}{k}$ with $\lim_{n\to\infty}\gamma_n = \gamma$, and $f:(0,\infty)\to(0,\infty)$ is a continuous function on $(0,\infty)$.

Solution: $\lim_{n\to\infty}\frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n} = \lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}{n^n}} =$

$$= \lim_{n\to\infty}\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}\cdot\sqrt[n+1]{(2n+1)!!}}{(n+1)^{n+1}}\cdot\frac{n^n}{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!}} =$$

$$= \lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{1}{e}\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}} = \frac{1}{e}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!} =$$

$$= \frac{1}{e}\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n = \frac{2}{e^2}, (1).$$

By (MVT) $\exists\xi_n\in(\gamma,\gamma_n)$ such that

$$\int_{\gamma}^{\gamma_n} f(x)dx = (\gamma_n-\gamma)f(\xi_n), \forall n\in N^*, \text{ so}$$

$$\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}}\int_{\gamma}^{\gamma_n} f(x)dx=\frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n}\cdot n(\gamma_n-\gamma)f(\xi_n),\ (2).$$

From (1) and (2) yields that

$$\lim_{n\to\infty}\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}}\int_{\gamma}^{\gamma_n} f(x)dx=$$

$$=\frac{2}{e^2}f(\gamma)\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}}\overset{Cesaoro-Stols}{\underset{\left(\frac{0}{0}\right)}{=}}\frac{2f(\gamma)}{e^2}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}}=\frac{2f(\gamma)}{e^2}\lim_{n\to\infty}\frac{-\gamma_{n+1}+\gamma_n}{\frac{1}{n(n+1)}}=$$

$$=\frac{2f(\gamma)}{e^2}\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}-\frac{1}{n+1}}{\frac{1}{n^2}}\overset{\left(\frac{1}{n}=x\right)}{=}\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{\ln(1+x)-\frac{x}{1+x}}{x^2}=$$

$$=\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{(x+1)\ln(1+x)-x}{x^2(1+x)}=\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{(x+1)\ln(1+x)-x}{x^2}=$$

$$=\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{\ln(1+x)+1-1}{2x}=\frac{2f(\gamma)}{2e^2}\lim_{\substack{x\to0\\x>0}}\ln(1+x)^{\frac{1}{x}}=\frac{f(\gamma)}{e^2}\ln e=\frac{f(\gamma)}{e^2}.$$

References: **[1]. Neculai Stanciu,** *Evaluating Limits of Integrals Using The Mean Value Theorem*, Crux Mathematicorum, Vol. 52(2), February, 2026, 83-87.

NEW IDENTITIES AND INEQUALITIES IN TRIANGLE-(III)

By Bogdan Fuștei-Romania

Lemma1: For x ≥ y ,real numbers and also t is a real number and x,y,t >0 ,x,y>t then:

$$\frac{x}{x-t}\le\frac{y}{y-t}.$$

Proof : $x(y-t)\le y(x-t)\to xy-xt\le xy-yt\to xt\ge yt\to x\ge y$

Lemma2: x,y,t,r_1 -real numbers and x,y,t,$r_1>0$; x>t , y>r_1. If $\frac{x}{y}=\frac{t}{r_1}$ then:

$$\frac{x}{y}=\frac{x-t}{y-r_1}.$$

Proof: $x(y-r_1)=y(x-t)\rightarrow xy-xr_1=xy-yt\rightarrow -xr_1=-yt\rightarrow xr_1=yt$

ABC triangle with usual notations:

$$\left(\frac{g_a}{r}\right)^2=\left(\frac{n_a}{r_a}\right)^2+4\frac{b+c}{a}\text{ (and analogous)[1]}$$

$$\frac{n_a}{r_a}=\frac{n_a-\sqrt{4r^2+(b-c)^2}}{r}\text{ (and analogous)[2]}$$

$$\frac{b+c}{a}=\frac{g_a^2-\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2}{4r^2}\textbf{ (and analogous)(1)}$$

$$\frac{b+c}{a}=\frac{\left(g_a+n_a-\sqrt{4r^2+(b-c)^2}\right)\left(g_a-n_a+\sqrt{4r^2+(b-c)^2}\right)}{4r^2}\textbf{ (and analogous) (2)}$$

From $2S=ah_a=2sr=(a+b+c)r\rightarrow\frac{h_a}{r}=1+\frac{b+c}{a}\rightarrow\frac{b+c}{a}=\frac{h_a-r}{r}$ (and analogous)

$\frac{h_a-r}{r}=\frac{\left(g_a+n_a-\sqrt{4r^2+(b-c)^2}\right)\left(g_a-n_a+\sqrt{4r^2+(b-c)^2}\right)}{4r^2}$ obtain:

$$\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}=\frac{g_a+n_a-\sqrt{4r^2+(b-c)^2}}{4r}\textbf{ (and analogous)(3)}$$

$$\frac{4(h_a-r)}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}=\frac{g_a}{r}+\frac{n_a}{r_a}\textbf{ (and analogous)(4)}$$

From (4) after summation:

$$4\sum\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}=\frac{g_a+g_b+g_c}{r}+\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\textbf{ (5)}$$

From $g_a^2=(s-a)^2+2rh_a$ (and analogous)[1]$\rightarrow g_a>s-a$ (and analogous)

$s-a=\sqrt{g_a^2-2rh_a}\rightarrow s-a=\sqrt{g_a-\sqrt{2rh_a}}\sqrt{g_a+\sqrt{2rh_a}}$

From $r_a=\frac{S}{s-a}$ (and analogous) ;$S=sr\rightarrow\frac{r_a}{r}=\frac{s}{s-a}$ (and analogous) we obtain:

$\frac{r_a}{r}=\frac{s}{\sqrt{g_a-\sqrt{2rh_a}}\sqrt{g_a+\sqrt{2rh_a}}}$ (and analogous)

We know that : $\frac{r_a}{r}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous)[2] ,we obtain:

$$\frac{n_a\sqrt{g_a+\sqrt{2rh_a}}}{n_a-\sqrt{4r^2+(b-c)^2}}=\frac{s}{\sqrt{g_a-\sqrt{2rh_a}}}\textbf{ (and analogous) (6)}$$

From (6) after summation :

$$\sum\frac{n_a\sqrt{g_a+\sqrt{2rh_a}}}{n_a-\sqrt{4r^2+(b-c)^2}}=s\sum\frac{1}{\sqrt{g_a-\sqrt{2rh_a}}}\ \textbf{(7)}$$

From $s^2=n_a^2+2h_ar_a$ (and analogous)[3]→ $2h_ar_a=s^2-n_a^2=(s-n_a)(s+n_a)$

$s-n_a=\frac{2h_ar_a}{s+n_a}\to\frac{2h_a}{s+n_a}=\frac{s-n_a}{r_a}$ (and analogous) $\to\frac{s}{r_a}=\frac{n_a}{r_a}+\frac{2h_a}{s+n_a}$

$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\to\frac{s}{r}=\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}+2\sum\frac{h_a}{s+n_a}\to\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}=\frac{s}{r}-2\sum\frac{h_a}{s+n_a}$ and (5) we obtain:

$$2\sum\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum\frac{h_a}{s+n_a}=\frac{g_a+g_b+g_c+s}{2r}\ \textbf{(8)}$$

$\frac{r_a}{r}=\frac{s}{s-a}=\frac{s^2}{s(s-a)}$; $s(s-a)=r_br_c$ (and analogous);

$s^2=n_a^2+2h_ar_a$ (and analogous)

$\frac{r_a}{r}=\frac{2s^2}{2r_br_c}=\frac{n_b^2+2h_br_b+n_c^2+2h_cr_c}{2r_br_c}\geq\frac{2n_bn_c+2h_br_b+2h_cr_c}{2r_br_c}=\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}$

$$\frac{r_a}{r}\geq\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\ \textbf{(and analogous) (9)}$$

From $\frac{r_a}{r}=\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous) and (9):

$$\frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}\geq\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\ \textbf{(and analogous)(10)}$$

Is well-known that $r_ar_br_c=Ss=s^2r$ and (9) we obtain:

$$\frac{r_ar_br_c}{r^3}=\left(\frac{s}{r}\right)^2\geq\prod\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)\textbf{(11)}$$

From (8) and (11) :

$$2\sum\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum\frac{h_a}{s+n_a}\geq\frac{g_a+g_b+g_c}{2r}+\frac{1}{2}\prod\left(\frac{n_bn_c}{r_br_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)^{\frac{1}{2}}\ \textbf{(12)}$$

We know that: $n_a\geq p_a\sqrt{\frac{l_a}{g_a}}$ (and analogous)[4] and using lemma1 :

$$\frac{r_a}{r}\leq\frac{p_a\sqrt{\frac{l_a}{g_a}}}{p_a\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\ \textbf{(and analogous)(13)}$$

From(13) :

$$\left(\frac{s}{r}\right)^2\leq\sqrt{\frac{l_al_bl_c}{g_ag_bg_c}}\prod\frac{p_a}{p_a\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\ \textbf{(14)}$$

From(14) and (11):

$$\sqrt{\frac{l_a l_b l_c}{g_a g_b g_c}}\prod\frac{p_a}{p_a\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\ge\prod\left(\frac{n_b n_c}{r_b r_c}+\frac{h_b}{r_c}+\frac{h_c}{r_b}\right)(15)$$

From (14) and (8):

$$\frac{g_a+g_b+g_c}{2r}+\frac{1}{2}\sqrt[4]{\frac{l_a l_b l_c}{g_a g_b g_c}}\prod\left(\frac{p_a}{p_a\sqrt{\frac{l_a}{g_a}}-\sqrt{4r^2+(b-c)^2}}\right)^{\frac{1}{2}}\ge 2\sum\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum\frac{h_a}{s+n_a}\ (16)$$

From $\frac{h_a-r}{r}=\frac{g_a^2-\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2}{4r^2}$ (and analogous) we obtain:

$$g_a^2=4r(h_a-r)+\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2\text{ (and analogous)(17)}$$

From (17) after summation:

$$g_a^2+g_b^2+g_c^2=4r(h_a+h_b+h_c-3r)+\sum\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2(18)$$

From C.B.S :

$$\sum\left(n_a-\sqrt{4r^2+(b-c)^2}\right)^2\ge\frac{1}{3}\left(n_a+n_a+n_a-\sum\sqrt{4r^2+(b-c)^2}\right)^2\text{ and (19):}$$

$$g_a^2+g_b^2+g_c^2\ge 4r(h_a+h_a+h_a-3r)+\frac{1}{3}\left(n_a+n_a+n_a-\sum\sqrt{4r^2+(b-c)^2}\right)^2(20)$$

From (3) ,(4) and $g_a+n_a\ge 2m_a$ (and analogous)[5]:

$$\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}\ge\frac{2m_a-\sqrt{4r^2+(b-c)^2}}{4r}\text{ (and analogous)(21)}$$

$$\frac{g_a}{r}+\frac{n_a}{r_a}\ge\frac{2m_a-\sqrt{4r^2+(b-c)^2}}{r}\text{ (and analogous)(22)}$$

From (21) and (22) after summation:

$$4\sum\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}\ge\sum\frac{2m_a-\sqrt{4r^2+(b-c)^2}}{r}(23)$$

$$\frac{n_a}{r_a}+\frac{n_b}{r_b}+\frac{n_c}{r_c}\ge\sum\frac{2m_a-g_a+\sqrt{4r^2+(b-c)^2}}{r}(24)$$

From (22) : $\frac{n_a}{r_a}\ge\frac{2m_a-g_a+\sqrt{4r^2+(b-c)^2}}{r}\to\frac{r}{r_a}\ge\frac{2m_a-g_a+\sqrt{4r^2+(b-c)^2}}{n_a}$ (and analogous);

$$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}=\frac{1}{r}\to 1\ge\sum\frac{2m_a-g_a+\sqrt{4r^2+(b-c)^2}}{n_a}(25)$$

From $g_a^2 = (s-a)^2 + 2rh_a$ (and analogous) and (17) we obtain:

$$\mathbf{(s-a)^2 = \left(n_a - \sqrt{4r^2+(b-c)^2}\right)^2 + 2r(h_a - 2r)} \textbf{ (and analogous) (26)}$$

From (26) : $2r(h_a - 2r) = \left(s - a + n_a - \sqrt{4r^2+(b-c)^2}\right)\left(s - a - n_a + \sqrt{4r^2+(b-c)^2}\right)$

$\frac{2(h_a-2r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s-a+n_a-\sqrt{4r^2+(b-c)^2}}{r}$ (*)

$\frac{n_a}{r_a} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{r}$ (and analogous) $\rightarrow$

$$\mathbf{\frac{2(h_a-2r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s-a}{r} + \frac{n_a}{r_a}} \textbf{ (and analogous) (27)}$$

From (27) after summation :

$$\mathbf{2\sum\frac{h_a-2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s}{r} + \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c}} \textbf{ (28)}$$

From (28) and $\frac{s}{r} = \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} + 2\sum\frac{h_a}{s+n_a}$ we obtain:

$$\mathbf{\sum\frac{h_a-2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} + \sum\frac{h_a}{s+n_a}} \ \textbf{(29)}$$

From (*) $\rightarrow \frac{2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s-a+n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r} = \frac{s-a}{h_a-2r} + \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}$

From $\frac{s}{h_a} = \frac{a}{2r}$ (and analogous) and lemma2 : $\frac{s}{h_a} = \frac{a}{2r} = \frac{s-a}{h_a-2r}$

From $\frac{n_a}{h_a} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r}$ (and analogous) [2] and $\frac{s}{h_a} = \frac{a}{2r} = \frac{s-a}{h_a-2r}$:

$$\mathbf{\frac{2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{a}{2r} + \frac{n_a}{h_a} = \frac{n_a+s}{h_a}} \textbf{ (and analogous) (30)}$$

From $s - n_a = \frac{2h_ar_a}{s+n_a} \rightarrow \frac{2r_a}{s+n_a} = \frac{s-n_a}{h_a} = \frac{a}{2r} - \frac{n_a}{h_a}$ and (30):

$$\mathbf{\frac{r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a}{h_a} + \frac{r_a}{s+n_a}} \textbf{ (and analogous)(31)}$$

From $\frac{a}{2r} = \frac{n_a}{h_a} + \frac{2r_a}{s+n_a}$, $\frac{a}{2r} = \frac{s-a}{h_a-2r}$ and $\frac{n_a}{h_a} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r} \rightarrow \frac{s-a}{h_a-2r} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r} + \frac{2r_a}{s+n_a}$

$$\mathbf{\frac{2r_a}{s+n_a} = \frac{s-a-n_a+\sqrt{4r^2+(b-c)^2}}{h_a-2r}} \textbf{ (and analogous) (32)}$$

From (32) after summation :

$$\mathbf{2\sum\frac{r_a}{s+n_a} = \sum\frac{s-a-n_a+\sqrt{4r^2+(b-c)^2}}{h_a-2r}} \textbf{ (33)}$$

From (32) because $r_a > 0$ and $s + n_a > 0 \to \frac{s-a-n_a+\sqrt{4r^2+(b-c)^2}}{h_a-2r} > 0$ and because $h_a - 2r > 0$ we obtain: $s - a - n_a + \sqrt{4r^2 + (b-c)^2} > 0$

$$\mathbf{s + \sqrt{4r^2 + (b-c)^2} > a + n_a} \textbf{ (and analogous) (34)}$$

From (34) : $s - a > n_a - \sqrt{4r^2 + (b-c)^2} \to \frac{s-a}{r} > \frac{n_a-\sqrt{4r^2+(b-c)^2}}{r}$ and $\frac{r_a}{r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous) we obtain :

$$\frac{s-a}{r} > \frac{n_a}{r_a} \textbf{ (and analogous) (35)}$$

From (27): $\frac{2(h_a-2r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s-a}{r} + \frac{n_a}{r_a}$ and (35) we obtain:

$$\frac{s-a}{r} > \frac{h_a-2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} > \frac{n_a}{r_a} \textbf{ (and analogous)(36)}$$

From $\frac{r_a}{r} = \frac{s}{s-a} \to \frac{s-a}{r} = \frac{s}{r_a}$ and (36) :

$$\frac{s}{r_a} > \frac{h_a-2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} > \frac{n_a}{r_a} \textbf{ (and analogous) (37)}$$

From (27) and $\frac{s-a}{r} = \frac{s}{r_a}$ (and analogous) :

$$\frac{2(h_a-2r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a+s}{r_a} \textbf{ (and analogous) (38)}$$

From (38) after summation :

$$2\sum\frac{h_a-2r}{n_a+s} = \sum\frac{s-a-n_a+\sqrt{4r^2+(b-c)^2}}{r_a} \textbf{ (39)}$$

From (38) after summation :

$$2(h_a + h_b + h_c - 6r) = \sum\frac{(n_a+s)\left(s-a-n_a+\sqrt{4r^2+(b-c)^2}\right)}{r_a} \textbf{ (40)}$$

From (30) and (38) after summation:

$$\frac{2(h_a-r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = (n_a + s)\left(\frac{1}{r_a} + \frac{1}{h_a}\right) \textbf{(and analogous)(41)}$$

From (30) : $\frac{2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{a}{2r} + \frac{n_a}{h_a} = \frac{n_a+s}{h_a}$ (and analogous) we obtain :

$$2\sum\frac{r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s}{r} + \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \textbf{ (42)}$$

$\frac{h_a}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a+s}{2r}$ and after summation :

$$\sum\frac{h_a}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a+n_b+n_c+3s}{2r} \textbf{ (43)}$$

From (8) and (43):

$$2\sum\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum\frac{h_a}{s+n_a}+\sum\frac{h_a}{s-a-n_a+\sqrt{4r^2+(b-c)^2}}=\frac{n_a+n_b+n_c+g_a+g_b+g_c+4s}{2r}\ \textbf{(44)}$$

From (44) and $g_a+n_a\ge 2m_a$ (and analogous) , we obtain:

$$2\sum\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum\frac{h_a}{s+n_a}+\sum\frac{h_a}{s-a-n_a+\sqrt{4r^2+(b-c)^2}}\ge\frac{2s+m_a+m_b+m_c}{r}\ \textbf{(45)}$$

From R$\ge 2r(\text{Euler})$ and (42):

$$R\sum\frac{1}{s-a-n_a+\sqrt{4r^2+(b-c)^2}}\ge\frac{s}{r}+\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}\ \textbf{(46)}$$

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ROMANIAN MATHEMATICAL MAGAZINE CHALLENGES-I

By Daniel Sitaru-Romania

01. In ΔABC the following relationship holds:

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}\ge 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}=\sum_{cyc}\frac{r_a}{b}=\sum_{cyc}\frac{F}{b(s-a)}=F\sum_{cyc}\frac{1}{b(s-a)}\overset{AM-GM}{\ge}\frac{3F}{\sqrt[3]{abc(s-a)(s-b)(s-c)}}=$$

$$=\frac{3F}{\sqrt[3]{4Rrs(s-a)(s-b)(s-c)}}=\frac{3F}{\sqrt[3]{4RrF^2}}=\frac{3F}{\sqrt[3]{4Rrr^2s^2}}=\frac{3F}{r\cdot\sqrt[3]{4Rs^2}}=\frac{3rs}{r\cdot\sqrt[3]{4Rs^2}}\ge$$

$$\overset{MITRINOVIC}{\ge}\frac{3s}{\sqrt[3]{4R\left(\frac{3\sqrt{3}R}{2}\right)^2}}\overset{MITRINOVIC}{\ge}\frac{3\cdot 3\sqrt{3}r}{\sqrt[3]{4R\cdot\frac{27R^2}{4}}}=\frac{9\sqrt{3}r}{3R}=3\sqrt{3}\cdot\frac{r}{R}$$

Equality holds for $a=b=c$.

02. In ΔABC the following relationship holds:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\ge\frac{3r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\overset{JENSEN}{\ge}3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}=3\sin\frac{\pi}{6}=3\cdot\frac{1}{2}\ge 3\cdot\frac{r}{R}$$

Equality holds for $A=B=C$.

03. In ΔABC the following relationship holds:

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=$$

$$=\sum_{cy}\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=\sum_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{ac\cdot ab}{s(s-b)\cdot s(s-c)}}=$$

$$=\sum_{cyc}\sqrt{\frac{a^2}{s^2}}=\frac{1}{s}\sum_{cyc}a=\frac{a+b+c}{s}=\frac{2s}{s}=2$$

04. In ΔABC the following relationship holds:

$$\sin^2 A\sin 2B+\sin^2 B\sin 2A=2\sin A\sin B\sin C$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin^2 A\sin 2B+\sin^2 B\sin 2A=\sin^2 A\cdot 2\sin B\cos B+\sin^2 B\cdot 2\sin A\cos A=$$

$$=2\sin A\sin B\,(\sin A\cos B+\sin B\cos A)=2\sin A\sin B\cdot\sin(A+B)=$$

$$=2\sin A\sin B\cdot\sin(\pi-C)=2\sin A\sin B\sin C$$

05. In ΔABC the following relationship holds:

$$\sin A + \sin B + \sin C \geq 3\sqrt{3} \cdot \frac{r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin A + \sin B + \sin C = \sum_{cyc} \sin A = \sum_{cyc} \frac{a}{2R} = \frac{1}{2R}(a+b+c) = \frac{2s}{2R} = \frac{s}{R} \overset{MITRINOVIC}{\geq} \frac{3\sqrt{3}r}{R}$$

Equality holds for $a = b = c$.

06. In ΔABC the following relationship holds:

$$\cot A + \cot B + \cot C = \frac{2(\sin^2 A + \sin^2 B + \sin^2 C)}{\sin 2A + \sin 2B + \sin 2C}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin 2A + \sin 2B + \sin 2C = 2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2} + \sin 2C =$$

$$= 2\sin(A+B)\cos(A-B) + 2\sin C\cos C =$$

$$= 2\sin(\pi - C)\cos(A-B) + 2\sin C\cos C = 2\sin C\,(\cos(A-B) + \cos C) =$$

$$= 2\sin C \cdot 2\cos\frac{A-B+C}{2}\cos\frac{A-B-C}{2} =$$

$$= 4\sin C\cos\frac{\pi-2B}{2}\cos\frac{2A-\pi}{2} = 4\sin C\cos\left(\frac{\pi}{2} - B\right)\cos\left(\frac{\pi}{2} - A\right) =$$

$$= 4\sin C\sin B\sin A$$

$$\sin 2A + \sin 2B + \sin 2C = 4\sin A\sin B\sin C \quad (1)$$

$$\cot A + \cot B + \cot C = \sum_{cyc}\cot A = \sum_{cyc}\frac{\cos A}{\sin A} = \sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A} =$$

$$= \frac{1}{4F}\sum_{cyc}(b^2+c^2-a^2) = \frac{1}{4F}\sum_{cyc}a^2 = \frac{1}{4F}\sum_{cyc}4R^2\sin^2 A =$$

$$= \frac{4R^2(\sin^2 A + \sin^2 B + \sin^2 C)}{4\cdot 2R^2\sin A\sin B\sin C} = \frac{2(\sin^2 A + \sin^2 B + \sin^2 C)}{4\sin A\sin B\sin C} \overset{(1)}{=}$$

$$= \frac{2(\sin^2 A + \sin^2 B + \sin^2 C)}{\sin 2A + \sin 2B + \sin 2C}$$

07. In ΔABC the following relationship holds:

$$\boldsymbol{h_a h_b h_c \geq 27r^3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$h_a h_b h_c = \frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c} = \frac{8F^3}{abc} = \frac{8F^3}{4RF} = \frac{2F^2}{R} = \frac{2r^2s^2}{R}$$

We must prove that:

$$\frac{2r^2s^2}{R} \geq 27r^3 \Leftrightarrow \frac{2s^2}{R} \geq 27r \Leftrightarrow 27r \Leftrightarrow 2s^2 \geq 27Rr$$

$$2s^2 \overset{GERRETSEN}{\geq} 2(16Rr - 5r^2) \geq 27Rr$$

$$32Rr - 10r^2 \geq 27Rr,\ 5Rr \geq 10r^2,\ R \geq 2r \quad \text{(Euler)}$$

Equality holds for $a = b = c$.

08. In ΔABC the following relationship holds:

$$\boldsymbol{\frac{h_a + h_b}{a} + \frac{h_b + h_c}{b} + \frac{h_c + h_a}{c} \geq 6\sqrt{3} \cdot \frac{r}{R}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Lemma: In ΔABC holds:

$$h_a h_b h_c \geq 27r^3$$

Proof:

$$h_a h_b h_c = \frac{8F^3}{abc} = \frac{8F^3}{4RF} = \frac{2F^2}{R} = \frac{2r^2s^2}{R}$$

$$\frac{2r^2s^2}{R} \geq 27r^3 \Leftrightarrow \frac{2s^2}{R} \geq 27r \Leftrightarrow 2s^2 \geq 27Rr$$

$$2s^2 \overset{GERRETSEN}{\geq} 2(16Rr - 5r^2) \geq 27Rr$$

$$32Rr - 10r^2 \geq 27Rr \Leftrightarrow R \geq 2r \quad \text{(Euler)}$$

Back to the problem:

$$\sum_{cyc} \frac{h_a + h_b}{a} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{(h_a + h_b)(h_b + h_c)(h_c + h_a)}{abc}} \geq$$

$$\overset{CESARO}{\geq} 3\sqrt[3]{\frac{8h_ah_bh_c}{4RF}} \overset{Lemma}{\geq} 3\sqrt[3]{\frac{8\cdot 27r^3}{4Rrs}} = 3\cdot 2\cdot 3r\cdot\sqrt[3]{\frac{1}{4rs}} = \frac{18r}{\sqrt[3]{2R^2s}} \overset{EULER}{\geq}$$

$$\geq \frac{18r}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot s}} = \frac{18r}{\sqrt[3]{2R^2s}} \overset{MITRINOVIC}{\geq}$$

$$\geq \frac{18r}{\sqrt[3]{2R^2\cdot\frac{3\sqrt{3}}{2}\cdot R}} = \frac{18r}{\sqrt[3]{\left(R\sqrt{3}\right)^3}} = \frac{18r}{\sqrt{3}R} = \frac{18\sqrt{3}r}{3R} = 6\sqrt{3}\cdot\frac{r}{R}$$

Equality holds for $a=b=c$.

09. In ΔABC the following relationship holds:

$$\frac{AI\cdot BI\cdot CI}{abc} \leq \frac{1}{3\sqrt{3}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{AI\cdot BI\cdot CI}{abc} = \frac{\frac{r}{\sin\frac{A}{2}}\cdot\frac{r}{\sin\frac{B}{2}}\cdot\frac{r}{\sin\frac{C}{2}}}{abc} = \frac{r^3}{abc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} =$$

$$= \frac{r^3}{abc\sqrt{\frac{(s-b)(s-c)(s-a)(s-c)(s-a)(s-b)}{bc\cdot ac\cdot ab}}} = \frac{r^3}{abc\cdot\frac{(s-a)(s-b)(s-c)}{abc}} = \frac{r^3s}{s(s-a)(s-b)(s-c)} =$$

$$= \frac{r^3s}{F^2} = \frac{r^3s}{r^2s^2} = \frac{r}{s} \overset{MITRINOVIC}{\leq} \frac{r}{3\sqrt{3}r} = \frac{1}{3\sqrt{3}}$$

Equality holds for $a=b=c$.

10. In ΔABC the following relationship holds:

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c} \geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c} = \sum_{cyc}\frac{a+b}{h_a} \overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{h_ah_bh_c}} \overset{CESARO}{\geq} 3\sqrt[3]{\frac{8abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}} =$$

$$= 3\sqrt[3]{\frac{(abc)^2}{F^3}} = 3\sqrt[3]{\frac{16R^2F^2}{F^3}} = 3\sqrt[3]{\frac{16R^2}{F}} \ge$$

$$\overset{MITRINOVIC}{\ge} 3\sqrt[3]{\frac{16\cdot\frac{4}{27}s^2}{rs}} = 3\cdot\frac{4}{3}\sqrt[3]{\frac{s}{r}} \overset{MITRINOVIC}{\ge} 4\sqrt[3]{\frac{3\sqrt{3}r}{r}} = 4\sqrt[3]{\left(\sqrt{3}\right)^3} = 4\sqrt{3}$$

Equality holds for $a = b = c$.

11. In ΔABC the following relationship holds:

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \ge 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \overset{AM-GM}{\ge} 3\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} =$$

$$= 3\sqrt[3]{\sqrt{\frac{s(s-a)\cdot s(s-b)\cdot s(s-c)}{bc\cdot ac\cdot ab}}} = 3\sqrt[6]{\frac{s^3(s-a)(s-b)(s-c)}{(abc)^2}} = 3\sqrt[6]{\frac{s^2F^2}{(abc)^2}} =$$

$$= 3\sqrt[3]{\frac{sF}{abc}} = 3\sqrt[3]{\frac{sF}{4RF}} = 3\sqrt[3]{\frac{s}{4R}}$$

Remains to prove:

$$3\sqrt[3]{\frac{s}{4R}} \ge 3\sqrt{3}\cdot\frac{r}{R} \Leftrightarrow \frac{s}{4R} \ge 3\sqrt{3}\cdot\frac{r^3}{R^3}$$

$$sR^3 \ge 12\sqrt{3}Rr^3 \Leftrightarrow sR^2 \ge 12\sqrt{3}r^3 \quad \text{(to prove)}$$

$$sR^2 \overset{EULER}{\ge} s\cdot(2r)^2 \overset{MITRINOVIC}{\ge} 3\sqrt{3}r\cdot 4r^2 = 12\sqrt{3}r^3$$

Equality holds for $A = B = C$.

12. If $a, b > 0$ and $a + b = 2ab$ then:

$$\frac{a}{b^2} + \frac{b}{a^2} \ge 2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{b^2} + \frac{b}{a^2} \ge 2 \Leftrightarrow a^3 + b^3 \ge 2a^2b^2, \qquad (a+b)(a^2 - ab + b^2) \ge 2a^2b^2$$

$$2ab(a^2-ab+b^2)\geq 2a^2b^2,\qquad a^2-ab+b^2\geq ab$$

$$a^2-2ab+b^2\geq 0,\qquad (a-b)^2\geq 0$$

Equality holds for $a=b$.

13. If $\boldsymbol{a},\boldsymbol{b}>0; a+b=2\boldsymbol{a}^2\boldsymbol{b}^2$ then:

$$\frac{\boldsymbol{a}+\mathbf{1}}{\boldsymbol{b}^2}+\frac{\boldsymbol{b}+\mathbf{1}}{\boldsymbol{a}^2}\geq \mathbf{4}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Denote $S=a+b; P=ab; S>0; P>0$

$$a+b\overset{AM-GM}{\geq}2\sqrt{ab}\Rightarrow S\geq 2\sqrt{P}\Rightarrow 2P^2\geq 2\sqrt{P}$$

$$\Rightarrow P^2\geq P\Rightarrow P^4\geq P\Rightarrow P^3\geq 1\Rightarrow P\geq 1\Rightarrow P-1\geq 0\quad (1)$$

We used the hypothesis: $S=2P^2$.

$$\frac{a+1}{b^2}+\frac{b+1}{a^2}\geq 4\Leftrightarrow a^2(a+1)+b^2(b+1)\geq 4a^2b^2$$

$$a^3+b^3+a^2+b^2-4a^2b^2\geq 0$$

$$S^3-3SP+S^2-2P-4P^2\geq 0$$

$$8P^6-6P^3+4P^4-2P-4P^2\geq 0,\qquad 4P^5-3P^2+2P^3-1-2P\geq 0$$

$$4P^5+2P^3-3P^2-2P-1\geq 0$$

$$4P^5-4P^4+4P^4-4P^3+6P^3-6P^2+3P^2-3P+P-1\geq 0$$

$$4P^4(P-1)+4P^3(P-1)+6P^2(P-1)+3P(P-1)+(P-1)\geq 0$$

$$(P-1)(4P^4+4P^3+6P^2+3P+1)\geq 0$$

$$P-1\geq 0\quad \text{(True by (1))}$$

Equality holds for $a=b=1$.

14. If $\boldsymbol{a},\boldsymbol{b}>0; ab=a+b$ then:

$$\frac{\boldsymbol{a}+\mathbf{1}}{\boldsymbol{b}^3}+\frac{\boldsymbol{b}+\mathbf{1}}{\boldsymbol{a}^3}\geq \frac{\mathbf{3}}{\mathbf{4}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Denote: $S=a+b; P=ab; S>0; P>0$

$$a+b\overset{AM-GM}{\geq}2\sqrt{ab}\Rightarrow S\geq 2\sqrt{P}\Rightarrow S\geq 2\sqrt{S}\Rightarrow S^2\geq 4S\Rightarrow S\geq 4\Rightarrow S-4\geq 0\qquad (1)$$

$$\frac{a+1}{b^3}+\frac{b+1}{a^3}\geq\frac{3}{4}\Leftrightarrow 4[a^3(a+1)+b^3(b+1)]\geq 3a^3b^3$$

$$4(a^4+b^4)+4(a^3+b^3)\geq 3a^3b^3$$

$$4[(S^2-2P)^2-2P^2]+4(S^3-3SP)-3P^3\geq 0$$

$$S=P\Rightarrow 4(S^2-2S)^2-8S^2+4S^3-12S^2-3S^3\geq 0$$

$$4S^4-16S^3+16S^2-8S^2+S^3-12S^2\geq 0$$

$$4S^4-15S^3-4S^2\geq 0\Leftrightarrow 4S^2-15S-4\geq 0$$

$$4S^2-16S+S-4\geq 0\Leftrightarrow 4S(S-4)+(S-4)\geq 0$$

$$(S-4)(4S+1)\geq 0\Leftrightarrow S-4\geq 0\ \text{(True by (1)).}$$

Equality holds for $a=b=2$.

15. In ΔABC the following relationship holds:

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}\geq 8\sqrt{3}R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{\sin^2 C}+\frac{b+c}{\sin^2 A}+\frac{c+a}{\sin^2 B}=\sum_{cyc}\frac{a+b}{\sin^2 C}\geq$$

$$\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{\sin^2 A\sin^2 B\sin^2 C}}\overset{CESARO}{\geq}3\sqrt[3]{\frac{8abc}{\frac{a^2}{4R^2}\cdot\frac{b^2}{4R^2}\cdot\frac{c^2}{4R^2}}}=3\cdot 4R^2\sqrt[3]{\frac{8abc}{a^2b^2c^2}}=$$

$$=12R^2\cdot 2\cdot\frac{1}{\sqrt[3]{abc}}=\frac{24R^2}{\sqrt[3]{4Rrs}}\overset{EULER}{\geq}\frac{24R^2}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot s}}=\frac{24R^2}{\sqrt[3]{2R^2\cdot\frac{3\sqrt{3}}{2}R}}=$$

$$=\frac{24R^2}{\sqrt[3]{\left(\sqrt{3}R\right)^3}}=\frac{24R^2}{\sqrt{3}R}=\frac{24R}{\sqrt{3}}=\frac{24\sqrt{3}R}{3}=8\sqrt{3}R$$

Equality holds for $a=b=c$.

16. In ΔABC the following relationship holds:

$$\frac{a+b}{(\sin A+\sin B)^2}+\frac{b+c}{(\sin B+\sin C)^2}+\frac{c+a}{(\sin C+\sin A)^2}\geq 2\sqrt{3}R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{(\sin A+\sin B)^2}+\frac{b+c}{(\sin B+\sin C)^2}+\frac{c+a}{(\sin C+\sin A)^2}=$$

$$=\sum_{cyc}\frac{a+b}{(\sin A+\sin B)^2}=\sum_{cyc}\frac{a+b}{\left(\frac{a}{2R}+\frac{b}{2R}\right)^2}=4R^2\sum_{cyc}\frac{a+b}{(a+b)^2}=$$

$$=4R^2\cdot\sum_{cyc}\frac{1^2}{a+b}\overset{BERGSTROM}{\geq}4R^2\cdot\frac{(1+1+1)^2}{a+b+b+c+c+a}=$$

$$=4R^2\cdot\frac{9}{4s}=\frac{9R^2}{s}\overset{MITRINOVIC}{\geq}\frac{9R^2}{\frac{3\sqrt{3}}{2}R}=\frac{3R\cdot 2}{\sqrt{3}}=2\sqrt{3}R$$

Equality holds for $a=b=c$.

17. In ΔABC the following relationship holds:

$$\frac{a+b}{(\sin A+\sin B)^3}+\frac{b+c}{(\sin B+\sin C)^3}+\frac{c+a}{(\sin C+\sin A)^3}\geq 2R$$

Proposed by Nguyen Hung Cuong - Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{(\sin A+\sin B)^3}+\frac{b+c}{(\sin B+\sin C)^3}+\frac{c+a}{(\sin C+\sin A)^3}=$$

$$=\sum_{cyc}\frac{a+b}{(\sin A+\sin B)^3}=\sum_{cyc}\frac{a+b}{\left(\frac{a}{2R}+\frac{b}{2R}\right)^3}=8R^3\sum_{cyc}\frac{a+b}{(a+b)^3}=8R^3\sum_{cyc}\frac{1}{(a+b)^2}\overset{RADON}{\geq}$$

$$\geq 8R^3\cdot\frac{(1+1+1)^3}{(a+b+b+c+c+a)^2}=\frac{8R^3\cdot 27}{4\cdot 4s^2}=\frac{27R^3}{2s^2}\overset{MITRINOVICI}{\geq}\frac{27R^3}{2\cdot\left(\frac{3\sqrt{3}}{2}R\right)^2}=\frac{27R^3}{2\cdot\frac{27R^2}{4}}=2R$$

Equality holds for $a=b=c$.

18. If $a,b,c>0$ then:

$$\sum_{cyc}\frac{\sqrt{(1+b^2)(1+c^2)}}{a}\geq 6$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}\frac{\sqrt{(1+b^2)(1+c^2)}}{a}\overset{AM-GM}{\geq}\sum_{cyc}\frac{\sqrt{2b\cdot 2c}}{a}=$$

$$= 2\sum_{cyc}\frac{\sqrt{bc}}{a} \overset{AM-GM}{\geq} 2\cdot 3\sqrt[3]{\frac{\sqrt{bc}}{a}\cdot\frac{\sqrt{ca}}{b}\cdot\frac{\sqrt{ab}}{c}} = 6\sqrt[3]{\frac{abc}{abc}} = 6\cdot 1 = 6$$

Equality holds for $a = b = c = 1$.

19. If $a, b > 0, ab = 1$ then:

$$\boldsymbol{a^{a^2+2b^2}\cdot b^{b^2+2a^2}\leq 1}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$a^{a^2+2b^2}\cdot b^{b^2+2a^2}\leq 1 \Rightarrow \ln\left(a^{a^2+2b^2}\cdot b^{b^2+2a^2}\right)\leq \ln 1$$

$$(a^2+2b^2)\ln a + (b^2+2a^2)\ln b \leq 0, \qquad \left(a^2+\frac{2}{a^2}\right)\ln a + \left(\frac{1}{a^2}+2a^2\right)\ln\frac{1}{a}\leq 0$$

$$a^2\ln a + \frac{2}{a^2}\ln a + \frac{1}{a^2}\ln a^{-1} + 2a^2\ln a^{-1}\leq 0, \qquad a^2\ln a + \frac{2}{a^2}\ln a - \frac{1}{a^2}\ln a - 2a^2\ln a \leq 0$$

$$\left(a^2+\frac{2}{a^2}-\frac{1}{a^2}-2a^2\right)\ln a\leq 0, \qquad \left(\frac{1}{a^2}-a^2\right)\ln a \leq 0 \Leftrightarrow \frac{a^4-1}{a^2}\ln a\geq 0$$

$$\frac{(a-1)(a+1)(a^2+1)}{a^2}\ln a \geq 0 \Leftrightarrow (a-1)\ln a\geq 0$$

If $a\geq 1 \Rightarrow a-1\geq 0; \ln a\geq 0 \Rightarrow (a-1)\ln a\geq 0$

If $0 < a\leq 1 \Rightarrow a-1\leq 0; \ln a\leq 0 \Rightarrow (a-1)\ln a\geq 0$

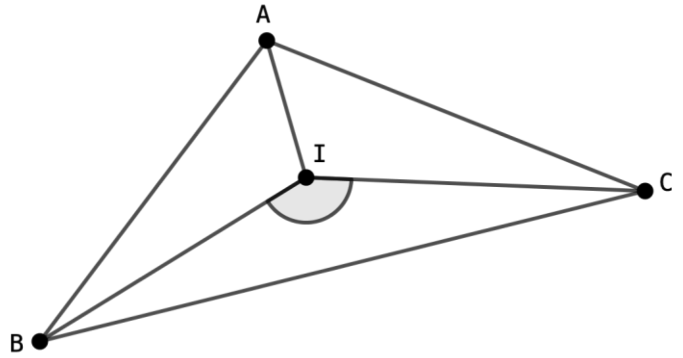
Equality holds for $a = b = 1$.

20. In $\Delta ABC, I$ – incenter, the following relationship holds:

$$\boldsymbol{\sin A+\sin B+\sin C = 4\sin(\widehat{AIB})\sin(\widehat{BIC})\sin(\widehat{CIA})}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania



$$\sphericalangle BIC = 180° - \frac{\hat{B}}{2} - \frac{\hat{C}}{2} = 180° - \frac{\hat{B}+\hat{C}}{2} =$$

$$= 180° - \frac{180° - \hat{A}}{2} = 180° - 90° + \frac{\hat{A}}{2} = 90° + \frac{\hat{A}}{2}$$

$$\sin(\widehat{BIC}) = \sin\left(90° + \frac{A}{2}\right) = \sin 90° \cos\frac{A}{2} + \sin\frac{A}{2}\cos 90° = \cos\frac{A}{2}$$

Analogous: $\sin(\widehat{AIB}) = \cos\frac{C}{2}; \sin(\widehat{CIA}) = \sin\frac{B}{2}$

$$\sin A + \sin B + \sin C = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2} + \sin\left(2\cdot\frac{C}{2}\right) =$$

$$= 2\sin\frac{180° - C}{2}\cos\frac{A-B}{2} + 2\sin\frac{C}{2}\cos\frac{C}{2} = 2\cos\frac{C}{2}\cos\frac{A-B}{2} + 2\cos\frac{C}{2}\cdot\cos\left(90° - \frac{C}{2}\right) =$$

$$= 2\cos\frac{C}{2}\left(\cos\frac{A-B}{2} + \cos\frac{180° - C}{2}\right) =$$

$$= 2\cos\frac{C}{2}\cdot 2\cos\frac{A-B+180°-C}{4}\cos\frac{A-B-180°+C}{4} =$$

$$= 4\cos\frac{C}{2}\cos\frac{A-B-C+A+B+C}{4}\cos\frac{A-B-A-B-C+C}{4} =$$

$$= 4\cos\frac{C}{2}\cos\frac{2A}{4}\cos\left(-\frac{2B}{4}\right) = 4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4\sin(\widehat{AIB})\sin(\widehat{BIC})\sin(\widehat{CIA})$$

21. In ΔABC the following relationship holds:

$$\mathbf{\sin 2026A + \sin 2026B + \sin 2026C = 4\sin 1013A \sin 1013B \sin 1013C}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Denote:

$$1013A = x; 1013B = y; 1013C = z$$

$$x + y + z = 1013(A+B+C) = 1013\pi \quad (1)$$

$$\cos\left(\frac{1013\pi}{2} - y\right) = \cos\frac{1013\pi}{2}\cos y + \sin\frac{1013\pi}{2}\sin y =$$

$$= \cos\left(506\pi + \frac{\pi}{2}\right)\cos y + \sin\left(506\pi + \frac{\pi}{2}\right)\sin y = \cos\frac{\pi}{2}\cos y + \sin\frac{\pi}{2}\sin y = \sin y$$

$$\cos\left(\frac{1013\pi}{2} - y\right) = \sin y \quad (2)$$

Analogous:

$$\cos\left(x - \frac{1013\pi}{2}\right) = \cos\left(\frac{1013\pi}{2} - x\right) = \sin x \quad (3)$$

$$\sin 2026A + \sin 2026B + \sin 2026C = \sin 2x + \sin 2y + \sin 2z =$$

$$= 2\sin\frac{2x+2y}{2}\cos\frac{2x-2y}{2} + 2\sin z\cos z = 2\sin(x+y)\cos(x-y) + 2\sin z\cos z \stackrel{(1)}{=}$$

$$= 2\sin(1013\pi - z)\cos(x-y) + 2\sin z\cos z =$$

$$= 2(\sin 1013\pi\cos z - \sin z\cos 1013\pi)\cos(x-y) + 2\sin z\cos z =$$

$$= 2(0\cdot\cos z - \sin z\cdot(-1)^{1013})\cos(x-y) + 2\sin z\cos z =$$

$$= 2\sin z\cos(x-y) + 2\sin z\cos z = 2\sin z\,(\cos(x-y) + \cos z) =$$

$$= 2\sin z\cdot 2\cos\frac{x-y+z}{2}\cdot\cos\frac{x-y-z}{2} = 4\sin z\cos\frac{x+z-y}{2}\cos\frac{y+z-x}{2} \stackrel{(1)}{=}$$

$$= 4\sin z\cos\left(\frac{1013\pi-2y}{2}\right)\cos\left(\frac{1013\pi-2x}{2}\right) \stackrel{(2),(3)}{=} = 4\sin z\sin y\sin x =$$

$$= 4\sin 1013A\sin 1013B\sin 1013C$$

22. If in ΔABC we have: $\tan\frac{A}{2}\tan\frac{B}{2} = \frac{1}{3}$ then:

$$\mathbf{2\sin C = \sin A + \sin B}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\tan\frac{A}{2}\tan\frac{B}{2} = \frac{1}{3} \Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \frac{1}{3}$$

$$\Rightarrow \frac{s-c}{s} = \frac{1}{3} \Rightarrow 3s - 3c = s \Rightarrow 2s - 3c = 0 \Rightarrow a+b+c-3c = 0 \Rightarrow 2c = a+b$$

$$2\cdot 2R\sin C = 2R\sin A + 2R\sin B, \qquad 2\sin C = \sin A + \sin B$$

23. In ΔABC the following relationship holds:

$$\mathbf{\cot A + \cot B + \cot C = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2\sin A\sin B\sin C}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\cot A + \cot B + \cot C = \sum_{cyc}\cot A = \sum_{cyc}\frac{\cos A}{\sin A} =$$

$$= \sum_{cyc}\frac{\frac{b^2+c^2-a^2}{2bc}}{\sin A} = \sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A} = \sum_{cyc}\frac{b^2+c^2-a^2}{4F} =$$

$$=\frac{1}{4F}\left(\sum_{cyc}b^2+\sum_{cyc}c^2-\sum_{cyc}a^2\right)=\frac{1}{4F}\left(\sum_{cyc}a^2+\sum_{cyc}a^2-\sum_{cyc}a^2\right)=\frac{1}{4F}\sum_{cyc}a^2=$$

$$=\frac{1}{4F}(a^2+b^2+c^2)=\frac{1}{4\cdot 2R^2\sin A\sin B\sin C}\sum_{cyc}4R^2\sin^2 A=$$

$$=\frac{4R^2}{8R^2}\cdot\frac{\sin^2 A+\sin^2 B+\sin^2 C}{\sin A\sin B\sin C}=\frac{\sin^2 A+\sin^2 B+\sin^2 C}{2\sin A\sin B\sin C}$$

24. In ΔABC the following relationship holds:

$$\frac{a}{\sin^3 B}+\frac{b}{\sin^3 C}+\frac{c}{\sin^3 A}\geq 8R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a}{\sin^3 B}+\frac{b}{\sin^3 C}+\frac{c}{\sin^3 A}=\sum_{cyc}\frac{a}{\sin^3 B}\overset{AM-GM}{\geq}3\cdot\sqrt[3]{\frac{abc}{\sin^3 A\sin^3 B\sin^3 C}}=$$

$$=3\sqrt[3]{\frac{abc}{\left(\frac{abc}{2R}\right)^3}}=3\sqrt[3]{\frac{(8R^3)^3}{(abc)^2}}=\frac{24R^3}{\sqrt[3]{(abc)^2}}=\frac{24R^3}{\sqrt[3]{16R^2r^2s^2}}\overset{EULER}{\geq}\frac{24R^3}{\sqrt[3]{16R^2\cdot\frac{R^2}{4}s^2}}\overset{MITRINOVIC}{\geq}$$

$$\geq\frac{24R^3}{\sqrt[3]{4R^4\cdot\frac{27R^2}{4}}}=\frac{24R^3}{3R^2}=8R$$

Equality holds for $a=b=c$.

25. In ΔABC the following relationship holds:

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}\leq\frac{3\sqrt{3}}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}=\sum_{cyc}\frac{h_a}{b}=\sum_{cyc}\frac{\frac{2F}{a}}{b}=2F\sum_{cyc}\frac{1}{ab}=2F\cdot\frac{a+b+c}{abc}=$$

$$=2F\cdot\frac{2s}{4RF}=\frac{s}{R}\overset{MITRINOVICI}{\leq}\frac{1}{R}\cdot\frac{3\sqrt{3}}{2}R=\frac{3\sqrt{3}}{2}$$

Equality holds for $a=b=c$.

26. In ΔABC the following relationship holds:

$$\frac{a+b}{\sin A}+\frac{b+c}{\sin B}+\frac{c+a}{\sin C}\geq 12R$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{\sin A}+\frac{b+c}{\sin B}+\frac{c+a}{\sin C}=\sum_{cyc}\frac{a+b}{\sin A}=\sum_{cyc}\frac{a+b}{\frac{a}{2R}}=2R\sum_{cyc}\frac{a+b}{a}\overset{AM-GM}{\geq}$$

$$\geq 2R\cdot 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{abc}}\overset{CESARO}{\geq}6R\cdot\sqrt[3]{\frac{8abc}{abc}}=6R\sqrt[3]{8}=6R\cdot 2=12R$$

Equality holds for $a=b=c$.

27. In ΔABC the following relationship holds:

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}\leq\frac{3\sqrt{3}}{4}\cdot\frac{R}{r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}=\sum_{cyc}\frac{h_a}{b}=\sum_{cyc}\frac{\frac{2F}{a}}{b}=2F\sum_{cyc}\frac{1}{ab}=2F\sum_{cyc}\frac{c}{abc}=$$

$$=\frac{2F}{abc}\sum_{cyc}c=\frac{2F}{4RF}(a+b+c)=\frac{1}{2R}\cdot 2s=\frac{s}{r}\overset{EULER}{\leq}\frac{s}{2r}\overset{MITRINOVIC}{\leq}\frac{3\sqrt{3}}{2}R\cdot\frac{1}{2r}=\frac{3\sqrt{3}}{4}\cdot\frac{R}{r}$$

Equality holds for $a=b=c$.

28. In ΔABC the following relationship holds:

$$r_ar_b+r_br_c+r_cr_a\geq h_ah_b+h_bh_c+h_ch_a$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc}r_ar_b\geq\sum_{cyc}h_ah_b$$

$$\sum_{cyc}\frac{F}{s-a}\cdot\frac{F}{s-b}\geq\sum_{cyc}\frac{2F}{a}\cdot\frac{2F}{b}$$

$$\sum_{cyc}\frac{1}{(s-a)(s-b)}\geq 4\sum_{cyc}\frac{1}{ab},\quad\frac{s-c+s-b+s-a}{(s-a)(s-b)(s-c)}\geq 4\cdot\frac{a+b+c}{abc}$$

$$\frac{s}{(s-a)(s-b)(s-c)} \ge 4 \cdot \frac{2s}{4RF}, \quad \frac{s^2}{s(s-a)(s-b)(s-c)} \ge \frac{2s}{R \cdot rs}$$

$$\frac{s^2}{F^2} \ge \frac{2}{Rr}, \quad \frac{s^2}{r^2s^2} \ge \frac{2}{Rr}, \quad \frac{1}{r^2} \ge \frac{2}{Rr}$$

$$\Leftrightarrow R \ge 2r \quad \text{(Euler)}$$

Equality holds for $a = b = c$.

29. In ΔABC the following relationship holds:

$$\frac{r_a r_b r_c}{h_a h_b h_c} = \frac{R}{2r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a r_b r_c}{h_a h_b h_c} = \frac{\frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}}{\frac{2F}{a} \cdot \frac{2F}{b} \cdot \frac{2F}{c}} = \frac{abc}{8(s-a)(s-b)(s-c)} =$$

$$= \frac{abcs}{8s(s-a)(s-b)(s-c)} = \frac{4RFs}{8F^2} = \frac{4Rr}{8F} = \frac{4Rs}{8rs} = \frac{R}{2r}$$

30. In ΔABC the following relationship holds:

$$216r^3 \le (h_a + h_b)(h_b + h_c)(h_c + h_a) \le 27R^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(h_a + h_b)(h_b + h_c)(h_c + h_a) = \prod_{cyc}(h_a + h_b) = \prod_{cyc}\left(\frac{2F}{a} + \frac{2F}{b}\right) = 8F^3 \prod_{cyc}\left(\frac{1}{a} + \frac{1}{b}\right) =$$

$$= 8F^3 \prod_{cyc}\left(\frac{a+b}{ab}\right) = 8F^3 \cdot \frac{(a+b)(b+c)(c+a)}{abc \cdot abc} \overset{CESARO}{\ge}$$

$$\ge 8F^3 \cdot \frac{8abc}{abc \cdot abc} = \frac{64F^3}{abc} = \frac{64F^3}{4RF} = \frac{16F^2}{R} = \frac{16r^2s^2}{R}$$

Remains to prove:

$$216r^3 \le \frac{16r^2s^2}{R} \le 27R^3$$

$$\frac{16r^2s^2}{R} \overset{EULER}{\le} \frac{16 \cdot \left(\frac{R}{2}\right)^2 \cdot s^2}{R} = 4Rs^2 \overset{MITRINOVIC}{\le} 4R \cdot \left(\frac{3\sqrt{3}}{2}R\right)^2 = 4R \cdot \frac{27R^2}{4} = 27R^3$$

$$\frac{16r^2s^2}{R} \ge 216r^3 \Leftrightarrow \frac{2r^2s^2}{R} \ge 27r^3 \Leftrightarrow \frac{2s^2}{R} \ge 27r \Leftrightarrow 2s^2 \ge 27Rr \quad \text{(to prove)}$$

$$2s^2 \overset{GERRESTEN}{\geq} 2(16Rr-5r^2)=32Rr-10r^2 \geq 27Rr \Leftrightarrow 5Rr \geq 10r^2 \Leftrightarrow R \geq 2r \quad \text{(Euler)}$$

Equality holds for $a=b=c$.

31. In ΔABC the following relationship holds:

$$\mathbf{216r^3 \leq (r_a+r_b)(r_b+r_c)(r_c+r_a) \leq 27R^3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(r_a+r_b)(r_b+r_c)(r_c+r_a)=\prod_{cyc}(r_a+r_b)=$$

$$=\prod_{cyc}\left(\frac{F}{s-a}+\frac{F}{s-b}\right)=F^3\prod_{cyc}\left(\frac{1}{s-a}+\frac{1}{s-b}\right)=$$

$$=F^3\prod_{cyc}\frac{s-b+s-a}{(s-a)(s-b)}=F^3\prod_{cyc}\frac{2s-a-b}{(s-a)(s-b)}=$$

$$=F^3\prod_{cyc}\frac{a+b+c-a-b}{(s-a)(s-b)}=F^3\prod_{cyc}\frac{c}{(s-a)(s-b)}=$$

$$=F^3\cdot\frac{abc}{(s-a)^2(s-b)^2(s-c)^2}=\frac{F^3\cdot abc\cdot s^2}{s^2(s-a)^2(s-b)^2(s-c)^2}=\frac{F^3\cdot 4RF\cdot s^2}{F^4}=4Rs^2$$

Remains to prove:

$$216r^3 \leq 4Rs^2 \leq 27R^3$$

$$4Rs^2 \overset{MITRINOVIC}{\leq} 4R\cdot\left(\frac{3\sqrt{3}}{2}R\right)^2=4R\cdot\frac{27R^2}{4}=27R^3$$

$$4Rs^2 \overset{MITRINOVIC}{\geq} 4R\cdot\left(3\sqrt{3}r\right)^2=4R\cdot 27r^2 \overset{EULER}{\geq} 4\cdot 2r\cdot 27r^2=216r^3$$

Equality holds for $a=b=c$.

32. In ΔABC the following relationship holds:

$$\mathbf{(h_a+h_b)^{\cos\frac{C}{2}}\cdot(h_b+h_c)^{\cos\frac{A}{2}}\cdot(h_c+h_a)^{\cos\frac{B}{2}} \leq (3R)^{\frac{3\sqrt{3}}{2}}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

WLOG: $a \leq b \leq c \Rightarrow \frac{1}{a} \geq \frac{1}{b} \geq \frac{1}{c} \Rightarrow \frac{2F}{a} \geq \frac{2F}{b} \geq \frac{2F}{c} \Rightarrow h_a \geq h_b \geq h_c \Rightarrow \begin{cases} h_a+h_c \geq h_b+h_c \\ h_b+h_a \geq h_c+h_a \end{cases} \Rightarrow$

$$\Rightarrow h_a+h_b \geq h_a+h_c \geq h_b+h_c \quad (1)$$

$$a\le b\le c\Rightarrow \cos\frac{A}{2}\ge\cos\frac{B}{2}\cos\frac{C}{2}\Rightarrow\Rightarrow\cos\frac{C}{2}\le\cos\frac{B}{2}\le\cos\frac{A}{2}\quad (2)$$

By (1); (2) and Cebyshev's inequality:

$$(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{A}{2}\le$$

$$\le\frac{1}{3}(h_a+h_b+h_b+h_c+h_c+h_a)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)=$$

$$=\frac{2}{3}(h_a+h_b+h_c)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)\quad (3)$$

$$\prod_{cyc}(h_a+h_b)\cos\frac{C}{2}\overset{WEIGHTED\ AM-GM}{\le}$$

$$\le\left(\frac{(h_a+h_b)\cos\frac{C}{2}+(h_b+h_c)\cos\frac{A}{2}+(h_c+h_a)\cos\frac{B}{2}}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{(3)}{\le}\left(\frac{\frac{2}{3}(h_a+h_b+h_c)\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)}{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\right)^{\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}}\le$$

$$\overset{Jensen}{\le}\left(\frac{2}{3}\left(\frac{2F}{a}+\frac{2F}{b}+\frac{2F}{c}\right)\right)^{3\cos\frac{A+B+C}{6}}=\left(\frac{4F}{3}\cdot\frac{ab+bc+ca}{abc}\right)^{3\cos\frac{\pi}{6}}=$$

$$=\left(\frac{4F}{3}\cdot\frac{s^2+r^2+4Rr}{4RF}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{s^2+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\overset{MITRINOVIC}{\le}\left(\frac{\frac{27R^2}{4}+r^2+4Rr}{3R}\right)^{\frac{3\sqrt{3}}{2}}\overset{EULER}{\le}$$

$$\le\left(\frac{\frac{27R^2}{4}+\frac{R^2}{4}+4R\cdot\frac{R}{2}}{3R}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{7R^2+2R^2}{3R}\right)^{\frac{3\sqrt{3}}{2}}=(3R)^{\frac{3\sqrt{3}}{2}}$$

Equality holds for $a=b=c$.

33. In ΔABC the following relationship holds:

$$(ab)^{\sin C}\cdot(bc)^{\sin A}\cdot(ca)^{\sin B}\le(3R^2)^{\frac{3\sqrt{3}}{2}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(ab)^{\sin C}\cdot(bc)^{\sin A}\cdot(ca)^{\sin B}\overset{WEIGHTED\ AM-GM}{\le}\left(\frac{ab\sin C+bc\sin A+ca\sin B}{\sin A+\sin B+\sin C}\right)^{\sin A+\sin B+\sin C}=$$

$$=\left(\frac{2F+2F+2F}{\frac{s}{R}}\right)^{\frac{s}{R}}\overset{MITRINOVIC}{\leq}\left(\frac{6F}{\frac{s}{R}}\right)^{\frac{1}{R}\cdot\frac{3\sqrt{3}}{2}R}=\left(\frac{6FR}{s}\right)^{\frac{3\sqrt{3}}{2}}=\left(\frac{6rsR}{s}\right)^{\frac{3\sqrt{3}}{2}}=$$

$$=(6rR)^{\frac{3\sqrt{3}}{2}}\leq\left(6\cdot\frac{R}{2}\cdot R\right)^{\frac{3\sqrt{3}}{2}}=(3R^2)^{\frac{3\sqrt{3}}{2}}$$

Equality holds for $a=b=c$.

34. In ΔABC the following relationship holds:

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}\geq 9$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a+2r_b}{h_c}+\frac{r_b+2r_c}{h_a}+\frac{r_c+2r_a}{h_b}=\sum_{cyc}\frac{r_a+2r_b}{h_c}=\sum_{cyc}\frac{r_a}{h_c}+2\sum_{cyc}\frac{r_b}{h_c}\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}+2\cdot 3\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}=9\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\geq 9\Leftrightarrow\sqrt[3]{\frac{r_ar_br_c}{h_ah_bh_c}}\geq 1\Leftrightarrow r_ar_br_c\geq h_ah_bh_c$$

$$\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\geq\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c},\qquad abc\geq 8(s-a)(s-b)(s-c)$$

$$abc\cdot s\geq 8s(s-a)(s-b)(s-c),\qquad 4RF\cdot s\geq 8F^2$$

$$Rs\geq 2F,\ Rs\geq 2rs,\ R\geq 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.

35. If $a,b,c>0; abc=1$ then:

$$(a^3+1)^a(b^3+1)^b(c^3+1)^c\geq 8$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$(a^3+1)^a(b^3+1)^b(c^3+1)^c\geq 8$$

$$\ln[(a^3+1)^a(b^3+1)^b(c^3+1)^c]\geq\ln 8$$

$$a\ln(a^3+1)+b\ln(b^3+1)+c\ln(c^3+1)\geq 3\ln 2\quad\text{(to prove)}$$

Let be $f:(0,\infty)\to\mathbb{R}; f(x)=x\ln(1+x^3)$

$$f'(x)=\ln(1+x^3)+x\cdot\frac{3x^2}{1+x^3}=\ln(1+x^3)+\frac{3x^3}{1+x^3}>0$$

$f'(x)>0\Rightarrow f$ increasing

$$f''(x)=\frac{3x^2}{1+x^3}+\frac{9x^2(1+x^3)-3x^3\cdot 3x^2}{(1+x^3)^2}=\frac{3x^2}{1+x^3}+\frac{9x^2}{(1+x^3)^2}>0$$

$$f''(x)>0\Rightarrow f \text{ convex}$$

$$a\ln(a^3+1)+b\ln(b^3+1)+c\ln(c^3+1)=$$

$$=f(a)+f(b)+f(c)\overset{JENSEN}{\geq}3f\left(\frac{a+b+c}{3}\right)\overset{AM-GM}{\geq}3f\left(\sqrt[3]{abc}\right)=3f\left(\sqrt[3]{1}\right)=3f(1)=3\ln 2$$

Equality holds for $a=b=c=1$.

36. In ΔABC the following relationship holds:

$$\frac{(a+b)^2}{h_c}+\frac{(b+c)^2}{h_a}+\frac{(c+a)^2}{h_b}\geq 48r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(a+b)^2}{h_c}+\frac{(b+c)^2}{h_a}+\frac{(c+a)^2}{h_b}=\sum_{cyc}\frac{(a+b)^2}{h_c}\geq$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\left((a+b)(b+c)(c+a)\right)^2}{h_ah_bh_c}}\overset{CESARO}{\geq}3\sqrt[3]{\frac{(8abc)^2}{h_ah_bh_c}}=3\sqrt[3]{\frac{64\cdot(4RF)^2}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=$$

$$=\frac{3}{2F}\cdot 4\sqrt[3]{(4RF)^2\cdot abc}=\frac{6}{F}\sqrt[3]{(4RF)^3}=\frac{6}{F}\cdot 4RF=24R\overset{EULER}{\geq}24\cdot 2r=47r$$

Equality holds for $a=b=c$.

37. In ΔABC the following relationship holds:

$$\frac{(r_a+r_b)^2}{h_c}+\frac{(r_b+r_c)^2}{h_a}+\frac{(r_c+r_a)^2}{h_b}\geq 36r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(r_a+r_b)^2}{h_c}+\frac{(r_b+r_c)^2}{h_a}+\frac{(r_c+r_a)^2}{h_b}=\sum_{cyc}\frac{(r_a+r_b)^2}{h_c}\geq$$

$$\overset{AM-GM}{\geq}3\sqrt[3]{\frac{\left((r_a+r_b)(r_b+r_c)(r_c+r_a)\right)^2}{h_ah_bh_c}}\overset{CESARO}{\geq}3\sqrt[3]{\frac{(8r_ar_br_c)^2}{h_ah_bh_c}}=3\sqrt[3]{\frac{64\cdot\left(\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}\right)^2}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=$$

$$=12\sqrt[3]{\frac{F^6}{\left((s-a)(s-b)(s-c)\right)^2}\cdot\frac{abc}{8F^3}}=\frac{12}{2}\sqrt[3]{\frac{F^3\cdot 4RF\cdot s^2}{\left(s(s-a)(s-b)(s-c)\right)^2}}=6\sqrt[3]{\frac{4Rs^2F^4}{F^4}}=$$

$$= 6\sqrt[3]{4Rs^2} \overset{MITRINOVIC}{\geq} 6\sqrt[3]{4R \cdot 27r^2} \overset{EULER}{\geq} 6 \cdot 3\sqrt[3]{8r \cdot r^2} = 36r$$

Equality holds for $a = b = c$.

38. In ΔABC the following relationship holds:

$$\frac{(a+b)^2}{r_c} + \frac{(b+c)^2}{r_a} + \frac{(c+a)^2}{r_b} \geq 48r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{(a+b)^2}{r_c} + \frac{(b+c)^2}{r_a} + \frac{(c+a)^2}{r_b} = \sum_{cyc} \frac{(a+b)^2}{r_c} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{\left((a+b)(b+c)(c+a)\right)^2}{r_a r_b r_c}} \overset{CESARO}{\geq}$$

$$\geq 3\sqrt[3]{\frac{(8abc)^2}{\frac{F}{s-a} \cdot \frac{F}{s-b} \cdot \frac{F}{s-c}}} = 3\sqrt[3]{\frac{64(abc)^2}{\frac{F^3 s}{F^2}}} = 12\sqrt[3]{\frac{(4RF)^2}{Fs}} = 12\sqrt[3]{\frac{16R^2F^2}{Fs}} =$$

$$= 12\sqrt[3]{\frac{16R^2F}{s}} = 12\sqrt[3]{\frac{16R^2rs}{s}} = 12\sqrt[3]{16R^2r} \overset{EULER}{\geq} 12\sqrt[3]{16 \cdot 4r^2 \cdot r} = 12\sqrt[3]{64r^3} = 12 \cdot 4r = 48r$$

Equality holds for $a = b = c$.

39. In ΔABC the following relationship holds:

$$\frac{r_a + r_b}{h_c} + \frac{r_b + r_c}{h_a} + \frac{r_c + r_a}{h_b} \leq \frac{3R}{r}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sum_{cyc} \frac{r_a + r_b}{h_c} = \sum_{cyc} \frac{\frac{F}{s-a} + \frac{F}{s-b}}{\frac{2F}{c}} = \sum_{cyc} \frac{\frac{1}{s-a} + \frac{1}{s-b}}{\frac{2}{c}} =$$

$$= \frac{1}{2}\sum_{cyc} \frac{c(s-a+s-b)}{(s-a)(s-b)} = \frac{1}{2}\sum_{cyc} \frac{c(a+b+c-a-b)}{(s-a)(s-b)} =$$

$$= \frac{1}{2}\sum_{cyc} \frac{c^2}{(s-a)(s-b)} = \frac{1}{2(s-a)(s-b)(s-c)}\sum_{cyc} c^2(s-c) =$$

$$= \frac{s}{2s(s-a)(s-b)(s-c)} \cdot 4rs(R+r) = \frac{4rs^2(R+r)}{2F^2} = \frac{2rs^2(R+r)}{r^2s^2} = \frac{2(R+r)}{r} \leq$$

$$\leq \frac{2\left(R + \frac{R}{2}\right)}{r} = \frac{2 \cdot \frac{3R}{2}}{r} = \frac{3R}{r}$$

Equality holds for $a=b=c$.

40. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}\ge 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{m_a^2+m_b^2}{c}+\frac{m_b^2+m_c^2}{a}+\frac{m_c^2+m_a^2}{b}=$$

$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c}=$$

$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c}+\sum_{cyc}c\ge$$

$$\overset{BERGSTROM}{\ge}\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+\frac{1}{4}\cdot\frac{(a+b+c)^2}{a+b+c}+2s=$$

$$=\frac{a+b+c}{4}+\frac{a+b+c}{4}+2s=\frac{2s}{4}+\frac{2s}{4}+2s=\frac{4s}{4}+2s=3s\overset{MITRINOVIC}{\ge}3\cdot 3\sqrt{3}r=9\sqrt{3}r$$

Equality holds for $a=b=c$.

41. In ΔABC the following relationship holds:

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}\ge\frac{9}{2}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{m_a^2+m_b^2}{c^2}+\frac{m_b^2+m_c^2}{a^2}+\frac{m_c^2+m_a^2}{b^2}=$$

$$=\sum_{cyc}\frac{m_a^2+m_b^2}{c^2}=\sum_{cyc}\frac{2(b^2+c^2)-a^2+2(a^2+c^2)-b^2}{4c^2}=$$

$$=\sum_{cyc}\frac{b^2+a^2+4c^2}{4c^2}=\frac{1}{4}\sum_{cyc}\frac{b^2}{c^2}+\frac{1}{4}\sum_{cyc}\frac{a^2}{c^2}+\sum_{cyc}\frac{4c^2}{4c^2}\ge$$

$$\overset{AM-GM}{\ge}\frac{1}{4}\cdot 3\sqrt[3]{\frac{b^2}{c^2}\cdot\frac{c^2}{a^2}\cdot\frac{a^2}{b^2}}+\frac{1}{4}\cdot 3\sqrt[3]{\frac{a^2}{c^2}\cdot\frac{b^2}{a^2}\cdot\frac{c^2}{b^2}}+3=\frac{3}{4}+\frac{3}{4}+3=\frac{6}{4}+3=\frac{3}{2}+3=\frac{9}{2}$$

42. In ΔABC the following relationship holds:

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}\ge 9\sqrt{3}r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a^2+r_b^2}{c}+\frac{r_b^2+r_c^2}{a}+\frac{r_c^2+r_a^2}{b}=\sum_{cyc}\frac{r_a^2+r_b^2}{c}=\sum_{cyc}\frac{r_a^2}{c}+\sum_{cyc}\frac{r_b^2}{c}\overset{BERGSTROM}{\ge}$$

$$\ge\frac{(r_a+r_b+r_c)^2}{a+b+c}+\frac{(r_a+r_b+r_c)^2}{a+b+c}=\frac{2(r_a+r_b+r_c)^2}{a+b+c}=$$

$$=\frac{2(4R+r)^2}{2s}=\frac{(4R+r)^2}{s}\overset{DOUCET}{\ge}\frac{(4R+r)^2}{\frac{4R+r}{\sqrt{3}}}=\sqrt{3}(4R+r)\overset{EULER}{\ge}\sqrt{3}(4\cdot 2r+r)=9\sqrt{3}r$$

Equality holds for $a=b=c$.

43. In ΔABC the following relationship holds:

$$\frac{a^2+b^2}{r_c}+\frac{b^2+c^2}{r_a}+\frac{c^2+a^2}{r_b}\ge 24r$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a^2+b^2}{r_c}+\frac{b^2+c^2}{r_a}+\frac{c^2+a^2}{r_b}=\sum_{cyc}\frac{a^2+b^2}{r_c}\overset{AM-GM}{\ge}\sum_{cyc}\frac{2ab}{r_c}=2\sum_{cyc}\frac{ab}{r_c}\overset{AM-GM}{\ge}$$

$$=2\cdot 3\sqrt[3]{\frac{(abc)^2}{r_ar_br_c}}=6\sqrt[3]{\frac{16r^2F^2}{Fs}}=6\cdot 2\sqrt[3]{\frac{2R^2F}{s}}=12\sqrt[3]{\frac{2R^2rs}{s}}=12\sqrt[3]{2R^2r}\ge$$

$$\ge 12\sqrt[3]{2\cdot 4r^2\cdot r}=12\cdot 2r=24r$$

Equality holds for $a=b=c$.

44. In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}\ge 288r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$r_ar_br_c=\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}=\frac{F^3s}{s(s-a)(s-b)(s-c)}=\frac{F^3s}{F^2}=Fs$$

$$\frac{a^4+b^4}{r_c}+\frac{b^4+c^4}{r_a}+\frac{c^4+a^4}{r_b}=\sum_{cyc}\frac{a^4+b^4}{r_c}\overset{AM-GM}{\geq}2\sum_{cyc}\frac{a^2b^2}{r_c}\overset{AM-GM}{\geq}2\cdot 3\cdot\sqrt[3]{\frac{(abc)^4}{r_ar_br_c}}=$$

$$=6abc\cdot\sqrt[3]{\frac{abc}{Fs}}=6\cdot 4RF\cdot\sqrt[3]{\frac{4RF}{Fs}}=24RF\sqrt[3]{\frac{4R}{s}}\overset{MITRINOVIC}{\geq}24RF\cdot\sqrt[3]{\frac{4}{s}\cdot\frac{2}{3\sqrt{3}}s}=$$

$$=24Rrs\cdot\frac{2}{\sqrt{3}}\overset{MITRINOVIC}{\geq}24R\cdot 3\sqrt{3}r\cdot\frac{2}{\sqrt{3}}=24\cdot 6Rr\overset{EULER}{\geq}144\cdot 2r\cdot r=288r^3$$

Equality holds for $a=b=c$.

ABOUT A RMM INEQUALITY-X

By Marin Chirciu-Romania

1) If $x,y,z>0$ such that $\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=3$ then:

$$4\sum x^3+7\left(\sum x^2+\sum\frac{1}{x^2}\right)\geq 54$$

Proposed by Daniel Sitaru-Romania

Solution Using means inequality AM-AH we obtain:

$$\sum x^3\geq\frac{9}{\sum\frac{1}{x^3}}=\frac{9}{3}=3 \quad (1)$$

$$\sum x^2+\sum\frac{1}{x^2}=\sum\left(x^2+\frac{1}{x^2}\right)\geq\sum 2=6 \quad (2)$$

From (1) and (2) we obtain the conclusion:

$$Ms=4\sum x^3+7\left(\sum x^2+\sum\frac{1}{x^2}\right)\geq 4\cdot 3+7\cdot 6=54$$

Equality holds if and only if $x=y=z$.

Remark: The problem can be developed.

2) If $x,y,z>0$ such that $\frac{1}{x^3}+\frac{1}{y^3}+\frac{1}{z^3}=3$ and $\lambda\geq 0,\mu\geq 0$ then:

$$\lambda\sum x^3+\mu\left(\sum x^2+\sum\frac{1}{x^2}\right)\geq 3(\lambda+2\mu)$$

Marin Chirciu

Solution Using the means inequality AM-AH (1) and AM-GM (2) we obtain:

$$\sum x^3 \geq \frac{9}{\sum\frac{1}{x^3}} = \frac{9}{3} = 3 \ \ (1)$$

$$\sum x^2 + \sum\frac{1}{x^2} = \sum\left(x^2 + \frac{1}{x^2}\right) \geq \sum 2 = 6 \ \ (2)$$

From (1) and (2) we obtain the conclusion:

$$Ms = \lambda\sum x^3 + \mu\left(\sum x^2 + \sum\frac{1}{x^2}\right) \geq \lambda\cdot 3 + \mu\cdot 6 = 3\lambda + 6\mu = 3(\lambda + 2\mu)$$

Equality holds if and only if $x = y = z$ and $\lambda = \mu = 0$

Note: For $\lambda = 4$ and $\mu = 7$ we obtain Cyclic Inequality – 857 from RMM 10/2020, proposed by Daniel Sitaru, Romania

3) If $x, y, z > 0$ such that $\frac{1}{x^n} + \frac{1}{y^n} + \frac{1}{z^n} = 3$ and $\lambda \geq 0, \mu \geq 0, n, k \in \mathbb{R}$ then:

$$\lambda\sum x^n + \mu\left(\sum x^k + \sum\frac{1}{x^k}\right) \geq 3(\lambda + 2\mu)$$

Marin Chirciu

Solution: Using means inequality AM-AH (1) and AM-GM (2) we obtain:

$$\sum x^n \geq \frac{9}{\sum\frac{1}{x^n}} = \frac{9}{3} = 3 \ \ (1)$$

$$\sum x^k + \sum\frac{1}{x^k} = \sum\left(x^k + \frac{1}{x^k}\right) \geq \sum 2 = 6 \ \ (2)$$

From (1) and (2) we obtain the conclusion:

$$Ms = \lambda\sum x^n + \mu\left(\sum x^k + \sum\frac{1}{x^k}\right) \geq \lambda\cdot 3 + \mu\cdot 6 = 3\lambda + 6\mu = 3(\lambda + 2\mu)$$

Equality holds if and only if $x = y = z$ or $\lambda = \mu = 0$ or $n = k = 0$

Note: For $\lambda = 4$ and $\mu = 7, n = 3$ and $k = 2$ we obtain Cyclic Inequality – 857 from RMM 10/2020, proposed by Daniel Sitaru, Romania

ABOUT A RMM INEQUALITY-XV

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$4(a\cdot m_b m_c + b\cdot m_c m_a + c\cdot m_a m_b) \geq 9abc$$

Proposed by Daniel Sitaru-Romania

Solution. Lemma. **2) In ΔABC the following relationship holds:**

$$\boldsymbol{a\cdot m_bm_c+b\cdot m_cm_a+c\cdot m_am_b\ge \frac{2}{3}s^3}$$

Proof. The triplets (a,b,c) and (m_bm_c,m_cm_a,m_am_b) are same ordered, from Chebyshev's inequality and form well-known inequality $\sum m_bm_c\ge s^2$, we get:

$$a\cdot m_bm_c+b\cdot m_cm_a+c\cdot m_am_b\ge \frac{1}{3}\left(\sum a\right)\left(\sum m_bm_c\right)\ge \frac{1}{3}\cdot 2s\cdot s^2=\frac{2}{3}s^3$$

Let's get back to the main problem. Using Lemma it is enough to prove that

$4\cdot\frac{2}{3}s^3\ge 9abc\Leftrightarrow \frac{8}{3}s^3\ge 9\cdot 4Rrs\Leftrightarrow 2s^2\ge 27Rr$ (Coșniță − Turtoiu. 1965), which follows from Gerretsen inequality $s^2\ge 16Rr-5r^2$ and Euler inequality $R\ge 2r$.

Equality holds if and only if triangle is equilateral.

Second solution. Hayashi-Cocea's inequality:

Let P be point in plane of ΔABC. If $PA=x, PB=y, PC=z$ then

$$\boldsymbol{a\cdot yz+b\cdot zx+c\cdot xy\ge abc}$$

Putting $P=G$, we get: $a\cdot GB\cdot GC+b\cdot GC\cdot GA+c\cdot GA\cdot GB\ge abc\Leftrightarrow$

$$a\cdot\frac{2}{3}m_b\cdot\frac{2}{3}m_c+b\cdot\frac{2}{3}m_c\cdot\frac{2}{3}m_a+c\cdot\frac{2}{3}m_a\cdot\frac{2}{3}m_b\ge abc\Leftrightarrow$$

$$4(a\cdot m_bm_c+b\cdot m_cm_a+c\cdot m_am_b)\ge 9abc$$

Equality holds if and only if triangle is equilateral.**Remark.** Let's find an opposite inequality.

3) In ΔABC the following relationship holds:

$$\boldsymbol{a\cdot m_bm_c+b\cdot m_cm_a+c\cdot m_am_b\le sR(4R+r)}$$

Marin Chirciu

Solution. Lemma. 4) **In ΔABC the following relationship holds:**

$$\boldsymbol{a\cdot m_bm_c+b\cdot m_cm_a+c\cdot m_am_b\le s(s^2-3r^2-3Rr)}$$

Proof. Using well-known inequality $m_bm_c\le\frac{2a^2+bc}{4}$, we get:

$$\sum a\cdot m_bm_c\le\sum a\cdot\frac{2a^2+bc}{4}=\frac{1}{4}(2\sum a^3+\sum abc)=$$

$$=\frac{1}{4}(2\cdot 2s(s^2-3r^2-6Rr)+3\cdot 4Rrs)=\frac{1}{4}\cdot 4s(s^2-3r^2-6Rr+3Rr)=$$

$=s(s^2-3r^2-3Rr)$. Let's get back to the main problem.

Using Lemma and Gerretsen inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$, we get: $\sum a \cdot m_b m_c \leq$

$$\leq s(s^2 - 3r^2 - 3Rr) \leq s(4R^2 + 4Rr + 3r^2 - 3r^2 - 3Rr) = s(4R^2 + Rr) = sR(4R + r)$$

Equality holds if and only if triangle is equilateral.

Remark. Inequality can be double.

5) In ΔABC the following relationship holds:

$$\frac{2}{3}s^3 \leq \sum_{cyc} a \cdot m_b m_c \leq sR(4R + r)$$

Marin Chirciu

Solution. See inequalities 1) and 3). Equality holds if and only if triangle is equilateral.

Remark. We can write:

6) In ΔABC the following relationship holds:

$$9Rrs \leq \frac{2}{3}s^3 \leq \sum_{cyc} a \cdot m_b m_c \leq s(s^2 - 3r^2 - 3Rr) \leq sR(4R + r)$$

Marin Chirciu

Solution. See up these inequalities. Equality holds if and only if triangle is equilateral.

References: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT A RMM INEQUALITY-XX

By Marin Chirciu-Romania

1) **If $a, b, c > 0$ then:**

$$\sum_{cyc} \frac{a^4}{b^4} \cdot \sum_{cyc} \frac{a^3}{b^3} \geq \left(\sum_{cyc} \frac{a^2}{b^2}\right)^2$$

Proposed by Daniel Sitaru-Romania

Solution. Using AM-GM and CBS Inequality, we get:

$$\sum_{cyc} \frac{a^4}{b^4} \cdot \sum_{cyc} \frac{a^3}{b^3} \geq \sum_{cyc} \frac{a^4}{b^4} \cdot 3\sqrt[3]{\prod_{cyc} \frac{a^3}{b^3}} = \sum_{cyc} \frac{a^4}{b^4} \cdot 3 = \sum_{cyc} \frac{a^4}{b^4} \cdot \sum_{cyc} 1^2 \geq$$

$$\geq \left(\sum_{cyc} \frac{a^2}{b^2} \cdot 1\right)^2 = \left(\sum_{cyc} \frac{a^2}{b^2}\right)^2$$

Equality holds if and only if $a=b=c$.

Remark. The problem can be developed.

2) If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}>\boldsymbol{0}$ and $\boldsymbol{n}\in\mathbb{N}^*$ then

$$\sum_{cyc}\frac{\boldsymbol{a}^{2n}}{\boldsymbol{b}^{2n}}\cdot\sum_{cyc}\frac{\boldsymbol{a}^{2n-1}}{\boldsymbol{b}^{2n-1}}\ge\left(\sum_{cyc}\frac{\boldsymbol{a}^{n}}{\boldsymbol{b}^{n}}\right)^{2}$$

Marin Chirciu

Solution. Using AM-GM and CBS Inequality, we get:

$$\sum_{cyc}\frac{a^{2n}}{b^{2n}}\cdot\sum_{cyc}\frac{a^{2n-1}}{b^{2n-1}}\ge\sum_{cyc}\frac{a^{2n}}{b^{2n}}\cdot 3\sqrt[3]{\prod_{cyc}\frac{a^{2n-1}}{b^{2n-1}}}=\sum_{cyc}\frac{a^{2n}}{b^{2n}}\cdot\sum_{cyc}1^2\ge$$

$$\ge\left(\sum_{cyc}\frac{a^n}{b^n}\cdot 1\right)^2=\left(\sum_{cyc}\frac{a^n}{b^n}\right)^2$$

Equality holds for $a=b=c$.

3) If $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}>\boldsymbol{0}$ and $\boldsymbol{n}\in\mathbb{N}^*$, $\boldsymbol{k}\in\mathbb{N}$ then

$$\sum_{cyc}\frac{\boldsymbol{a}^{2n}}{\boldsymbol{b}^{2n}}\cdot\sum_{cyc}\frac{\boldsymbol{a}^{k}}{\boldsymbol{b}^{k}}\ge\left(\sum_{cyc}\frac{\boldsymbol{a}^{n}}{\boldsymbol{b}^{n}}\right)^{2}$$

Marin Chirciu

Solution. Using AM-GM and CBS Inequality, we get:

$$\sum_{cyc}\frac{a^{2n}}{b^{2n}}\cdot\sum_{cyc}\frac{a^{k}}{b^{k}}\ge\sum_{cyc}\frac{a^{2n}}{b^{2n}}\cdot 3\sqrt[3]{\prod_{cyc}\frac{a^{k}}{b^{k}}}=\sum_{cyc}\frac{a^{2n}}{b^{2n}}\cdot 3=\sum_{cyc}\frac{a^{2n}}{b^{2n}}\cdot\sum_{cyc}1^2\ge$$

$$\ge\left(\sum_{cyc}\frac{a^n}{b^n}\cdot 1\right)^2=\left(\sum_{cyc}\frac{a^n}{b^n}\right)^2$$

Equality holds for $a=b=c$.

Note. For $n=2$ and $k=3$ we get the proposed problem by Daniel Sitaru-JP.348 from R.M.M.-24, Spring Edition 2022.

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT A RMM INEQUALITY-XXII

By Marin Chirciu-Romania

1) If $a, b, c > 0$ such that $a + b + c = 2$, then

$$\sum_{cyc} \frac{b^2 + bc + c^2}{b + c} \geq 3$$

Proposed by Daniel Sitaru-Romania

Solution. Lemma. **2) If $x, y > 0$ then**

$$\frac{x^2 + xy + y^2}{x + y} \geq \frac{3(x + y)}{4}$$

Proof. We have:

$$\frac{x^2 + xy + y^2}{x + y} \geq \frac{3(x + y)}{4} \Leftrightarrow (x - y)^2 \geq 0, \text{equality holds for } x = y.$$

Let's get back to the main problem. Using Lemma, we have:

$$LHS = \sum_{cyc} \frac{b^2 + bc + c^2}{b + c} \geq \sum_{cyc} \frac{3(b + c)}{4} = \frac{3}{2} \sum_{cyc} a = \frac{3}{2} \cdot 2 = 3$$

Equality holds for $a = b = c = \frac{2}{3}$.

Remark. The problem can be developed.

3) If $a, b, c \geq 0$ such that $a + b + c = 2$ and $\lambda \leq 2$ then

$$\sum_{cyc} \frac{b^2 + \lambda bc + c^2}{b + c} \geq \lambda + 2$$

Marin Chirciu

Solution. Lemma. **4) If $x, y > 0$ and $\lambda \leq 2$ then**

$$\frac{x^2 + \lambda xy + y^2}{x + y} \geq \frac{\lambda + 2}{4}(x + y)$$

Proof. We have:

$$\frac{x^2 + \lambda xy + y^2}{x + y} \geq \frac{\lambda + 2}{4}(x + y) \Leftrightarrow (2 - \lambda)(x - y)^2 \geq 0$$

which is true from $(x - y)^2 \geq 0$ and $\lambda \leq 2$. Equality holds for $x = y$.

Let's get back to the main problem. Using Lemma, we have:

$$LHS = \sum_{cyc} \frac{b^2 + \lambda bc + c^2}{b + c} \geq \frac{\lambda + 2}{4} \sum_{cyc} (b + c) = \frac{\lambda + 2}{4} \cdot 2 \sum_{cyc} a = \lambda + 2$$

Equality holds for $a = b = c = \frac{2}{3}$.

Note. For $\lambda = 1$ we get the proposed problem by Daniel Sitaru in R.M.M. 10/2020.

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro.

PROBLEMS FOR JUNIORS

J.3301 If $m, u, v, x, y \geq 0$ and $u + v = 2m, x + y = 2m + 4$ then in ΔABC with the area F and M is an interior point in the triangle, and d_a, d_b, d_c are the distances of M to the sides BC, CA respectively AB, the following inequality holds:

$$\left(\frac{a^x}{d_a^u} + \frac{b^x}{d_b^u} + \frac{c^x}{d_c^u}\right) \cdot \left(\frac{a^y}{d_a^v} + \frac{b^y}{d_b^v} + \frac{c^y}{d_c^v}\right) \geq 4^{m+2} \cdot 3^{m+1} \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3302 If $x, y > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^2b^2}{c(xa + yb)} + \frac{b^2c^2}{a(xb + yc)} + \frac{c^2a^2}{b(xc + ya)} \geq \frac{4\sqrt{3}}{x + y} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3303 If $m \geq 0, x, y \in \mathbb{R}$ and $x + y = 4m + 4$ then in any triangle with the area F the following inequality holds: $(a^x + b^x + c^y) \cdot (a^y + b^y + c^x) \geq 16^{m+1} \cdot 3^{1-m} \cdot F^{2m+2}$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3304 If $m \geq 0$ and F is the area of ΔABC, then:

$$(a^{3m+1} + b^{m+3} + c^{2m+2}) \cdot (a^{m+3} + b^{3m+1} + c^{2m+2}) \geq 16^{m+1} \cdot 3^{1-m} \cdot F^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3305 If $t, x, y \in \mathbb{R}$ and $x + y = 4$ then in any triangle with the area F the following inequality holds:

$$\left(a^{x\sin^2 t + y\cos^2 t} + b^{x\cos^2 t + y\sin^2 t + x\sin^2 t + y\cos^2 t}\right) \cdot$$

$$\cdot \left(a^{y\sin^2 t + x\cos^2 t} + b^{ycos^2 t + x\sin^2 t} + c^{y\sin^2 t + x\cos^2 t}\right) \geq 48F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3306 If $x, y, z > 0$, then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^3x}{(y + z)h_a} + \frac{b^3y}{(y + z)h_b} + \frac{c^3z}{(x + y)h_c} \geq 4F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3307 Let be $t, u > 0, X, Y$ the interior points of triangle ABC with the area F and x_a the distances from X to BC, y_a the distances from Y to BC and x_b, x_c, y_b, y_c the analogs of x_a respectively y_a. Prove that:

$$\frac{a^3}{tx_a + uy_a} + \frac{b^3}{tx_b + uy_b} + \frac{c^3}{tx_c + uy_c} \geq \frac{24}{t+u} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3308 In any triangle ABC with the area F the following inequality holds:

$$\frac{a}{h_a} + \frac{b}{h_b} + \frac{c}{h_c} \geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3309 Let be M and N the interior point in triangle ABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CD, AB and u_a, u_b, u_c the distances of point N to the sides BC, CA, AB. Prove that:

$$\frac{a^3}{d_a + u_a} + \frac{b^3}{d_b + u_b} + \frac{c^3}{d_c + u_c} \geq 12 \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3310 If $x, y > 0$, then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^3}{bx + cy} + \frac{b^3}{cx + ay} + \frac{c^3}{ax + by} \geq \frac{4\sqrt{3}}{x+y} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3311 In any triangle ABC the following inequality holds:

$$\frac{a}{h_b} + \frac{b}{h_c} + \frac{c}{h_a} \geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3312 In triangle ABC with the area F the following inequality holds:

$$\frac{a^3}{b+c} + \frac{b^3}{c+a} + \frac{c^3}{a+b} \geq 2\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3314 If $x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{ax}{(y+z)h_a} + \frac{by}{(z+x) \cdot h_b} + \frac{cz}{(x+y)h_c} \geq \sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3315 Let ABC be a triangle with the semiperimeter s, then:

$$\frac{1}{a^2+b^2+c^2}+\frac{2}{ab+bc+ca}\geq\frac{9}{4s^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3316 If $x,y,z>0$, then in any triangle ABC with the area F the following inequality holds:

$$\frac{xa^3}{(y+z)h_a}+\frac{yb^3}{(z+x)h_b}+\frac{zc^3}{(x+y)h_c}\geq 4F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3317 In any triangle ABC with the area F and any $x,y>0$ the following inequality holds:

$$\frac{a^5}{bx+cy}+\frac{b^5}{cx+ay}+\frac{c^5}{ax+by}\geq\frac{16}{x+y}\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

J.3318 In any triangle ABC with the area F, the following inequality holds:

$$\frac{a^5}{b+c}+\frac{b^5}{c+a}+\frac{c^5}{a+b}\geq 8F^2$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

J.3319 If $x,y,z>0$ then in triangle ABC with the area F the following inequality holds:

$$\frac{y+z}{x}\cdot a^4+\frac{z+x}{y}\cdot b^4+\frac{x+y}{z}\cdot c^4\geq 32F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3320 If $x,y,z>0$ then in triangle ABC with the area F the following inequality holds:

$$\frac{y+z+2t}{x+t}r_a^4+\frac{z+x+2t}{y+t}r_b^4+\frac{x+y+2t}{z+t}c^4\geq 18F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3321 If ABC is a triangle with the semiperimeter s and the area F then:

$$\frac{1}{(s-a)h_a}+\frac{1}{(s-b)h_b}+\frac{1}{(s-c)h_c}\geq\frac{3}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3322 Let be $x \geq 0, y, z > 0$ and ΔABC with the semiperimeter s such that $zs > y\max\{a, b, c\}$, then:

$$\frac{xs+ya}{zs-ya}+\frac{xs+yb}{zs-yb}+\frac{xs+yc}{zs-yc} \geq \frac{9x+15y-9z}{3z-2y}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3323 Let be $x, y > 0$ and ABC a triangle with the area F and semiperimeter s. If

$xs > y\max\{a, b, c\}$ then

$$\frac{1}{(xs-ya)h_a}+\frac{1}{(xs-yb)h_b}+\frac{1}{(xs-yc)h_c} \geq \frac{3}{(3x-2y)F}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

J.3324 With the usual notations in triangle ABC and for $x, y > 0$ with $x + y = 4$ the following inequality holds:

$$(a^{2x}+2)(b^{2y}+2)(c^{2x}+2)(a^{2y}+2)(b^{2x}+2)(c^{2y}+2) \geq 20736S^4$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

J.3325 If $x, y > 0, x \geq y$ and M is an interior point in triangle ABC with the area F and d_a, d_b, d_c are the distances of point M to the sides BC, CA, AB, then:

$$\frac{a^2}{xh_a^2-yd_a^2}+\frac{b^2}{xh_b^2-yd_b^2}+\frac{c^2}{xh_c^2-yd_c^2} \geq \frac{36}{9x-y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3326 Let $ABCD$ be a convexe quadrilateral with the sides $a = AB, b = BC, c = CD, d = DA$ and the semiperimeter s. Prove that:

$$\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}+\frac{d}{s-d} \geq 4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3327 If s is the semiperimeter of ΔABC and the other notations are the usual ones, then:

$$\left(\frac{a^2}{(s-a)^2}+2\right)\left(\frac{b^2}{(s-b)^2}+2\right)\left(\frac{c^2}{(s-c)^2}+2\right) \geq 108$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

J.3328 Let be $t \geq 0$ and ABC a triangle with the semiperimeter s. Prove that:

$$\frac{a}{(s-a)^t}+\frac{b}{(s-b)^t}+\frac{c}{(s-c)^t} \geq \frac{2\cdot 3^t}{s^{t-1}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3329 Let be $s,t,u \geq 0, x,y > 0$ such that $s+t=2, u+v=4$ then in any ΔABC with the area F the following inequality holds:

$$\frac{x^s}{(y+z)^t}\cdot a^u+\frac{y^s}{(z+x)^t}\cdot b^v+\frac{z^s}{(x+y)^t}\cdot c^u+\frac{x^t}{(y+z)^s}\cdot a^v+\frac{y^t}{(z+x)^s}\cdot b^u+\frac{z^t}{(x+y)^s}\cdot c^v \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

J.3330 If $x,y,z > 0$ then in triangle ABC with the area F the following inequality holds:

$$((x+y)^2+(y+z)^2+(z+x)^2)\cdot(a^2b^2+b^2c^2+c^2a^2) \geq$$

$$\geq 4F^2\left(\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}}+4\cdot\sqrt{xy\sin^2\frac{C}{2}+yz\sin^2\frac{A}{2}+zx\sin^2\frac{B}{2}}\right)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3331 Let ABC be a triangle with the area F and M is an interior point in the triangle and

$F_a=$ area $MBC, F_b=$ area $MCA, F_c=$ area MAB. Prove the inequality: $\frac{a^2b^2}{F_c}+\frac{b^2c^2}{F_a}+\frac{c^2a^2}{F_c} \geq 48F$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

J.3332 In any ΔABC with the area F the following inequality holds:

$$\frac{m_a^4+m_b^4}{m_c^2}+\frac{m_b^4+m_c^4}{m_a^2}+\frac{m_c^4+m_a^4}{m_b^2} \geq 6\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

J.3333 Let be $s,t \geq 0, M$ an interior point in ΔABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CA respectively AB. Prove that:

$$\frac{a\cdot b^{s+t+1}}{d_a^{s+t}}+\frac{b\cdot c^{s+t+1}}{d_b^{s+t}}+\frac{c\cdot a^{s+t+1}}{d_c^{s+t}} \geq 2^{s+t+2}\cdot\left(\sqrt{3}\right)^{s+t+1}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

J.3334 In ΔABC the following relationship holds:

$$\left(\frac{6r}{Rp}\right)^2 \leq \frac{1}{m_a w_a}+\frac{1}{m_b w_b}+\frac{1}{m_c w_c} \leq \frac{1}{27r^2}\left(1+\frac{4R}{r}\right)$$

Proposed by Marin Chirciu – Romania

J.3335 If $a,b,c > 0$ such that $abc=1$ then

$$\frac{1}{2}(a+b)^3+\frac{b^2+c^2}{a^2}+\frac{c^2+a^2}{b^2} \geq 8$$

Proposed by Marin Chirciu – Romania

J.3336 In ΔABC the following relationship holds:

$$48Fr \leq \sum a\,(b+c)^2 \cos A \leq 4F(5R+2r)$$

Proposed by Marin Chirciu – Romania

J.3337 In ΔABC the following relationship holds:

$$\sqrt{\sum \frac{\csc^4 \frac{A}{2}}{\left(\csc \frac{B}{2} + \csc \frac{C}{2}\right)^4}} \geq \frac{3}{4}\sqrt{\frac{r}{6R}}$$

Proposed by Marin Chirciu – Romania

J.3338 In ΔABC the following relationship holds:

$$\sum \frac{m_a^2}{h_b h_c} \geq \sum \frac{m_a^2}{r_b r_c}$$

Proposed by Marin Chirciu – Romania

J.3339 In ΔABC the following relationship holds:

$$\sum \frac{m_a^2}{h_a^2} \leq \sum \frac{m_a^2}{r_a^2}$$

Proposed by Marin Chirciu – Romania

J.3340 In ΔABC the following relationship holds:

$$5 - \frac{4r}{R} \leq \sum \frac{h_b \cos B + h_c \cos C}{h_a} \leq \frac{2R}{r} - 1$$

Proposed by Marin Chirciu – Romania

J.3341 In ΔABC the following relationship holds:

$$\frac{2\sqrt{3}}{R} \leq \sum \frac{\sin B + \sin C}{r_a} \leq \frac{\sqrt{3}R}{2r^2}$$

Proposed by Marin Chirciu – Romania

J.3342 In ΔABC the following relationship holds:

$$\frac{r_a}{a} + \frac{r_b}{b} + \frac{r_c}{c} \geq \frac{3}{R}\sqrt{\frac{3Rr}{2}}$$

Proposed by Marin Chirciu – Romania

J.3343 In ΔABC the following relationship holds:

$$9r\sqrt[3]{\frac{2r}{R}} \le \frac{s_b s_c}{s_a} + \frac{s_c s_a}{s_b} + \frac{s_a s_b}{s_c} \le 9r\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu – Romania

J.3344 In ΔABC the following relationship holds:

$$\frac{\lambda}{\lambda - 3 + \cos(A-B) + \cos(B-C) + \cos(C-A)} \le \frac{R}{2r}, \lambda \ge 4$$

Proposed by Marin Chirciu – Romania

J.3345 In ΔABC the following relationship holds:

$$\frac{a^3}{h_a} + \frac{b^3}{h_b} + \frac{c^3}{h_c} \ge \frac{1}{3}(a^3 + b^3 + c^3) \ge \frac{a^3}{r_a} + \frac{b^3}{r_b} + \frac{c^3}{r_c}$$

Proposed by Marin Chirciu – Romania

J.3346 In ΔABC the following relationship holds:

$$\frac{1}{2}\left(\frac{3r}{R}\right)^2 \le \sum \frac{s_a^2}{b^2 + c^2} \le \frac{1}{2}\left(\frac{3R}{4r}\right)^2$$

Proposed by Marin Chirciu – Romania

J.3347 In ΔABC the following relationship holds:

$$\sum a^4 + \lambda abc(a + b + c) \ge \frac{\lambda + 1}{3}(ab + bc + ca)^2, 0 \le \lambda \le 5$$

Proposed by Marin Chirciu – Romania

J.3348 In ΔABC the following relationship holds:

$$\frac{1}{R} \le \sum \frac{h_a}{w_b^2 + w_c^2} \le \frac{R}{4r^2}$$

Proposed by Marin Chirciu – Romania

J.3349 In ΔABC the following relationship holds:

$$\frac{1}{R}\left(\frac{2r}{R}\right)^{\frac{4}{3}} \le \sum \frac{h_a}{m_b^2 + m_c^2} \le \frac{1}{2r}$$

Proposed by Marin Chirciu – Romania

J.3350 In ΔABC the following relationship holds:

$$\frac{1}{R}\left(\frac{2r}{R}\right)^{\frac{4}{3}} \le \sum \frac{h_a}{s_b^2 + s_c^2} \le \frac{1}{R}\left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu – Romania

J.3351 In ΔABC the following relationship holds:

$$\sum \frac{h_a^2}{h_c} \le \frac{9R}{2}\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu – Romania

J.3352 If $a, b, c > 0$, such that $a + b + c = 1$ and $\lambda \le 2$ then

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + 3\lambda \ge 9 + \lambda\sqrt{\frac{a}{bc} + \frac{b}{ca} + \frac{c}{ab}}$$

Proposed by Marin Chirciu – Romania

J.3353 In all nonisosceles triangle ABC is true the following identity:

$$\sum \frac{\sin^4 A \sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}} = \frac{r(3s^2 - r^2 - 4Rr)}{4R^3}$$

Proposed by Neculai Stanciu – Romania

J.3354 In all nonisosceles triangle ABC holds:

$$\sum \frac{\sin^3 A \sin\frac{A}{2}}{\sin\frac{A-B}{2}\sin\frac{A-C}{2}} = \frac{sr}{R^2}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3355 Let a, b, c be three positive real numbers. Prove that:

$$\frac{a}{2b + 7\cdot\sqrt[4]{ab^3}} + \frac{b}{2c + 7\cdot\sqrt[4]{bc^3}} + \frac{c}{2a + 7\cdot\sqrt[4]{ca^3}} \ge \frac{1}{3}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3356 If $x_k > 0$ $(k = 1,2,\dots,n)$ and $\sum_{k=1}^{n} x_k = 1$, then prove that:

$$\sum_{k=1}^{n} \frac{n^2 + 1}{n^3(1 + x_k^2)} \le 1$$

Proposed by Neculai Stanciu – Romania

J.3357 In all triangle ABC holds:

$$\sum \frac{m_a m_b}{m_a + m_b - m_c} \ge \sum m_a$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3358 If $a_k > 0$ $(k = 1,2,\dots,n)$ then:

$$\sum_{cyclic} \frac{a_1^2}{a_1 + a_2} \ge \frac{1}{2}\sum_{k=1}^{n} a_k \ge \sum_{cyclic} \frac{a_1 a_2^2}{a_1^2 + a_2^2}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3359 If $a_k > 0$ $(k = 1,2,\dots,n)$ and $\sum_{k=1}^{n} a_k^2 = n$, then prove that:

$$\sum_{k=1}^{n} \frac{1}{n+1-a_k} \le 1$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3360 In all nonisoceles triangle ABC holds:

$$\sum \frac{\sin\frac{A}{2}}{\sin^2 A \sin\frac{A-B}{2}\sin\frac{A-C}{2}} = \frac{R(s^2 + r^2 + 4Rr)}{s^2 r}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3361 If $a_k > 0$ $(k = 1,2,\dots,n)$, then prove that:

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} a_k^{n-1}\right) - \sum_{k=1}^{n} a_k^n - n(n-1)\prod_{k=1}^{n} a_k \ge 0$$

Proposed by Neculai Stanciu – Romania

J.3362 If $a_k > 0$ $(k = 1,2,\dots,n)$ and $\sum_{k=1}^{n} \frac{1}{n+1-a_k} = 1$, then prove that:

$$\sum_{k=1}^{n} \frac{1}{n+1-a_k^2} \le 1$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3363 If $x, y, z > 0$, then prove that:

$$\sum \frac{(x+y)^5 - x^5 - y^5}{5((x+y)^3 - x^3 - y^3)} \le \sum x^2$$

Proposed by Neculai Stanciu – Romania

J.3364 If $x, y, z > 0$, then prove that:

$$\prod \frac{(x+y)^7 - x^7 - y^7}{(x+y)^5 - x^5 - y^5} \ge \frac{343}{125}\left(\sum xy\right)^3$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3365 If $0 < 2a_k \leq 1$ $(k = 1,2, \dots, n)$ and $\sum_{k=1}^{n} a_k^2 = 1$, then prove that:

$$\sum_{k=1}^{n} \frac{9}{2(1-a_k)} \leq 5n + 16$$

Proposed by Neculai Stanciu – Romania

J.3366 If $a, b, c > 0$, then prove that:

$$\sum \frac{3}{2a} + \frac{9}{\sum a} - \sum \frac{5}{a+b} \geq 0$$

Proposed by Neculai Stanciu – Romania

J.3367 If $a, b, c > 0$, then prove that:

$$\sum \frac{b}{a^2 - b^2 + c^2} \geq \frac{\sum a^2}{\prod a}$$

Proposed by Neculai Stanciu – Romania

J.3368 If $a, b, c > 0, n \in \mathbb{N}, n \geq 2$, then prove that:

$$\left(\sum \frac{a}{\sqrt[n]{a+2b}}\right)^n - \left(\sum a\right)^{n-1} \geq 0$$

Proposed by Neculai Stanciu – Romania

J.3369 If $a, b, c > 0, n \in \mathbb{N}, n \geq 2$, then prove that:

$$\left(\sum \frac{a}{\sqrt[n]{a^2 + 3ab + 3b^2 + 2bc}}\right)^n - \left(\sum a\right)^{n-2} \geq 0$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3370 In any triangle ABC with the area F the following inequality holds:

$$(a^3 + b^3 + c^3) \cdot \left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) \geq 24 \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3371 In any triangle ABC the following inequality holds:

$$(a^2 + b^2 + c^2) \cdot \left(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2}\right) \geq 12$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3372 If $u \in \mathbb{R}$ then in any triangle ABC with the area F the following inequality holds:

$$\left(a^{4\sin^2 u} + b^{4\cos^2 u} + c^2\right) \cdot \left(a^{\cos^2 u} + b^{4\sin^2 u} + c^2\right) \geq 48F^2$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

J.3373 Let be $t\in\mathbb{R}$, then in triangle ABC with the area F the following inequality holds:

$$\left(a^{4\sin^2 t}+b^{4\cos^2 t}+c^{4\sin^2 t}\right)\cdot\left(a^{4\cos^2 t}+b^{4\sin^2 t}+c^{4\cos^2 t}\right)\geq 48\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

J.3374 If $t\in\mathbb{R}$ then in triangle ABC with the area F the following inequality holds:

$$\left(a^3+b^{4\sin^2 t}+c^{4\cos^2 t}\right)\cdot\left(a+b^{4\cos^2 t}+c^{4\sin^2 t}\right)\geq 48\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3375 If $t\in\mathbb{R}$, then in any triangle ABC with the area F the following inequality holds:

$$\left(a^{4\cos^2 t}+b^{4\sin^2 t}+c^{4\cos^2 t}\right)\cdot\left(a^{4\sin^2 t}+b^{4\cos^2 t}+c^{4\sin^2 t}\right)\geq 48\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3376 Let be $x,y\geq 0$, such that $x+y=4$ and $A_1A_2\dots A_n, n\geq 3$ a convexe polygon with the sides $A_kA_{k+1}=a_k, k=\overline{1,n}, A_{n+1}=A_1$ then:

$$(a_1^x+a_2^x+\cdots+a_n^x)\left(a_1^y+a_2^y+\cdots+a_n^y\right)\geq 16\cdot F^2\cdot\tan^2\frac{\pi}{n}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3377 In any triangle ABC with the area F the following inequality holds:

$$m_a^4+m_b^4+m_c^4+3\geq 6\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3378 If $a,b,c,x,y>0$ then:

$$\frac{a^2b^2}{c(xa+yb)}+\frac{b^2c^2}{a(xb+yc)}+\frac{c^2a^2}{b(xc+ya)}\geq\frac{ab+bc+ca}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3379 Let be $x,y,z>0$ and ABC a triangle with the area F, then:

$$\frac{x+y}{h_ah_b}+\frac{y+z}{h_bh_c}+\frac{z+x}{h_ch_a}-\left(\frac{x}{h_a^2}+\frac{y}{h_b^2}+\frac{z}{h_c^2}\right)\geq\frac{2}{F}\sqrt{\sum_{cyc}xy\sin^2\frac{C}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3380 Let be $x,y>0$ and M an interior point in ΔABC with the area F and d_a,d_b,d_c are the distances of point M to the sides BC, CA respectively AB. Prove that:

$$\frac{a^2b}{xd_b+yh_b}+\frac{b^2c}{xd_c+yh_c}+\frac{c^2a}{xd_a+yh_a}\geq\frac{24}{x+3y}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3381 In ΔABC the following relationship holds:

$$\sum w_a AI \geq 18r^2$$

Proposed by Marin Chirciu – Romania

J.3382 In ΔABC the following relationship holds:

$$12\sqrt{3}r(R-r) \leq \sum a^2 \tan\frac{A}{2} \leq 6\sqrt{3}R(R-r)$$

Proposed by Marin Chirciu – Romania

J.3383 In ΔABC the following relationship holds:

$$3\sum a^2 \tan\frac{A}{2} \geq \sum a^2 \cot\frac{A}{2}$$

Proposed by Marin Chirciu – Romania

J.3384 In ΔABC the following relationship holds:

$$\sum \frac{AI}{r_a} \geq 2$$

Proposed by Marin Chirciu – Romania

J.3385 In ΔABC the following relationship holds:

$$3\sqrt[6]{\frac{27r^5}{2R^2}} \leq \sum \sqrt{\frac{h_a r_a}{r_b + r_c}} \leq \frac{3}{2}\sqrt{3R}$$

Proposed by Marin Chirciu – Romania

J.3386 In ΔABC the following relationship holds:

$$\frac{16r}{R} \leq \sum \frac{b^2 + c^2}{w_a^2} \leq \frac{2R^2}{r^2}$$

Proposed by Marin Chirciu – Romania

J.3387 In ΔABC the following relationship holds:

$$\sum \frac{\sin A}{\sin A + n\sin B} \leq \frac{3}{n+1}\left(1 + n\frac{R}{2r}\right), \text{ where } n \in \mathbb{N}.$$

Proposed by Marin Chirciu – Romania

J.3388 Let be x, y real positive numbers. Prove that:

$$(x^3 + y^3)^4 \leq 2(x^4 + y^4)^3$$

Proposed by Marin Chirciu – Romania

J.3389 Let be $\lambda > 0$ a real fixed number. Solve in real set numbers the equation:

$$\lambda x(\lambda x + 1) + \lambda y(\lambda + 1) = \lambda^2 xy - 1$$

Proposed by Marin Chirciu – Romania

J.3390 If $a, b, c > 0$ such that $ab + bc + ca = 1$ and $\lambda \geq \frac{2}{3}$ then find the maximum value of

$$P = \frac{1}{\lambda + a^2} + \frac{1}{\lambda + b^2} + \frac{1}{\lambda + c^2}$$

Proposed by Marin Chirciu – Romania

J.3391 In $\Delta ABC, I -$ incenter, $AD, BE, CF -$ bisectors

$$\left(\frac{AI}{AD}\right)^n + \left(\frac{BI}{BD}\right)^n + \left(\frac{CI}{CD}\right)^n \geq 3 \cdot \left(\frac{2}{3}\right)^n, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

J.3392 If $x, y, z > 0$ such that $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 3$ and $k \in \mathbb{N}^*$ then find the maximum value of

$$A = \frac{1}{(x + y + 2k - 2)^k} + \frac{1}{(y + z + 2k - 2)^k} + \frac{1}{(z + x + 2k - 2)^k}$$

Proposed by Marin Chirciu – Romania

J.3393 If $x, y, z, t > 0$ and $\lambda \geq 1$ then:

$$\sum \frac{x}{\lambda x + y + z + t} \leq \frac{4}{\lambda + 3}$$

Proposed by Marin Chirciu – Romania

J.3394 Solve in $\left(0, \frac{\pi}{2}\right)$ the equation:

$$\frac{1}{\left(1 + \sqrt{2}\sin x\right)^2} + \frac{1}{\left(1 + \sqrt{2}\cos x\right)^2} = \frac{1}{1 + \sin 2x}$$

Proposed by Marin Chirciu – Romania

J.3395 If $a, b, c > 0$ with $a + b + c \geq 3$ and $\lambda \geq 0$ then:

$$\frac{a}{\sqrt{\lambda + b}} + \frac{b}{\sqrt{\lambda + c}} + \frac{c}{\sqrt{\lambda + a}} \geq \frac{3}{\sqrt{\lambda + 1}}$$

Proposed by Marin Chirciu – Romania

J.3396 If $a, b, c > 0$ with $a + b + c = 1$ and $\lambda \geq 4$ then:

$$\frac{1}{abc} + \frac{\lambda}{a^2 + b^2 + c^2} \geq \frac{\lambda + 9}{ab + bc + ca}$$

Proposed by Marin Chirciu – Romania

J.3397 If $a, b, c > 0$ and $2a + b + c = 2$ and $n \in \mathbb{N}$ then:

$$(a+b)^n + (b+c)^n + (c+a)^n + 2na \geq n + 3$$

Proposed by Marin Chirciu – Romania

J.3398 Solve in $\mathbb{R}$ the equation:

$$\frac{1}{(1+e^x)^2} + \frac{1}{(1+e^y)^2} = \frac{1}{1+e^{x+y}}$$

Proposed by Marin Chirciu – Romania

J.3399 Let $m, n, x, y, z > 0$. In ΔABC:

$$\sum \frac{my + nz}{x} \cdot IA^4 \geq 48(m+n)r^4$$

Proposed by Marin Chirciu – Romania

J.3400 If $m, n, p, x, y, z > 0$, then in ΔABC:

$$\sum \frac{m+n}{p} \cdot \frac{y+z}{x} r_b^2 r_c^2 \geq 36F^2$$

Proposed by Marin Chirciu – Romania

J.3401 In any triangle ABC the following inequality holds:

$$\frac{a^2}{h_b^2} + \frac{b^2}{h_c^2} + \frac{c^2}{h_a^2} \geq 4$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3402 In triangle ABC with the area F the following inequality holds:

$$(a^3 + b^3 + c^3)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 12 \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.3403 If $m \geq 0$ and ABC is a triangle with the area F, then:

$$a^{m+2} + b^{m+2} + c^{m+2} \geq 2^{m+2} \cdot r^m \left(\sqrt{3}\right)^{m+1} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3404 If $m \geq 0$, then in triangle ABC with the area F the following inequality holds:

$$\frac{a^{2m}}{h_a^2} + \frac{b^{2m}}{h_b^2} + \frac{c^{2m}}{h_c^2} \geq 2^m \left(\sqrt{3}\right)^{m+1} \cdot F^{m-1}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3405 Let be $m \geq 0$ then in triangle ABC with the area F the following inequality holds:

$$\frac{a^{m+2}}{h_a^m}+\frac{b^{m+2}}{h_b^m}+\frac{c^{m+2}}{h_c^m} \geq 2^{m+2} \cdot \left(\sqrt{3}\right)^{1-m} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3406 Let be $x, y, z > 0$ and ABC a triangle with the area F, then:

$$\frac{xa}{y+z}+\frac{yb}{z+x}+\frac{zc}{x+y} \geq \sqrt[4]{27} \cdot \sqrt{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3407 If $x, y, z > 0$ then in any triangle ABC the following inequality holds:

$$\left(\frac{x^2\sqrt{a}}{y+z}+\frac{yb}{z+x}+\frac{x\sqrt{c}}{(x+y)^2}\right) \cdot \left(\frac{a\sqrt{a}}{y+z}+\frac{yb}{z+x}+zc\sqrt{c}\right) \geq 27r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3408 If $x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{x^2 a}{y+z}+\frac{yb^2}{z+x}+\frac{zc}{(x+y)^2}\right) \cdot \left(\frac{a^3}{y+z}+\frac{yb^2}{z+x}+zc^3\right) \geq 12 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3409 Let be M an interior point in triangle ABC with the area F and $x_a = MA, xb = MB$,

$x_c = MC$. Prove that: $2\left(x_a^2 + x_b^2 + x_c^2\right) + a^2 + b^2 + c^2 \geq 4\sqrt{3}F$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3410 If G is the centroid of the triangle ABC with the area F and $x_a = GA, x_b = GB, x_c = GC$ then:

$$x_a^2 + x_b^2 + x_c^2 \geq \frac{4\sqrt{3}}{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.3411 Let $x, y, z > 0$ and ABC a triangle with the area F then:

$$\left(\frac{x^2 a^4}{(y+z)^2}+2\right) \cdot \left(\frac{y^2 b^4}{(z+x)^2}+2\right) \cdot \left(\frac{z^2 c^4}{(x+y)^2}+2\right) \geq 36 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.3412 In any triangle ABC with the area F the following inequality holds:

$$a^2b + b^2c + c^2a \geq 2F \cdot \sqrt{\frac{bc}{\sin^2\frac{C}{2}} + \frac{ca}{\sin^2\frac{A}{2}} + \frac{ab}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3413 In any triangle ABC with the area F the following inequality holds:

$$a^3 + b^3 + c^3 \geq 2F \cdot \sqrt{\frac{ab}{\sin^2\frac{C}{2}} + \frac{bc}{\sin^2\frac{A}{2}} + \frac{ca}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3414 If $x, y, z > 0$ then in triangle ABC the following inequality holds:

$$\frac{x\sqrt{a}}{(y+z)\sqrt{h_a}} + \frac{y\sqrt{b}}{(z+x)\sqrt{h_b}} + \frac{z\sqrt{c}}{(x+y)\sqrt{h_c}} \geq \frac{\sqrt[4]{27}}{\sqrt{2}} = \frac{\sqrt{2}\cdot\sqrt[4]{27}}{2} = \frac{\sqrt[4]{108}}{2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3415 If $x, y \geq 0$ and $x + y = 6$ and ABC is a triangle having the area F and the sides of lengths a, b, c and s is the semiperimeter of the triangle then:

$$(2s + a^x + b^y + c^x)(2s + a^y + b^x + c^y) \geq 192 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3416 In any triangle ABC with the area F the following inequality holds:

$$\frac{ab}{h_a h_b} + \frac{bc}{h_b h_c} + \frac{ca}{h_c h_a} \geq 4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3417 Let M be an interior point in triangle ABC and d_a, d_b, d_c the distances of point M to the sides BC, CA, AB, then:

$$\frac{a^3}{h_a + d_a} + \frac{b^3}{h_b + d_b} + \frac{c^3}{h_c + d_c} \geq 6S$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3418 In any triangle ABC the following inequality holds:

$$\frac{a}{h_b} + \frac{b}{h_c} + \frac{c}{h_a} \geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3419 Let X, Y be interior points in ΔABC with the area F and x_a, x_b, x_c the distances of point X to the sides BC, CA, respectively AB and y_a, y_b, y_c the distances of point Y to the sides BC, CA, AB, then:

$$\frac{a^3}{x_a+y_a}+\frac{b^3}{x_b+y_b}+\frac{c^3}{x_c+y_c}\geq 12\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3420 If $m \geq 0$ and M an interior point in triangle ABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CA, AB, then:

$$\left(\frac{a^3}{d_a}\right)^{m+1}+\left(\frac{b^3}{d_b}\right)^{m+1}+\left(\frac{c^3}{d_c}\right)^{m+1}\geq 2^{3m+3}\cdot 3\cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3421 Let be $m \geq 0$ and $x, y, z > 0$, then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^{m+2}\cdot x^{m+1}}{(y+z)^{m+1}\cdot h_a^m}+\frac{b^{m+2}\cdot y^{m+1}}{(z+x)^{m+1}\cdot h_b^m}+\frac{c^{m+2}\cdot z^{m+1}}{(x+y)^{m+1}\cdot h_c^m}\geq 2\left(\sqrt{3}\right)^{1-m}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.3422 In any triangle ABC the following inequality holds:

$$\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}\geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3423 Let be $a_k \in \mathbb{R}_+^* = (0,\infty), \forall k=\overline{1,n}, n\geq 2$ and $s=\sum_{k=1}^{n} a_k$. If $x, y > 0, xs > y \max\limits_{1\leq k\leq n} a_k$ and $m \in \mathbb{N}, m \geq 2$, then:

$$\sum_{k=1}^{n}\frac{\sqrt[m]{a_k^m+(s-a_k)^m}}{xs-ya_k}\geq\frac{n^2}{(nx-y)\cdot 2^{\frac{m-1}{m}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.3424 Let be $m \geq 0$ and ABC a triangle ABC with the area F. Prove that:

$$a^{2m+4}+b^{2m+4}+c^{2m+4}\geq 4^{m+2}\cdot\left(\sqrt{3}\right)^{-m}\cdot F^{m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu– Romania

J.3425 If $x, y > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{x^2a^4+y^2b^4}{c^2}+\frac{x^2b^4+y^2c^4}{a^2}+\frac{x^2c^4+y^2a^4}{b^2}\geq 2(x+y)^2F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu– Romania

J.3426 Let be $m \geq 1$ and ABC a triangle with the area F, then:

$$a^m + b^m + c^m + a^{m+2} + b^{m+2} + c^{m+2} \geq 2^{m+2} \cdot \left(\sqrt[4]{3}\right)^{m+1} \cdot \left(\sqrt{F}\right)^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

J.3427 In any triangle ABC with the area F the following inequality holds:

$$\frac{a^4+b^4}{m_c^2} + \frac{b^4+c^4}{m_a^2} + \frac{c^4+a^4}{m_b^2} \geq \frac{32\sqrt{3}}{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.3428 Let $m \geq 0$ and ABC a triangle with the area F and semiperimeter s, then:

$$\frac{(a^2+b^2+2s^2)^{m+1}}{(c^2+s^2)^m} + \frac{(b^2+c^2+2s^2)^{m+1}}{(a^2+s^2)^m} + \frac{(c^2+a^2+2s^2)^{m+1}}{(b^2+s^2)^m} \geq 2^{m+1} \cdot 13 \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.3429 In any triangle ABC with the semiperimeter s with the area F the following inequality holds:

$$s \cdot (a^3 + b^3 + c^3) \geq 24F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

J.3430 In triangle ABC with the semiperimeter s and the area F the following inequality holds:

$$\frac{(a^2+b^2+2s^2)^3}{(c^2+s^2)^2} + \frac{(b^2+c^2+2s^2)^3}{(a^2+s^2)^2} + \frac{(c^2+a^2+2s^2)^3}{(b^2+s^2)^2} \geq 104 \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3431 If $x, y, z > 0, t, u, v, w \geq 0$ and $t + u = 2, v + w = 4$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\left(\frac{x}{y+z}\right)^t \cdot a^v + \left(\frac{y}{z+x}\right)^u \cdot b^w + \left(\frac{z}{x+y}\right)^u \cdot c^v\right) \cdot$$

$$\cdot \left(\left(\frac{x}{y+z}\right)^u \cdot a^w + \left(\frac{y}{z+x}\right)^t \cdot b^v + \left(\frac{z}{x+y}\right)^t \cdot c^w\right) \geq 12 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3432 Let be $m \geq 0$, then in any triangle ABC with the area F, the following inequality holds:

$$r_a^{2m+2} + r_b^{2m+2} + r_c^{2m+2} \geq \left(\sqrt{3}\right)^{m+3} \cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3433 In any triangle with the area F and the semiperimeter s the following inequality holds:

$$s \geq \sqrt[4]{27} \cdot \sqrt{F} + \frac{1}{4}\left(\left(\sqrt{a}-\sqrt{b}\right)^2 + \left(\sqrt{b}-\sqrt{c}\right)^2 + \left(\sqrt{c}-\sqrt{a}\right)^2\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3434 If $x, y > 0$ and ABC is a triangle with the area F, then:

$$\frac{a^3}{xb+yc} + \frac{b^3}{xc+ya} + \frac{c^3}{xa+yb} \geq \frac{4\sqrt{3}}{x+y} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3435 Let be $t \in \left(0, \frac{\pi}{2}\right)$ and ABC a triangle with the area F, then:

$$\frac{a^3}{b\sin^2 t + c\cos^2 t} + \frac{b^3}{c\sin^2 t + a\cos^2 t} + \frac{c^3}{a\sin^2 t + b\cos^2 t} \geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3436 If $m \geq 0, x, y > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{(ax+by)^{2m+2}}{(ab)^m} + \frac{(bx+cy)^{2m+2}}{(bc)^m} + \frac{(cx+ay)^{2m+2}}{(ca)^m} \geq 4\sqrt{3}(x+y)^{2m+2} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3437 Let be $m \geq 0, x, y > 0$ and M an interior point in triangle ABC with the area F and

$F_a =$ area $MBC, F_b =$ area $MCA, F_c =$ area MAB, then:

$$(x(ab+bc)+2yF_a)^{m+1} + (x(bc+ca)+2yF_b)^{m+1} + (x(ca+ab)+2yF_c)^{m+1}$$
$$\geq \frac{2^{m+1}}{3^m}\left(4\sqrt{3}x+y\right)^{m+1} \cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3438 In any triangle ABC with the area F and the length sides a, b, c the following inequality holds:

$$\frac{(a+b)^4}{bc} + \frac{(b+c)^4}{ca} + \frac{(c+a)^4}{ab} \geq 64\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3439 In triangle ABC with the area F, with $u, z, y > 0, x + y = 4$ the following inequality holds:

$$\left(m_a^x + m_b^y + m_c^x\right) \cdot \left(m_a^y + m_b^x + m_c^y\right) \geq 27F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3440 If $x, y > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{(ax+by)^4}{ab}+\frac{(bx+cy)^4}{bc}+\frac{(cx+ay)^4}{ca}\geq 4\sqrt{3}\cdot(x+y)^4\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3441 If $m \geq 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{(a+b)^{2m+2}}{(ab)^m}+\frac{(b+c)^{2m+2}}{(bc)^m}+\frac{(c+a)^{2m+2}}{(ca)^m}\geq 4^{m+2}\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

J.3442 If $a, b, c, d, e, f > 0$ and $a+b+c=s, d+e+f=t$ then

$$\frac{(a+d)^3}{(b+e)^2}+\frac{(b+e)^3}{(c+f)^2}+\frac{(c+f)^3}{(a+d)^2}\geq s+t$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3443 If $x, y, z > 0$ and $x+y+z=1$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{(a^2+ab+b^2)^2}{1-x^2}+\frac{(b^2+bc+ca^2)^2}{1-y^2}+\frac{(c^2+ca+a^2)^2}{1-z^2}\geq 162\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3444 If $m \geq 0$, then in any triangle ABC the following inequality holds:

$$a^{m+2}+b^{m+2}+c^{m+2}\geq 2^{m+2}\cdot\left(\sqrt{3}\right)^{m+4}\cdot r^{m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3445 Let be $a, b, c > 0$ prove that:

$$\frac{1}{(a+b)^2+2025}+\frac{1}{(b+c)^2+2025}+\frac{1}{(c+a)^2+2025}\leq\frac{1}{90}\left(\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3446 In any triangle ABC with the area F the following inequality holds:

$$a^8+b^8+c^8\geq\frac{256}{3}\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

J.3447 Let be $m \geq 0$, then in any triangle ABC with the area F the following inequality holds:

$$a^{4m+4}+b^{4m+4}+c^{4m+4}\geq\frac{16^{m+1}}{3^m}\cdot F^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3448 Let M an interior point in triangle ABC with the area F and $F_a=$ are $MBC, F_b=$ area MCA, $F_c=$ area MAB. Prove that:

$$\frac{(x(ab+bc)+2yF_a)^2}{xca+yF_b}+\frac{(x(bc+ca)+2yF_b)^2}{xab+yF_c}+\frac{(x(ca+ab)+2yF_c)^2}{xbc+yF_a}\geq 4\left(4\sqrt{3}x+y\right)F, \forall x,y\in\mathbb{R}^*$$

Proposed by *D.M. Bătinețu – Giurgiu, Daniel Sitaru* ***– Romania***

J.3449 In any triangle ABC with the area F the following inequality holds:$a^8+b^8+c^8+9\geq 16\sqrt{3}F$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

J.3450 Let be M an interior point in triangle ABC with the area F and $F_a=$ area MBC,

$F_b=$ area MCA respectively $F_c=$ are MAB. Prove that:

$$((m_a^2+F_b)^2+2)\cdot((m_b^2+F_c)^2+2)\cdot((m_c^2+F_a)^2+2)\geq 3\cdot\left(3\sqrt{3}+1\right)^2\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.3301 In any triangle ABC with the area F the following inequality holds:

$$m_a + m_b + m_c + m_a^3 + m_b^3 + m_c^3 \geq 6\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3302 If $x, y, z > 0$ and $a + b + c = 18 \cdot abc$ then:

$$\left(\frac{1}{a^2} + 5\right)\left(\frac{1}{b^2} + 5\right)\left(\frac{1}{c^2} + 5\right) \geq 243$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3303 If $x, y > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{x^2 \cdot m_a^4 + y^2 m_b^4}{m_c^2} + \frac{x^2 \cdot m_b^4 + y^2 \cdot m_c^4}{m_a^2} + \frac{x^2 m_c^4 + y^2 m_a^4}{m_b^2} \geq \frac{3\sqrt{3}}{2} \cdot (x + y)^2 \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3304 If $a, b > 0$ and $x \in \mathbb{R}$, then:

$$(a \sin^2 x + b)^3 + (a \cos^2 x + b)^3 \geq \frac{1}{4}(a + 2b)^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3305 If $x, y \geq 0$ with $x + y = 4$ and ABC is a triangle with the area F, then:

$$\left(m_a^x + m_b^y + m_c^x\right)\left(m_a^y + m_b^x + m_c^y\right) \geq 27 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3306 If $a, b > 0$ and $t \in \left(0, \frac{\pi}{2}\right)$ then:

$$(a \tan^2 t + b \cot^2 t)^3 + (a \cot^2 t + b \tan^2 t)^3 \geq 2(a + b)^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3307 In any triangle ABC with the area F the following inequality holds:

$$a^4+b^4+c^4+27\geq 24\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3308 If $a,b>0$ and $x\in\mathbb{R}$ such that $\sin x\cdot\cos x\neq 0$, then:

$$(a\tan^2 x+b)^3+(a\cdot\cot^2 x+b)^3\geq 2(a+b)^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3309 If $x,y>0$ and $x+y=3$ then in triangle ABC with the area F and semiperimeter s the following inequality holds:

$$3+a^xb^y+b^xc^y+c^xa^y\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3310 If $a,b,c>0$ then:

$$\left(\frac{a\sqrt[3]{a}}{b+c}+\frac{b\sqrt[3]{b}}{c+a}+\frac{c\sqrt[3]{c}}{a+b}\right)\left(\frac{\sqrt[3]{a^2}}{b+c}+\frac{\sqrt[3]{b^2}}{c+a}+\frac{\sqrt[3]{c^2}}{a+b}\right)\geq\frac{9}{4}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3311 If $x,y>0$ and $x+y=4$ then in any triangle ABC with the area F the following inequality holds:

$$m_a^x+m_b^y+m_c^x+m_a^y+m_b^x+m_c^y\geq 6\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3312 If $a,b,c,d>0$, then:

$$\left(a\cdot\frac{c}{d}+b\right)^{2025}+\left(a\cdot\frac{d}{c}+b\right)^{2025}\geq 2(a+b)^{2025}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3313 In any ΔABC with the area F the following inequality holds:

$$\sqrt{a^2b^2+b^2c^2}+\sqrt{b^2c^2+c^2a^2}+\sqrt{c^2a^2+a^2b^2}\geq 4\sqrt{6}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3314 Let be $a,b>0, n\in\mathbb{N}^*, n\geq 2$ and $t_k,u_k>0\ \forall k=\overline{1,n}$. Prove that:

$$\sum_{k=1}^{n}\left(a\cdot\frac{t_k}{u_k}+b\cdot\frac{u_k}{t_k}\right)^3+\sum_{k=1}^{n}\left(a\cdot\frac{u_k}{t_k}+b\cdot\frac{t_k}{u_k}\right)^3\geq 2n(a+b)^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3315 If $m \geq 0$ and $a, b, t, u, x, y > 0$, then:

$$\left(a \cdot \frac{t}{u} + b \cdot \frac{x}{y}\right)^{m+1} + \left(a \cdot \frac{u}{t} + b \cdot \frac{y}{x}\right)^{m+1} \geq 2(a+b)^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3316 Let be $t \in \mathbb{R}$ and ABC a triangle with the area F, then:

$$\left((ab)^{2\cos^2 t} + (bc)^{2\sin^2 t} + (ca)^{2\sin^2 t}\right) \cdot \left((ab)^{2\sin^2 t} + (bc)^{2\cos^2 t} + (ca)^{2\cos^2 t}\right) \geq 48 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3317 If $m \geq 0$ and $t, x, y > 0$, then:

$$(x^{2m+2} + t^{2m+2})(y^{2m+2} + t^{2m+2}) \geq \frac{3^{m+1}}{4^{2m+1}} \cdot t^{2m+2}((x+y)^2 + t^2)^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3318 If $u \in \mathbb{R}$ then in any triangle ABC with the area F the following inequality holds:

$$\left(a^{4\sin^2 u} + b^{4\cos^2 u} + c^2\right) \cdot \left(a^{4\cos^2 u} + b^{4\sin^2 u} + c^2\right) \geq 48F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3319 If $a, b, c, d, u, v > 0$ prove that:

$$\left(a \cdot \frac{c}{d} + b \cdot \frac{u}{v}\right)^3 + \left(a \cdot \frac{d}{c} + b \cdot \frac{v}{u}\right) \geq 2(a+b)^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.3320 Let be $t \in \mathbb{R}$, then in triangle ABC with the area F the following inequality holds:

$$\left(a^{4\sin^2 t} + b^{4\cos^2 t} + c^{4\sin^2 t}\right) \cdot \left(a^{4\cos^2 t} + b^{4\sin^2 t} + c^{4\cdot\cos^2 t}\right) \geq 48 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3321 If $a, b > 0$ and $t \in \mathbb{R}$ then:

$$(a\sin^2 t + b \cdot \cos^2 t)^3 + (a\cos^2 t + b\sin^2 t) \geq \frac{1}{4}(a+b)^3$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.3322 Let be $t \in \left(0, \frac{\pi}{2}\right)$ and M an interior point in triangle ABC with the area F.

If $F_a =$ area MBC, $F_b =$ area MCA, $F_c =$ area MAB then:

$$\frac{a^4}{F_b \cdot \sin^2 t + F_c \cdot \cos^2 t} + \frac{b^4}{F_c \cdot \sin^2 t + F_a \cdot \cos^2 t} + \frac{c^4}{F_a \cdot \sin^2 t + F_b \cdot \cos^2 t} \geq 48F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3323 If $a,b,c,x,y>0$ then:

$$\left(\frac{a\cdot\sqrt[3]{a}}{bx+cy}+\frac{b\cdot\sqrt[3]{b}}{cx+ay}+\frac{c\cdot\sqrt[3]{c}}{ax+by}\right)\cdot\left(\frac{\sqrt[3]{a^2}}{bx+cy}+\frac{\sqrt[3]{b^2}}{cx+ay}+\frac{\sqrt[3]{c^2}}{ax+by}\right)\geq\frac{9}{(x+y)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3324 If $n\in\mathbb{N}, n\geq 3, a_k\in\mathbb{R}_+^*=(0,\infty),\forall k=\overline{1,n}$ and $s_n=\sum_{k=1}^{n}a_k$ and $x,y\in\mathbb{R}_+^*$ such that $xs_n>y\cdot\max\limits_{1\leq k\leq n}a_k$, then:

$$\left(\sum_{k=1}^{n}\frac{a_k^3}{xs_n-ya_k}\right)\cdot\left(\sum_{k=1}^{n}\frac{a_k}{xs_n-ya_k}\right)\geq\frac{s_n^2}{(nx-y)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

S.3325 If $n\geq 3$ and $a_k>0,\forall k=1,n$, and $s_n=\sum_{k=1}^{n}a_k$ and $x,y>0$ such that

$xs_n>y\cdot\max\limits_{1\leq k\leq n}a_k$ then:

$$\left(\sum_{k=1}^{n}\frac{a_k\cdot\sqrt[3]{a_k}}{xs_n-ya_b}\right)\left(\sum_{k=1}^{n}\frac{\sqrt[3]{a_k^2}}{xs_n-ya_k}\right)\geq\frac{n^2}{(nx-y)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

S.3326 In any triangle ABC with the area F the following inequality holds:

$$(a^3+b^3+c^3)\cdot\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\geq 24\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3327 In any triangle ABC with the area F the following inequality holds:

$$m_a^4+m_b^4+m_c^4+3\geq 6\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3328. If $t\in\mathbb{R}$, then in any triangle ABC with the area F the following inequality holds:

$$\left(a^{4\cos^2 t}+b^{4\sin^2 t}+c^{4\cos^2 t}\right)\cdot\left(a^{4\sin^2 t}+b^{4\cos^2 t}+c^{4\sin^2 t}\right)\geq 48\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3329 In any triangle ABC the following inequality holds:

$$(a^2+b^2+c^2)\cdot\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)\geq 12$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3330 In any triangle ABC with the semiperimeter s the following inequality holds:

$$a^3+b^3+c^3\geq 72\sqrt{3}\cdot r^3$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

S.3331 Let be $u,v\geq 0$ and ABC a triangle with the semiperimeter s and the area F such that

$u+v=2$. Prove that: $2s+a^ub^v+b^uc^v+c^ua^v\geq 8\sqrt{3}F$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3332 If $a,b,c,x,y>0$, then:

$$\frac{a^2b^2}{c(xa+yb)}+\frac{b^2c^2}{a(xb+yc)}+\frac{c^2a^2}{b(xc+ya)}\geq\frac{ab+bc+ca}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3333 If $x,y,z\in[0,4]$ then in any triangle ABC with the area F the following inequality holds:

$$a^x+b^y+c^z+a^{4-x}+b^{4-y}+c^{4-z}\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

S.3334 In any triangle ABC with the area F the following inequality holds:

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}+\frac{a^2}{(b+c)^4h_a}+\frac{b^2}{(c+a)^4h_b}+\frac{c^2}{(a+b)^4h_c}\geq\frac{3}{4F}$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

S.3335 Let be $x,y,z>0$ and ABC a triangle with the area F, then:

$$\frac{x+y}{h_ah_b}+\frac{y+z}{h_bh_c}+\frac{z+x}{h_ch_a}-\left(\frac{x}{h_a^2}+\frac{y}{h_b^2}+\frac{z}{h_c^2}\right)\geq\frac{2}{F}\sqrt{\sum_{cyc}xy\sin^2\frac{c}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3336 Let M be an interior point in ΔABC with the area F and d_a,d_b,d_c the distances of point M to the sides BC,CA respectively AB. Prove that:

$$\frac{a^2b}{d_b+h_b}+\frac{b^2c}{d_c+h_c}+\frac{c^2a}{d_a+h_a}\geq 6F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3337. If $t\in\mathbb{R}$ then in triangle ABC with the area F the following inequality holds:

$$\left(a^3+b^{4\sin^2t}+c^{4\cos^2t}\right)\cdot\left(a+b^{4\cdot\cos^2t}+c^{4\sin^2t}\right)\geq 48\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3338 Let be $x \geq y$, such that $x + y = 4$ and $A_1A_2 \dots A_n, n \geq 3$ a convexe polygon with the side lengths $A_kA_{k+1} = a_k, k = \overline{1,n}, A_{n+1} = A_1$ then:

$$(a_1^x + a_2^x + \cdots + a_n^x) \cdot (a_1^y + a_2^y + \cdots + a_n^y) \geq 16 \cdot F^2 \cdot \tan^2 \frac{\pi}{n}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3339 In any triangle ABC with the area F the following inequality holds:

$$m_a \cdot a^2 + m_b \cdot b^2 + m_c \cdot c^2 \geq 2F \cdot \sqrt{\frac{m_a \cdot m_b}{\sin^2 \frac{C}{2}} + \frac{m_b m_c}{\sin^2 \frac{A}{2}} + \frac{m_c m_a}{\sin^2 \frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

S.3340 In ΔABC the following relationship holds:

$$4 \leq \sum \sec \frac{A}{2} \sqrt{\sec \frac{B}{2} \sec \frac{C}{2}} \leq \frac{2R}{r}$$

Proposed by Marin Chirciu – Romania

S.3341 In ΔABC the following relationship holds:

$$3 \leq \frac{h_a}{a \sin A} + \frac{h_b}{b \sin B} + \frac{h_c}{c \sin C} \leq 3 \left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu – Romania

S.3342 In ΔABC the following relationship holds:

$$\frac{9}{4} \cdot \left(\frac{2r}{R}\right)^{\frac{4}{3}} \leq \sum \sin A \sqrt{\sin B \sin C} \leq \left(1 + \frac{r}{R}\right)^2$$

Proposed by Marin Chirciu – Romania

S.3343 In ΔABC the following relationship holds:

$$\frac{\sqrt[3]{\left(a^3 + \frac{1}{2}\right)\left(b^3 + \frac{1}{2}\right)\left(c^3 + \frac{1}{2}\right)}}{\csc \frac{A}{2} + \csc \frac{B}{2} + \csc \frac{C}{2}} \geq \frac{4r^3(4R + r)}{3R^2}$$

Proposed by Marin Chirciu – Romania

S.3344 In ΔABC the following relationship holds:

$$\frac{32r^3}{4R^3 + R^2r - 4r^3} \leq \frac{2m_bm_c}{(m_b + m_b)^2} + \frac{1}{2} \leq 1$$

Proposed by Marin Chirciu – Romania

S.3345 In ΔABC the following relationship holds:

$$\frac{r_a}{a\sin A}+\frac{r_b}{b\sin B}+\frac{r_c}{c\sin C}\le\frac{3R}{2r}$$

Proposed by Marin Chirciu – Romania

S.3346 In ΔABC the following relationship holds:

$$3\sqrt{3r}\le\sqrt{w_a}+\sqrt{w_b}+\sqrt{w_c}\le3\sqrt{\frac{3R}{2}}$$

Proposed by Marin Chirciu – Romania

S.3347 If $a,b,c>0$ and $n\in\mathbb{N}^*$ then:

$$\frac{a^{n+1}}{b^n}+\frac{b^{n+1}}{c^n}+\frac{c^{n+1}}{a^n}+(2n-1)(a+b+c)\ge n\sum\sqrt[n+1]{2^n(a^{n+1}+b^{n+1})}$$

Proposed by Marin Chirciu – Romania

S.3348 If $a,b,c>0$ such that $a+b+c=3$ then find the minimum of the expression:

$$P=abc+(a-\lambda)^2+(b-\lambda)^2+(c-\lambda)^2$$

Proposed by Marin Chirciu – Romania

S.3349 If $a,b,c>0,a^5+b^5+c^5=3$ then find the maximum of

$$F=(a^7+a^3)(b^7+b^3)(c^7+c^3)$$

Proposed by Marin Chirciu – Romania

S.3350 If $a,b,c>0,a^3+b^3+c^3=3$ then find the maximum of:

$$F=(a^5+a)(b^5+b)(c^5+c)$$

Proposed by Marin Chirciu – Romania

S.3351 If $x,y,z>0,x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ and $\lambda\ge0$ then:

$$x+y+z\ge\sqrt{\frac{xy+\lambda}{\lambda+1}}+\sqrt{\frac{yz+\lambda}{\lambda+1}}+\sqrt{\frac{zx+\lambda}{\lambda+1}}$$

Proposed by Marin Chirciu – Romania

S.3352 If $a,b,c>0,a+b+c=1$ and $0\le\lambda\le\frac{9}{4}$ then:

$$\frac{a^3}{b}+\frac{b^3}{c}+\frac{c^3}{a}+\lambda abc\ge\frac{\lambda+9}{27}$$

Proposed by Marin Chirciu – Romania

S.3353 If $x, y, z > 0, x + y + z = 1$ and $\frac{3}{2} \le \lambda \le 3$ then:

$$\frac{1}{\lambda x^2 + y + z} + \frac{1}{\lambda y^2 + z + x} + \frac{1}{\lambda z^2 + x + y} \le \frac{27}{\lambda + 6}$$

Proposed by Marin Chirciu – Romania

S.3354 If $a, b, c, d > 0, a^n + b^n + c^n + d^n = 2^n, n \in \mathbb{N}^*$ then:

$$a^{n+1} + b^{n+1} + c^{n+1} + d^{n+1} \le 2^{n+1}$$

Proposed by Marin Chirciu – Romania

S.3355 If $x, y, z \in \mathbb{R}$, then prove that:

$$2(x^2 + y^2 + z^2) + x^2y^2 + y^2z^2 + z^2x^2 - 3(xy + yz + zx - 1) \ge 0$$

Proposed by Neculai Stanciu – Romania

S.3356 If $a, b, c > 0, a + b + c = 3$ and $\lambda \ge 0$ then:

$$\sqrt{\frac{ab}{(\lambda + 1)a + b + c}} + \sqrt{\frac{bc}{(\lambda + 1)b + c + a}} + \sqrt{\frac{ca}{(\lambda + 1)c + a + b}} \le \frac{3}{\sqrt{\lambda + 3}}$$

Proposed by Marin Chirciu – Romania

S.3357 If $x, y, z > 0, xyz \ge 1$ and $\lambda \ge 0$ then:

$$(\lambda + 1)(x^2 + y^2 + z^2) \ge x + y + z + 3\lambda$$

Proposed by Marin Chirciu – Romania

S.3358 If $a, b, c > 0, (a + b + c)(ab + bc + ca) = abc + 1$ and $n \in \mathbb{N}$ then:

$$\left(\frac{1}{a + b}\right)^n + \left(\frac{1}{b + c}\right)^n + \left(\frac{1}{c + a}\right)^n \ge 3$$

Proposed by Marin Chirciu – Romania

S.3359 Prove that in all triangles ABC with usual notations holds the inequality:

$$\sum \tan\frac{A}{2}\sqrt{\frac{1}{3}\left(\tan^2\frac{B}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \tan^2\frac{C}{2}\right)} \ge 1$$

Proposed by Neculai Stanciu – Romania

S.3360 If $a, b, c > 0$, then prove that:

$$\left(\sum \frac{a}{\sqrt[3]{a + 2b}}\right)^{\frac{3}{2}} \ge \sum a$$

Proposed by Neculai Stanciu – Romania

S.3361 Prove that in all triangles ABC with usual notations holds the following inequality:

$$27\left(\sum a^2\right)^2 - 54\sum a^4 \le 16s^4$$

Proposed by Neculai Stanciu – Romania

S.3362 If $x, y, z \in \mathbb{R}$, then:

$$x^2y^2 + y^2z^2 + z^2x^2 + x^2 + y^2 + z^2 + x + y + z - 2(xy + yz + zx - 3) \ge 0$$

Proposed by Neculai Stanciu – Romania

S.3364 If $a, b, c > 0$, then prove that:

$$\left(\sqrt{3}\sum a^3\right)\left(\sqrt{3}\sum a^4\right) \ge abc\left(\sum a\right)^2\left(\sum a^2\right)$$

Proposed by Neculai Stanciu – Romania

S.3365 If ABC is a triangle, then prove that:

$$\prod(a+b) + \prod(a+b-c) \ge 9abc \Leftrightarrow s^2 + 5r^2 \ge 16Rr$$

and then the both inequalities are true.

Proposed by Neculai Stanciu – Romania

S.3366 If $a, b, c > 0$, then prove that:

$$3\sum ab\sum\frac{a}{b} - 7\sum ab \ge 2\sum a^2$$

Proposed by Neculai Stanciu – Romania

S.3367(a) If $a, b, c > 0$, then prove the inequality $\sum a^2 \ge \sum ab + \frac{1}{2}\sum(|a-c| + |b-c|)^2$ is false!

(b) If $a, b, c > 0$, then prove the inequality $\sum a^2 \ge \sum ab + \frac{1}{8}\sum(|a-c| + |b-c|)^2$is true!

Proposed by Neculai Stanciu – Romania

S.3368 Let be $m \ge 0$ and $t, x, y > 0$ and ABC a triangle with the area F, then:

$$\frac{(a^2+b^2+tc^2)^{m+1}}{(a^2x+b^2y)^m} + \frac{(b^2+c^2+ta^2)^{m+1}}{(b^2x+c^2y)^m} + \frac{(c^2+a^2+tb^2)^{m+1}}{(c^2x+a^2y)^m} \ge \frac{4(t+2)^{m+1}}{(x+y)^m}\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze – Romania

S.3369 If $x_k \in \mathbb{R}(k = 1,2,\ldots,n)$ and $\sum_{k=1}^{n} x_k^2 = 1$, then prove that:

$$\frac{3}{2}\left(\sum_{k=1}^{n} x_k\right) - 1 \le n$$

Proposed by Neculai Stanciu – Romania

S.3370 If $a, b > 0$ are such that its arithmetic, geometric and harmonic means are the sides of triangle $ABC(\angle 90°, \angle B < \angle C)$, then prove that:

(i) $\sin B = \cos^2 B$; (ii) $\cos C = \sin^2 C$; (iii) $30° < \angle B < 45°$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.3371 Let be $u, v \geq 0$ and ABC a triangle with the sermiperimeter s and the area F such that

$u + v = 2$. Prove that:

$$2s + a^u b^v + b^u c^v + c^u a^v \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

S.3372 If $x, y, z \in [0,4]$ then in any triangle ABC with the area F the following inequality holds:

$$a^x + b^y + c^z + a^{4-x} + b^{4-y} + c^{4-z} \geq 8\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3373 Let $m \geq 0, x, y \geq 0$ and ABC a triangle with the area F, then:

$$\frac{a^{m+2}}{(bx+cy)^m} + \frac{b^{m+2}}{(cx+ay)^m} + \frac{c^{m+2}}{(ax+by)^m} \geq \frac{4}{(x+y)^m} \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3374 In any triangle ABC with the area F the following inequality holds:

$$\frac{(a^2+b^2+2)^{m+1}}{(c^2+1)^m} + \frac{(b^2+c^2+2)^{m+1}}{(a^2+1)^m} + \frac{(c^2+a^2+2)^{m+1}}{(b^2+1)^m} \geq 2^{m+1}\left(4F+\sqrt{3}\right)\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3375 In any triangle ABC with the area F the following inequality holds:

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} + \frac{a^2}{(b+c)^4 h_a} + \frac{b^2}{(c+a)^4 h_b} + \frac{c^2}{(a+b)^4 h_c} \geq \frac{3}{4F}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3376 In any triangle ABC with the semiperimeter s the following inequality holds:

$$a^3 + b^3 + c^3 \geq 72\sqrt{3} \cdot r^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3377 Let be $m \geq 0$ and ABC a triangle with the area F then:

$$\frac{(a^2+b^2+2c^2)^{m+1}}{(a^2+b^2)^m} + \frac{(b^2+c^2+2a^2)^{m+1}}{(b^2+c^2)^m} + \frac{(c^2+a^2+2b^2)^{m+1}}{(c^2+a^2)^m} \geq 2^{m+3} \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3378 Let be $x, y, z > 0$ and M an interior point in triangle ABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CA respectively AB, then:

$$\frac{x^2a^3}{(y+z)^2d_a}+\frac{y^2b^3}{(z+x)^2d_b}+\frac{z^2c^3}{(x+y)^2d_c}\geq 6F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3379 In any triangle ABC with the area F the following inequality holds:

$$m_a\cdot a^2+m_b\cdot b^2+m_c\cdot c^2\geq 2F\cdot\sqrt{\frac{m_a\cdot m_b}{\sin^2\frac{C}{2}}+\frac{m_b\cdot m_c}{\sin^2\frac{A}{2}}+\frac{m_c\cdot m_a}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3380 If $s, t, x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{x^2a^8}{(sy+tz)^2}+\frac{y^2b^8}{(sz+tx)^2}+\frac{z^2c^8}{(sx+ty)^2}\geq\frac{256}{3(s+t)}\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

S.3381 Let be $x, y, z > 0$ and M an interior point in triangle ABC with the area F and

$F_a =$ area $MBC, F_b =$ area $MCA, F_c =$ area MAB. Prove that:

$$\frac{x^2\cdot a^4}{(y+z)^2F_b}+\frac{y^2\cdot b^4}{(z+x)^2F_c}+\frac{z^2\cdot c^4}{(x+y)^2F_a}\geq 12\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3382 In any triangle ABC with the area F the following inequality holds:

$$a^2r_a+b^2r_b+c^2r_c\geq 2F\cdot\sqrt{\frac{r_ar_b}{\sin^2\frac{C}{2}}+\frac{r_br_c}{\sin^2\frac{A}{2}}+\frac{r_cr_a}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3383 In any triangle ABC the following inequality holds:

$$a^3+b^3+c^3\geq 36\sqrt{3}r^3$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

S.3384 If $x, y, z > 0$ and ABC is a triangle with the area F, then:

$$\frac{x+y}{h_a}\cdot b+\frac{y+z}{h_b}\cdot c+\frac{z+x}{h_c}\cdot a-\left(\frac{xa}{h_a}+\frac{yb}{h_b}+\frac{zc}{h_c}\right)\geq 4\sqrt{xy\sin^2\frac{C}{2}+yz\sin^2\frac{A}{2}+zx\sin^2\frac{B}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

S.3385 Let be $s, t, x, y, z > 0$ and ABC a triangle with the area F, then:

$$\frac{x^2a^3}{(y+z)^2h_a} + \frac{y^2b^3}{(z+x)^2h_b} + \frac{z^2c^3}{(x+y)^2h_c} \geq 2F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3386 Let be $m \geq 0, x, y \geq 0, x + y \geq 4(m+1)$ and ABC a triangle with the area F then the following inequality holds: $(a^x + b^y + c^x) \cdot (a^y + b^x + c^y) \geq 4^{m+1} \cdot (\sqrt{3})^{1-m} \cdot F^{m+1}$

Proposed by D.M. Bătinețu – Giurgiu -Romania

S.3387 If $x, y, z \in \mathbb{R}_+^* = (0, \infty)$ and $(F_n)_{n\geq 1}$ is Fibonacci sequence, then in any triangle ABC with the area S the following inequality holds:

$$\frac{xa^4}{y \cdot F_n^2 + zF_{n+1}^2} + \frac{yb^4}{z \cdot F_n^2 + x \cdot F_{n+1}^2} + \frac{zc^4}{xF_n^2 + y \cdot F_{n+1}^2} \geq \frac{16}{F_{2n+1}} \cdot S^2, \forall n \in \mathbb{N}^*$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

S.3388 Let be M be an interior point in triangle ABC with the area F and $F_a =$ area MBC,

$F_b =$ area $MCA, F_c =$ area MAB. Prove that:

$$\frac{(ab+bc+2xF_a)^2}{ca+xF_b} + \frac{(bc+ca+2xF_b)^2}{ab+xF_c} + \frac{(ca+ab+2xF_c)^2}{bc+xF_a} \geq 4(\sqrt{3}+x) \cdot F, \forall x \in \mathbb{R}_+^* = (0, \infty)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

S.3389 If $x, y, z > 0$ and $t \geq 0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{x}{y+z}+t\right)\frac{a^3}{h_a} + \left(\frac{y}{z+x}+t\right)\frac{b^3}{h_b} + \left(\frac{z}{x+y}+t\right)\frac{c^3}{h_c} \geq 4(2t+1)F$$

Proposed by D.M. Bătinețu – Giurgiu ,Daniel Sitaru-Romania

S.3390 Let be M an interior point in triangle ABC with the area F and $F_a =$ area MBC,

$F_b =$ area $MCA, F_c =$ area MAB. Prove that:

$$((a^2+F_b)^2+2) \cdot ((b^2+F_c)^2+2) \cdot ((c^2+F_a)^2+2) \geq 3(4\sqrt{3}+1)^2 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu ,Daniel Sitaru-Romania

S.3391 If $x, y, z > 0$ and $t \geq 1$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{x}{y+z}+t\right)a^4 + \left(\frac{y}{z+x}+t\right)b^4 + \left(\frac{z}{x+y}+t\right)c^4 \geq 8(2t+1) \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu ,Daniel Sitaru-Romania

S.3392 If $x,y,z>0$ and $t\geq 0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{x}{y+z}+t\right)a^4+\left(\frac{y}{z+x}+t\right)b^4+\left(\frac{z}{x+y}+t\right)c^4\geq 8(2t+1)\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3393 In any triangle ABC with the area F the following inequality holds:

$$\prod_{cyc}(a^4+1)\cdot(b^4+1)\geq 1296\cdot F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3394 Let be ABC a triangle with the area F, then:

$$\frac{(a^2+b^2+2F)^2}{c^2+F}+\frac{(b^2+c^2+2F)^2}{a^2+F}+\frac{(c^2+a^2+2F)^2}{b^2+F}\geq 4\left(4\sqrt{3}+1\right)F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3395 If $x,y>0$ then in any triangle ABC with the area F the following inequality holds:

$$x^2(a^4+b^4+c^4)+y^2(a^2b^2+b^2c^2+c^2a^2)\geq 32xy\cdot F^2+\sum_{cyc}(ax-by)^2\cdot a^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3396 Let be $s,t,x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{x}{sy+tz}\cdot a^4+\frac{y}{sz+tx}\cdot b^4+\frac{z}{sx+ty}\cdot c^4\geq\frac{16}{s+t}\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3397 In any triangle ABC with the area F let be M an interior point and $F_a=$ area MBC,

$F_b=$ area $MCA, F_c=$ area MAB then:

$$\frac{(1936F+388F_2)^2}{F_b+F_c}+\frac{(1936F+388F_b)^2}{F_c+F_a}+\frac{(1936F+388F_c)^2}{F_a+F_b}\geq 18436593F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3398 With the usual notations in triangle ABC with the area F and $x,y\geq 0, x+y\geq 8$ the following inequality holds:

$$(a^x+b^y+c^x)\cdot(a^y+b^x+c^y)\geq 256F^4$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

S.3399 If $a,b,c>0, a+b+c\geq 3$ and $\lambda\geq 0$ then:

$$\frac{a^2}{b+\lambda bc}+\frac{b^2}{c+\lambda ca}+\frac{c^2}{a+\lambda ab}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu –Romania

S.3400 If $a,b,c>0, a+b+c\geq 3$ and $\lambda\geq 0, n\in\mathbb{N}, n\geq 2$ then:

$$\frac{a^n}{b+\lambda bc}+\frac{b^n}{c+\lambda ca}+\frac{c^n}{a+\lambda ab}\geq\frac{3}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.3401 If $x,y,z>0$ and $u,v\geq 0, u+v=4$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{xa^u}{y+z}+\frac{yb^v}{z+x}+\frac{zc^u}{x+y}\right)\left(\frac{xa^v}{y+z}+\frac{yb^u}{z+x}+\frac{zc^v}{x+y}\right)\geq 12\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru –Romania

S.3402 If $x,y>0$ then in any triangle ABC the following inequality holds:

$$\frac{1}{a(bx+cy)}+\frac{1}{b(cx+ay)}+\frac{1}{c(ax+by)}\geq\frac{27}{(x+y)(a+b+c)^2}$$

Proposed by D.M. Bătinețu – Giurgiu–Romania

S.3403 If $x,y\geq 0, x+y=4$ then in any acute-angled triangle ABC with the area F the following inequality holds: $(a^x\cot A+b^y\cot B+c^y\cot C)(a^y\cot A+b^x\cot B+c^x\cot C)\geq 16F^2$

Proposed by D.M. Bătinețu – Giurgiu–Romania

S.3404 If $x,y,z>0$ and ABC is a triangle with the area F, then:

$$\frac{x^2a^4}{(y+z)^2}+\frac{y^2b^4}{(z+x)^2}+\frac{z^2c^4}{(x+y)^2}+3\geq 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

S.3405 If $x,y>0$ then in triangle ABC with the area S the following inequality holds:

$$\frac{1}{a(bx+cy)}+\frac{1}{b(cx+ay)}+\frac{1}{c(ax+by)}\geq\frac{108}{(x+y+1)^2(a+b+c)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3406 If $x,y,z>0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(xa^4+\frac{yb^3}{z+x}+\frac{zc^2}{x+y}\right)\cdot\left(\frac{x}{(y+z)^2}+\frac{yb}{z+x}+\frac{zc^2}{x+y}\right)\geq 12\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3407 Let $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0$. Find:

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}^2} - \sqrt[n]{a_n^2}}{\sqrt[n]{n!}}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze –Romania

S.3408 Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n} = a > 0, \lim_{n\to\infty} \frac{b_{n+1}}{n^2 b_n} = b > 0$$

Find: $\lim\limits_{n\to\infty} \frac{1}{n^3} \cdot \sum_{k=2}^{n} \sqrt[k]{a_k} \cdot \left(\sqrt[k+1]{b_{k+1}} - \sqrt[k]{b_k}\right)$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze –Romania

S.3409 Let be $u, v > 0$ and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of reals strictly positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^u} = a > 0, \lim_{n\to\infty} \frac{b_{n+1}}{b_n n^v} = b > 0$$

and $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n} \frac{1}{k}$. Find: $\lim\limits_{n\to\infty} \sqrt[n]{a_n b_n} \cdot e^{-(u+v)H_n}$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu–Romania

S.3411 If $s \geq 0$, find: $\lim\limits_{n\to\infty} \left((n+1)\cdot\left(\sqrt[n+1]{n+1}\right)^{s+1} - n\left(\sqrt[n]{n}\right)^{s+1}\right)$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru –Romania

S.3412 Let be $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n} \frac{1}{k}$ and $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n \cdot e^{H_n}} = a > 0$. Find: $\lim\limits_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$

Proposed by D.M. Bătinețu – Giurgiu–Romania

S.3413 If $(a_n)_{n\geq 1}$ is a sequence of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{(2n-1)!!}} = a$$

Find: $\lim\limits_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$

Proposed by D.M. Bătinețu – Giurgiu –Romania

S.3414 If $(a_n)_{n\geq 1}, (\gamma_n)_{n\geq 1}$, are sequence of real strictly positive numbers such that

$\lim\limits_{n\to\infty} r_n = r > 0$ and $a_{n+1} = a_n + r_n, \forall n \geq 1$. Find:

$$\lim_{n\to\infty} \left(a_{n+1} \sqrt[n+1]{a_{n+1}} - a_n \cdot \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3415 If $t \geq 0$, find:

$$\lim_{n\to\infty}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{t+1}}{(n+1)^t}-\frac{\left(\sqrt[n]{n!}\right)^{t+1}}{n^t}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3416 Let be $(a_n)_{n\geq 1}$ a sequence with $a_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$ and

$$a_{n+1} = a_n + \sqrt[n]{n!}\cdot\tan\frac{\pi}{2}, \forall n \in \mathbb{N}^*. \text{ Find: } \lim_{n\to\infty}\left(a_{n+1}\cdot\sqrt[n+1]{a_{n+1}} - a_n\cdot\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3417 If $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty}\frac{a_{n+1}}{n^3\cdot a_n} = a > 0$, find: $\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{\left(\sqrt[n]{n!}\right)^2}$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3418 Find: $\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{\left((n+1)!\right)^2}}{n+1}-\frac{\sqrt[n]{(n!)^2}}{n}\right)$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3419 If $f: \mathbb{R}_+^* \to \mathbb{R}_+^* = (0,\infty)$ is a continuous function such that $\lim_{n\to\infty}\frac{f(x+1)}{x^{s+1}f(x)} = a > 0$. Find:

$$\lim_{x\to\infty}\frac{1}{x^s}\left(\left(f(x+1)\right)^{\frac{1}{x+1}}-\left(f(x)\right)^{\frac{1}{x}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3420 Let be $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n} = a > 0$. Find:

$$\lim_{n\to\infty}\sqrt[n]{(n!)^3}\cdot\left(\frac{1}{\sqrt[n]{a_n}}-\frac{1}{\sqrt[n+1]{a_{n+1}}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3421 Let be $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^n\frac{1}{k}$ and $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ such that

$$\lim_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n} = a > 0. \text{ Find: } \lim_{n\to\infty}e^{2H_n}\cdot\left(\sqrt[n]{a_n}-\sqrt[n+1]{a_{n+1}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

S.3422 Let be $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{na_n} = a > 0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!\,a_{n+1}}-\sqrt[n]{n!\,a_n}\right)\cdot\frac{1}{n}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3423 If $(a_n)_{n\geq1}, (b_n)_{n\geq1}, a_n, b_n > 0, \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a > 0$,

$$\lim_{n\to\infty} \frac{b_{n+1}}{n \cdot b_n} = b > 0.\text{ Find: } \lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}}{\sqrt[n]{b_n}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

S.3424 Let be $(a_n)_{n\geq1}$ a sequence of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{n!}} = a > 0$$

Find:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

S.3425 Let be $(H_n)_{n\geq1}, H_n = \sum_{k=1}^{n} \frac{1}{k}$ and $(a_n)_{n\geq1}$ a sequence of real strictly positive numbers such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n \cdot \sqrt[n]{n!}} = a > 0$. Find: $\lim\limits_{n\to\infty} \frac{e^{2H_n}}{n \cdot \sqrt[n]{a_n}}$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3427 Let be $(a_n)_{n\geq1}, a_n > 0, \forall n \in \mathbb{N}^*$ such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = a > 0$. Find:

$$\lim_{n\to\infty} \left(\sqrt[n+1]{(n+1)!\, a_{n+1}} - \sqrt[n]{n!\, a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3428 Let be $(a_n)_{n\geq1}, a_n = n!$. Find:

$$\lim_{n\to\infty} \left(\frac{\sqrt[n+1]{a_{n+1}^2}}{n+1} - \frac{\sqrt[n]{a_n^2}}{n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3429 Let be $(a_n)_{n\geq1}, (b_n)_{n\geq1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{a_n}{n} = a > 0 \text{ and } \lim_{n\to\infty} \frac{b_{n+1}}{a_n b_n} = b > 0.\text{ Find: } \lim_{n\to\infty} \frac{b_n^{\frac{1}{a_n}}}{a_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3430 Let be $(a_n)_{n\geq1}, a_n > 0, \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0$. Find:

$$\lim_{n\to\infty} \frac{\sqrt[n+1]{a_{n+1}^2} - \sqrt[n]{a_n^2}}{\sqrt[n]{(2n-1)!!}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3431 Let be $(H_n)_{n\ge1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$ and $(a_n)_{n\ge1}$ a sequence of real strictly positive numbers such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\cdot\sqrt[n]{n!}\cdot e^{H_n}} = a > 0$. Find: $\lim\limits_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1}}} - \frac{n^3}{\sqrt[n]{a_n}}\right)$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3432 Let be $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ sequences of real strictly positive numbers such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = a > 0$, $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = b > 0$. Find: $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}\cdot b_{n+1}} - \sqrt[n]{a_n\cdot b_n}\right)$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți–Romania

S.3433 Let be $s, t > 0$ and the sequences of real strictly positive numbers such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{n^s\cdot a_n} = a > 0$, $\lim\limits_{n\to\infty}\frac{n^{s+t}\cdot b_{n+1}}{b_n} = b$. Find: $\lim\limits_{n\to\infty}\sqrt[n]{a_n\cdot b_n}\cdot n^t$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3434 Let be $(a_n)_{n\ge1}, a_n > 0, \forall n \ge 1$ such that $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = a > 0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\cdot a_{n+1}} - \sqrt[n]{(2n-1)!!\cdot a_n}\right)$$

Proposed by** D.M. Bătinețu – Giurgiu, Claudia Nănuți **–Romania

S.3435 Let $n \in \mathbb{N}, n \ge 3$ and $a_k \in \mathbb{R}_+^* = (0,\infty), \forall k = \overline{1,n}$. If $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$\sum_{k=1}^{n}\frac{(a_k+1)^{m+1}}{(a_{k+1}+1)^m} \ge n+1,\ \forall m \ge 0$, where $a_{n+1} = a_1$.

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3436 Let $(a_n)_{n\ge1}$ a sequence of real strictly positive numbers and

$(H_n)_{n\ge1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$. If $\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = a > 0$. Find: $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)\cdot e^{H_n}$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3437 Let be $m \ge 0$ and ABC a triangle then:

$$\frac{a^{4m+4}}{(bc)^{m+1}} + \frac{b^{4m+4}}{(ca)^{m+1}} + \frac{c^{4m+4}}{(ab)^{m+1}} \ge 4^{m+1}\cdot 3^{m+2}\cdot r^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3438 If $x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\left(\frac{x^2}{(y+z)^2}\cdot a^2 + \frac{y^2}{(z+x)^2}\cdot b^2 + \frac{z^2}{(x+y)^2}\cdot c^2\right)\cdot(a^2+b^2+c^2) \ge 12\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3439 If $x, y > 0$ and ABC is a triangle with the area F then the following inequality holds:

$$\frac{ab^2}{bx+cy}+\frac{bc^2}{cx+ay}+\frac{ca^2}{ax+by}\geq\frac{4\sqrt{3}}{x+y}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3440 If $x, y > 0$ and ABC is a triangle with the area F, then:

$$\frac{a^4}{b^2x+c^2y}+\frac{b^4}{c^2x+a^2y}+\frac{c^4}{a^2x+b^2y}\geq\frac{4\sqrt{3}}{x+y}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3441 If $m \geq 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^{m+2}}{b^m}+\frac{b^{m+2}}{c^m}+\frac{c^{m+2}}{a^m}\geq 4\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

S.3442 In any triangle ABC with the area F the following inequality holds:

$$\frac{a^4+b^4}{c^2}+\frac{b^4+c^4}{a^2}+\frac{c^4+a^4}{b^2}\geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze –Romania

S.3443 If $x, y, z > 0$ and ABC is a triangle with the area F, then:

$$\frac{x+y}{z}a^2+\frac{y+z}{x}b^2+\frac{z+x}{y}c^2\geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze –Romania

S.3444 Let be $t \in \left(0, \frac{\pi}{2}\right)$ and ABC is a triangle with the area F, then:

$$a^2+b^2+c^2\geq 4\sqrt{3}\cdot F\sin 2t+(a\sin t-b\cos t)^2+(b\sin t-c\cos t)^2+(c\sin t-a\cos t)^2$$

Proposed by D.M. Bătinețu – Giurgiu –Romania

S.3445 In any triangle ABC with the area F the following inequality holds:

$$a^2b^2+b^2c^2+c^2a^2\geq 16\cdot F^2+\frac{1}{2}\left(a^2(b-c)^2+b^2(c-a)^2+c^2(a-b)^2\right)$$

Proposed by D.M. Bătinețu – Giurgiu –Romania

S.3446 In any triangle ABC with the area F the following inequality holds:

$$a^4+b^4+c^4\geq 16F^2+\frac{1}{2}\left((a^2-b^2)+(b^2-c^2)^2+(c^2-a^2)^2\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru–Romania

S.3447 If $m\in[1,\infty)$, then in any triangle with the area F and semiperimeter s the following inequality holds:

$$\frac{a^3}{2ms-(m+1)a}+\frac{b^3}{2ms-(m+1)b}+\frac{c^3}{2ms-(m+1)c}\geq\frac{4\sqrt{3}}{3(3m-1)}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru –Romania

S.3448 If $a,b,c>0,m\in[1,\infty)$ and $a+b+c=s$, then:

$$\frac{a}{ms-(m+1)b}+\frac{b}{ms-(m+1)c}+\frac{c}{ms-(m+1)a}\geq\frac{3}{2m-1}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu –Romania

S.3449 Let be $m\in[1,\infty)$ and $n\in\mathbb{N},n\geq 3$ and $a_1,a_2,\dots,a_n$ the lengths of the sides of a polygon with n sides and s and F the semiperimeter, respectively the area of the polygon, then:

$$\sum_{cyc}\frac{a_k^3}{2ms-(m+1)a_k}\geq\frac{4F}{mn-m-1}\cdot\tan\frac{\pi}{n}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu –Romania

S.3450 Let be $m\geq 0,n\in\mathbb{N},n\geq 3$ and $a_k\in\mathbb{R}_+^*,\forall k=\overline{1,n},s,t>0$ then:

$$\sum_{k=1}^{n}\frac{(a_k+s)^{m+1}}{(a_{k+1}+t)^m}\geq\frac{\left(\sum_{k=1}^{n}a_k+ns\right)^{m+1}}{\left(\sum_{k=1}^{n}a_k+nt\right)^m},\text{ where }a_{n+1}=a_1.$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți –Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.3301 Let be $(F_n)_{n\geq 1}, (L_n)_{n\geq 1}$ Fibonacci sequence and Lucas' sequence and $a, b, c > 0$.

Prove that:

$$\frac{(F_n a + F_{n+1} b)^2}{L_n b + L_{n+1} \cdot c} + \frac{(F_n b + F_{n+1} c)^2}{L_n c + L_{n+1} a} + \frac{(F_n c + F_{n+1} a)^2}{L_n a + L_{n+1} b} \geq \frac{F_{n+2}^2}{L_{n+2}}(a + b + c), \forall n \in \mathbb{N}^*$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3302 Prove that in any triangle ABC, with area S and usual notations is true the inequality:

$$\frac{a^{2(m+1)}}{(F_n m_a^2 + F_{n+1} m_b^2 + F_{n+2} m_c^2)^m} + \frac{b^{2(m+1)}}{(F_n m_b^2 + F_{n+1} m_c^2 + F_{n+2} m_a^2)} +$$

$$+\frac{c^{2(m+1)}}{(F_n m_c^2 + F_{n+1} m_a^2 + F_{n+2} m_b^2)} \geq \frac{2^{m+2}\sqrt{3}}{3^m F_{n+2}^m} S,$$

where $m > 0$ and F_k is the k-th Fibonacci number.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

U.3303 Determine all matrices $A \in M_2(\mathbb{R})$ such that:

$$A^3 = \begin{pmatrix} 19 & 30 \\ -45 & -71 \end{pmatrix}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

U.3304 Solve on $[0, \infty)$ the following system:

$$\begin{cases} (x+y+z)^3 = y^3 + z^3 + t^3 + 24yzt \\ (y+z+t)^3 = z^3 + t^3 + x^3 + 24ztx \\ (z+t+x)^3 = t^3 + x^3 + y^3 + 24txy \\ (t+x+y)^3 = x^3 + y^3 + z^3 + 24xyz \end{cases}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

U.3305 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$\frac{n_a - m_a}{h_a} + \frac{n_b - m_b}{h_b} + \frac{n_c - m_c}{h_c} \geq \frac{a^2 + b^2 + c^2 - ab - bc - ca}{2F}$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3306 If n_a, n_b, n_c – Nagel cevians in ΔABC then:

$$n_a + n_b + n_c \geq m_a + m_b + m_c + \frac{{n_a}^2 + {n_b}^2 + {n_c}^2 - s^2}{2s}$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3307 If n_a, n_b, n_c – Nagel cevians in ΔABC then:

$$n_a + n_b + n_c \geq 2s - 3\left(2\sqrt{3} - 3\right)r$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3308 If n_a, n_b, n_c – Nagel cevians in ΔABC then: $(n_a + n_b + n_c)^2 \geq 9s^2 - 80Rr - 2r^2$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3309 If n_a, n_b, n_c – Nagel cevians in ΔABC then: $n_a + n_b + n_c + 18r \geq 3(h_a + h_b + h_c)$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3310 If n_a, n_b, n_c – Nagel cevians in ΔABC then: $n_a n_b + n_b n_c + n_c n_a \geq 3s^2 - 32Rr + 10r^2$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3311 If n_a, n_b, n_c – Nagel cevians in ΔABC then:

$$n_a + n_b + n_c + \frac{2\sqrt{3} - 3}{3}.(h_a + h_b + h_c) \geq 2s$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3312 If n_a, n_b, n_c – Nagel cevians in ΔABC then:

$$3(n_a + n_b + n_c) + h_a + h_b + h_c \geq 4(m_a + m_b + m_c)$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3313 If n_a, n_b, n_c – Nagel cevians in ΔABC then: $3(n_a + n_b + n_c) + 9r \geq 4(m_a + m_b + m_c)$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3314 If n_a, n_b, n_c – Nagel cevians in ΔABC then: $n_a + n_b + n_c \geq h_a + h_b + h_c + 2\left(s - 3\sqrt{3}r\right)$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3315 If n_a, n_b, n_c – Nagel cevians in ΔABC then:

$$n_a + n_b + n_c \geq m_a + m_b + m_c + \frac{1}{2}\left(s - 3\sqrt{3}r\right)$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3316 If n_a, n_b, n_c − Nagel cevians in ΔABC then:

$$(2\sqrt{3}-3)(n_a+n_b+n_c)+(4-2\sqrt{3})(m_a+m_b+m_c)\geq\sqrt{3}s$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3317 If n_a, n_b, n_c − Nagel cevians in ΔABC then:

$$(2-\sqrt{3})(n_a+n_b+n_c)+\sqrt{3}s\geq(3-\sqrt{3})(m_a+m_b+m_c)$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3318 If n_a, n_b, n_c − Nagel cevians in ΔABC then:

$$n_a+n_b+n_c\geq\frac{(32-15\sqrt{3})(a^2+b^2+c^2)-(32-18\sqrt{3})(ab+bc+ca)}{2(a+b+c)}$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3319 If $\mathrm{g_a}, \mathrm{g_b}, \mathrm{g_c}$ − Gergonne cevians in ΔABC then:

$$(\mathrm{g_a}+\mathrm{g_b}+\mathrm{g_c})^2\geq\frac{8\mathrm{R}+29\mathrm{r}}{5}(\mathrm{h_a}+\mathrm{h_b}+\mathrm{h_c})$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3320 If g_a, g_b, g_c − Gergonne cevians in ΔABC then:

$$(g_a+g_b+g_c)^2\leq s^2+4(2\sqrt{5}-1)Rr+2(31-8\sqrt{5})r^2$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3321 If n_a, n_b, n_c − Nagel cevians in ΔABC then:

$$n_a(n_b+n_c-n_a)+n_b(n_c+n_a-n_b)+n_c(n_a+n_b-n_c)\geq{h_a}^2+{h_b}^2+{h_c}^2$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3322 If n_a, n_b, n_c − Nagel cevians in ΔABC then:

$$s(n_a+n_b+n_c)\geq(2-\sqrt{3})({n_a}^2+{n_b}^2+{n_c}^2)+2(\sqrt{3}-1)s^2$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3321 If g_a, g_b, g_c − Gergonne cevians in ΔABC then:

$$(g_a+g_b+g_c)^2\geq s^2+6r(h_a+h_b+h_c)$$

Proposed by Mohamed Amine Ben Ajiba − Morocco

U.3322 If g_a, g_b, g_c – Gergonne cevians in ΔABC then:

$$m_a + m_b + m_c \le g_a + g_b + g_c + 2(R - 2r).$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3323 If g_a, g_b, g_c – Gergonne cevians in ΔABC then:

$$2(m_a + m_b + m_c) + 9r \ge 3(g_a + g_b + g_c).$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3324 If n_a, n_b, n_c – Nagel cevians in ΔABC then:

$$n_a + n_b + n_c - 9r \ge 3 \cdot \frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{a+b+c}$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3325 If n_a, n_b, n_c – Nagel cevians, g_a, g_b, g_c – Gergonne cevians in ΔABC then:

$$n_a + n_b + n_c + 9r \ge 2(g_a + g_b + g_c).$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3326 If n_a, n_b, n_c – Nagel cevians, g_a, g_b, g_c – Gergonne cevians in ΔABC then:

$$\left(\sqrt{3} - 1\right)(n_a + n_b + n_c) + \left(2 - \sqrt{3}\right)(g_a + g_b + g_c) \ge \sqrt{3}s.$$

Proposed by Mohamed Amine Ben Ajiba – Morocco

U.3327 Find:

$$\Omega = \int_0^\infty \frac{x \ln^2(x) \arctan(x)}{\left(x^2+1\right)\left(x^2+x+1\right)} dx$$

Proposed by Vasile Mircea Popa-Romania

U.3328 Find:

$$\Omega = \lim_{n\to\infty}\left[n \int_0^1 \ln\left(1+e^{-nx}\right) dx\right]$$

Proposed by Vasile Mircea Popa-Romania

U.3329 In any acute triangle ABC holds:

$$bc\sqrt{\cot A}+ca\sqrt{\cot B}+ab\sqrt{\cot C}>\frac{8}{3}R^2$$

Proposed by Vasile Mircea Popa-Romania

U.3330 Find:

$$\Omega=\int_0^{\infty}\frac{x\,|\ln(x)|}{(x^2+1)(x+1)^2}dx$$

Proposed by Vasile Mircea Popa-Romania

U.3331 Find:

$$\Omega=\int_0^{\infty}\frac{x\ln^2(x)}{(x^2+1)(x^2-x+1)}dx$$

Proposed by Vasile Mircea Popa-Romania

U.3332 If $a,b>0$ and $n\in\mathbb{N},n\geq 2$ then:

$$(3a+3b)^n+(n-4)(4a)^{n-1}a\geq 3n(4a)^{n-1}b$$

Proposed by Marin Chirciu – Romania

U.3333 If $x,y,z>0,xyz=1$ and $\lambda\leq\frac{3}{2}$ then:

$$x^2+y^2+z^2+3(2\lambda-1)\geq\lambda(x+y+z+xy+yz+zx)$$

Proposed by Marin Chirciu – Romania

U.3334 If $x,y,z>0,xyz=\lambda^3,\lambda\geq 2$ then:

$$\frac{1}{x^2+\lambda^3}+\frac{1}{y^2+\lambda^3}+\frac{1}{z^2+\lambda^3}\leq\frac{1}{\lambda^2(\lambda+1)}$$

Proposed by Marin Chirciu – Romania

U.3335 If $x,y,z>0$ and $-2\leq\lambda\leq 2$ then:

$$\sum\left(\frac{x^2+\lambda xy+y^2}{\sqrt{x+y}}\right)\geq\frac{3}{2}(\lambda+2)^2xyz$$

Proposed by Marin Chirciu – Romania

U.3336 Solve in real numbers:

$$4^{x^2}+4^{1+\frac{1}{\sqrt{x}}}=20$$

Proposed by Marin Chirciu – Romania

U.3337 Let $a > 1$ fixed. Solve in real numbers:

$$(a^x + 2a + 1)^{\log_{a+1} a} - ((a+1)^x - 2a - 1)^{\log_a(a+1)} = (a+1)^x - a^x - 2(2a+1)$$

Proposed by Marin Chirciu – Romania

U.3338 If $a, b, c > 0, ab + bc + ca = 3$ and $-2 \leq \lambda \leq 2$ then:

$$\sum \left(\frac{a^2 + \lambda ab + b^2}{a+b}\right)^2 \geq \frac{3}{4}(\lambda + 2)^2$$

Proposed by Marin Chirciu – Romania

U.3339 Let $n \in \mathbb{N}^*$ fixed. Solve in real numbers:

$$n^{x^{\frac{n}{2}}} + n^{1+\frac{1}{\sqrt{x}}} = n(n+1)$$

Proposed by Marin Chirciu – Romania

U.3340 If $a, b, c > 0, abc = 1$ then:

$$\sum \frac{(a+b+1)^3}{a^4 + b^4 + 1} \leq 27$$

Proposed by Marin Chirciu – Romania

U.3342 In ΔABC the following relationship holds:

$$\sum \frac{y+z}{x \cdot \cot^4 \frac{A}{2}} \geq \frac{2}{3}$$

Proposed by Marin Chirciu – Romania

U.3343 Let $x, y, z > 0$. In ΔABC the following relationship holds:

$$\sum \frac{y+z}{x \cdot \tan^4 \frac{A}{2}} \geq 54$$

Proposed by Marin Chirciu – Romania

U.3344 In ΔABC the following relationship holds:

$$\sum \frac{y+z}{x \cdot r_a^4} \geq \frac{32}{37R^4}$$

Proposed by Marin Chirciu – Romania

U.3345 In ΔABC the following relationship holds:

$$\sum \frac{y+z}{x \cdot m_a^4} \geq \frac{32}{27R^4}$$

Proposed by Marin Chirciu – Romania

U.3346 In ΔABC the following relationship holds:

$$\sum \frac{y+z}{x\cdot IA^4} \geq \frac{3}{2R^2r^2}$$

Proposed by Marin Chirciu – Romania

U.3347 In ΔABC the following relationship holds:

$$3\cdot\left(\frac{2r}{R}\right)^2 \leq \frac{w_a}{w_b}+\frac{w_b}{w_c}+\frac{w_c}{w_a} \leq 3\cdot\left(\frac{R}{2r}\right)^{\frac{3}{2}}$$

Proposed by Marin Chirciu – Romania

U.3348 In ΔABC the following relationship holds:

$$3\cdot\left(\frac{2r}{R}\right)^2 \leq \frac{s_a}{s_b}+\frac{s_b}{s_c}+\frac{s_c}{s_a} \leq 3\cdot\left(\frac{R}{2r}\right)^{\frac{3}{2}}$$

Proposed by Marin Chirciu – Romania

U.3349 In ΔABC the following relationship holds:

$$\sum \frac{\sin^2\frac{A}{2}}{2\sin^6\frac{A}{2}+\sin^6\frac{B}{2}} \leq \frac{2R^3}{r^3}$$

Proposed by Marin Chirciu – Romania

U.3350 In ΔABC the following relationship holds:

$$\sum \frac{\sin B+\sin C}{h_b+h_c} \geq \sum \frac{\sin B+\sin C}{r_b+r_c}$$

Proposed by Marin Chirciu – Romania

U.3351 In ΔABC the following relationship holds:

$$\sum \frac{\sin B\cdot\sin C}{\csc\frac{A}{2}} \geq \frac{3\sqrt{3}}{2}\cdot\left(\frac{2r}{R}\right)^{\frac{4}{3}}$$

Proposed by Marin Chirciu – Romania

U.3352 In ΔABC the following relationship holds:

$$\sum \frac{\sin B+\sin C}{w_a} \geq \frac{2\sqrt{3}}{R}\left(\frac{2r}{R}\right)^{\frac{1}{3}}$$

Proposed by Marin Chirciu – Romania

U.3353 In acute ΔABC the following relationship holds:

$$\sum \frac{\sin B + \sin C}{HA} \ge \frac{3\sqrt{3}}{R} \cdot \left(\frac{2r}{R}\right)$$

Proposed by Marin Chirciu – Romania

U.3354 In ΔABC the following relationship holds:

$$\sum \frac{\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)^4}{\tan^2\frac{A}{2} + \lambda \tan\frac{A}{2}\tan\frac{B}{2} + \tan^2\frac{B}{2}} \ge \frac{16}{\lambda + 2}, \qquad \lambda \ge 0$$

Proposed by Marin Chirciu – Romania

U.3355 In ΔABC the following relationship holds:

$$\sum \frac{\sin B + \sin C}{s_a} \ge \frac{2\sqrt{3}}{R} \left(\frac{2r}{R}\right)^{\frac{2}{3}}$$

Proposed by Marin Chirciu – Romania

U.3356 In ΔABC the following relationship holds:

$$\sum \frac{\sin B + \sin C}{IA} \ge \frac{3\sqrt{3}}{R} \left(\frac{2r}{R}\right)^{\frac{1}{3}}$$

Proposed by Marin Chirciu – Romania

U.3357 In ΔABC the following relationship holds:

$$\sum b^2c^2\,(\lambda + \cos^2 A) \ge 4(4\lambda + 1)F^2,\ \ \lambda \ge 0.$$

Proposed by Marin Chirciu – Romania

U.3358 In ΔABC the following relationship holds:

$$p\left(1 + \frac{2r}{R}\right) \le (b + c)\sin\frac{A}{2} + (c + a)\sin\frac{B}{2} + (a + b)\sin\frac{C}{2} \le 2p$$

Proposed by Marin Chirciu – Romania

U.3359 In ΔABC the following relationship holds:

$$\frac{1}{2Rr} \le \sum \frac{1}{a(a + b - c)} \le \frac{1}{4r^2}\sqrt{\frac{R}{r} - 1}$$

Proposed by Marin Chirciu – Romania

U.3360 In ΔABC the following relationship holds:

$$\frac{\sqrt{3}}{2R} \le \sum \frac{b+c-a}{2a\sqrt{bc}} \le \frac{\sqrt{3}R}{8r^2}$$

Proposed by Marin Chirciu – Romania

U.3361 In ΔABC the following relationship holds:

$$\frac{6r^2}{R^2} \le \sum \frac{\sin^2 A}{\sin^2 B + \sin^2 C} \le \frac{2R-r}{2r}$$

Proposed by Marin Chirciu – Romania

U.3362 In ΔABC the following relationship holds:

$$\frac{\sqrt{3}}{R} \le \sum \frac{b+c-a}{a\sqrt{bc}} \le \frac{\sqrt{3}R}{4r^2}$$

Proposed by Marin Chirciu – Romania

U.3363 In ΔABC the following relationship holds:

$$18r^2 \le \frac{\sum (m_b^2 + m_c^2)a}{a+b+c} \le \frac{9R^3}{4r}$$

Proposed by Marin Chirciu – Romania

U.3364 In ΔABC the following relationship holds:

$$\frac{27r^2}{2R^2} \le \prod (1+\cos A) \le \frac{27}{8}$$

Proposed by Marin Chirciu – Romania

U.3365 In ΔABC the following relationship holds:

$$27\sqrt{3}r^2 \le \sum (m_b^2 + m_c^2)\sin A \le \frac{27\sqrt{3}R^3}{8r}$$

Proposed by Marin Chirciu – Romania

U.3366 In ΔABC the following relationship holds:

$$\frac{3S(4Rr - 2r^2 - R^2)}{R^4} \le \sum \sin^2 A \sin 2A \le \frac{9\sqrt{3}}{8} \cdot \frac{2r}{R}$$

Proposed by Marin Chirciu – Romania

U.3367 Let $\lambda > \frac{1}{3}$ fixed. If $a, b, c > 0, abc = 1$ then find minimum of

$$P = \frac{(a+b)(b+c)(c+a)}{\lambda(a+b+c) - 1}$$

Proposed by Marin Chirciu – Romania

U.3368 If $a, b, c > 0$ then:

$$\sum \frac{a+1}{\sqrt{4a^4+1}} + \sum \frac{2a^2+1}{a+1} \geq 6$$

Proposed by Marin Chirciu – Romania

U.3369 If $x, y, z > 0$ and $\frac{7}{4} \leq \lambda \leq 4$ then:

$$\sum \frac{x}{\sqrt{\lambda y^2 + yz + \lambda y^2}} \geq \frac{3}{\sqrt{2\lambda + 1}}$$

Proposed by Marin Chirciu – Romania

U.3370 Let $a \geq 1$. If $x > 0$ then:

$$\frac{1}{\sqrt{x+a}} + \frac{x}{2\sqrt{ax^2+1}} \leq \frac{3}{2\sqrt{a+1}}$$

Proposed by Marin Chirciu – Romania

U.3371 If $a, b, c > 0, a + b + c = 3$ and $\lambda \geq 0$ then:

$$\sum \frac{bc(c+\lambda)}{a+\lambda} \geq ab + bc + ca$$

Proposed by Marin Chirciu – Romania

U.3372 If $x_1, x_2, \ldots, x_n > 0, x_1 + x_2 + \cdots + x_n = 1$ then find the maximum value of $x_1 x_2^2 x_3^3 \ldots x_n^n$

Proposed by Marin Chirciu – Romania

U.3373 If $a, b, c > 0, ab + bc + ca = 1$ and $\lambda \geq 0$ then:

$$\frac{(a+b)^4}{a^2 + \lambda ab + b^2} + \frac{(b+c)^4}{b^2 + \lambda bc + c^2} + \frac{(c+a)^4}{c^2 + \lambda ca + a^2} \geq \frac{16}{\lambda + 2}$$

Proposed by Marin Chirciu – Romania

U.3374 If $x, y, z > 0$ and $\frac{7}{4} \leq \lambda \leq 4$ then:

$$\sum \frac{x}{\sqrt{\lambda y^2 + yz + \lambda z^2}} \geq \frac{3}{\sqrt{2\lambda + 1}}$$

Proposed by Marin Chirciu – Romania

U.3375 Let $a > 0, \lambda > 0$ fixed. Solve for real numbers:

$$(ax+1)\left(1+\sqrt{(ax+1)^2+\lambda}\right)+x\left(1+\sqrt{x^2+\lambda}\right)=0$$

Proposed by Marin Chirciu – Romania

U.3376 If $a, b, c, m, n > 0$ then:

$$\left(\sum \frac{a}{\sqrt[4]{mb+nc}}\right)^4 \geq \frac{(a+b+c)^5}{(m+n)(a^2+b^2+c^2)}$$

Proposed by Marin Chirciu – Romania

U.3377 If $0 \leq a_i \leq 1, i = \overline{1,n}$ and $\lambda \geq 1$ then:

$$\frac{1}{\lambda+a_1}+\frac{1}{\lambda+a_2}+\cdots+\frac{1}{\lambda+a_n} \leq \frac{n}{\lambda+\sqrt[n]{a_1a_2\ldots a_n}}$$

Proposed by Marin Chirciu – Romania

U.3378 If $a, b, c > 0, \frac{1}{a}+\frac{1}{b}+\frac{1}{c}=3$ and $x, y, z > 0, x+y+z=3n, n > 0$ and $\lambda \geq 0$ then:

$$\frac{1}{a(xa+\lambda)}+\frac{1}{b(yb+\lambda)}+\frac{1}{c(zc+\lambda)} \geq \frac{3}{n+\lambda}$$

Proposed by Marin Chirciu – Romania

U.3379 In $\Delta ABC, AA_1, BB_1, CC_1$ – angle bisectors. Prove that:

$$\left(\frac{BC}{B_1C_1}\right)^{2n}+\left(\frac{CA}{C_1A_1}\right)^{2n}+\left(\frac{AB}{A_1B_1}\right)^{2n} \geq 3\cdot 4^n, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

U.3380 Let $n \in \mathbb{N}$ fixed. Solve for real numbers:

$$[x]+\frac{n}{[x]}=\{x\}+\frac{n}{\{x\}}$$

Proposed by Marin Chirciu – Romania

U.3381 Solve for real numbers:

$$\sqrt{1-x}=1-2x^2-2x\sqrt{1-x^2}$$

Proposed by Marin Chirciu – Romania

U.3382 If $a, b, c \geq 0, a+b+c=3$ and $n \geq \lambda \geq 0$ then:

$$n(a^2+b^2+c^2)+\lambda\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right) \geq 3(n+\lambda)$$

Proposed by Marin Chirciu – Romania

U.3383 If $a,b,c \geq 0$ and $n \in \mathbb{N}, n \geq 2$ then:

$$\sum \sqrt[n]{\frac{a^{n-1}(b+c)}{2}} > \sqrt{ab}+\sqrt{bc}+\sqrt{ca}$$

Proposed by Marin Chirciu – Romania

U.3384 Write the number $72^{3n+1}, n \in \mathbb{N}$, as a sum of two perfect cubes.

Proposed by Marin Chirciu – Romania

U.3385 Prove that:

$$\frac{1}{\sin^2\frac{4\pi}{9}}-\frac{1}{\sin^2\frac{\pi}{9}}=\frac{8\sqrt{3}}{3}\left(-2\sin\frac{4\pi}{9}+\sin\frac{\pi}{9}\right)$$

Proposed by Vasile Mircea Popa-Romania

U.3386 Let be $s,t,x,y,z \in \mathbb{R}_+^* = (0,\infty)$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{xa^4}{sy+tz}+\frac{yb^4}{sz+tx}+\frac{zc^4}{sx+ty} \geq \frac{16}{s+t}\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3387 If $x,y,z > 0, (F_n)_{n\geq 1}$ is Fibonacci sequence then in any triangle ABC with the area S the following inequality holds:

$$\frac{xa^4}{yF_n+zF_{n+1}}+\frac{yb^4}{zF_n+xF_{n+1}}+\frac{zc^4}{xF_n+yF_{n+1}} \geq \frac{16}{F_{n+2}}\cdot S^2, \forall n \in \mathbb{N}^*$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3388 Let $m \geq 0$ and $a,b,c > 0, a+b+c=1$, then:

$$\frac{(a+b+2)^{m+1}}{(c+1)^m}+\frac{(b+c+2)^{m+1}}{(a+1)^m}+\frac{(c+a+2)^{m+1}}{(b+1)^m} \geq 2^{m+3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3389 If $a,b,c > 0$ and $a+b+c=1$ then:

$$\frac{(a+b+2)^3}{(c+1)^2}+\frac{(b+c+2)^3}{(a+1)^2}+\frac{(c+a+2)^3}{(b+1)^2} \geq 32$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3390 If $s,t,x,y,z\in\mathbb{R}_+^*=(0,\infty)$ then in any triangle ABC with the area F the following inequality:

$$\frac{xa^2}{sy+tz}+\frac{yb^2}{sz+tx}+\frac{zc^2}{sx+ty}\geq\frac{4}{s+t}\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3391 In any triangle ABC with the area F and the semiperimeter s the following inequality holds:

$$s+a^3+b^3+c^3\geq 4\sqrt{6}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3392 Let be M an interior point in triangle ABC with the area F and d_a,d_b,d_c the distances of point M to the sides BC,CA respectively AB, then:

$$\frac{a^2b}{h_ad_b^2}+\frac{b^2c}{h_bd_c^2}+\frac{c^2a}{h_cd_a^2}\geq 24\cdot\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3393 If $x,y,z>0$ then:

$$\frac{1936x+88y}{88x+1936y+2024z}+\frac{1936y+88z}{88y+1936z+2024x}+\frac{1936z+88x}{88z+1936z+2024y}\geq\frac{3}{2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3394 If $x,y,z>0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{xa}{y+z}+\frac{yb}{z+x}+\frac{zc}{x+y}\geq\sqrt[4]{27}\cdot\sqrt{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3395 Let $x,y,z>0$ and ABC a triangle with the area F then:

$$\left(\frac{xa}{y+z}+\frac{yb^2}{z+x}+\frac{zc}{x+y}\right)\cdot\left(\frac{xa^3}{y+z}+\frac{yb^2}{z+x}+\frac{zc^3}{x+y}\right)\geq 12F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3396 Prove that:

$$\sum_{k=1}^{n}\frac{k}{n(n+1)-k}\geq\frac{n}{2(n-1)},\forall n\in\mathbb{N}^*$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3397 Let $a_k \in \mathbb{R}_+^*, \forall k = \overline{1,n}$ and $s = \sum_{k=1}^{n} a_k$. If $x, y > 0$ and $xs > y \cdot \max\limits_{1\le k\le n} a_k$, then:

$$\sum_{k=1}^{n} \frac{\sqrt[3]{a_k^3 + (s-a_k)^3}}{xs - ya_k} \ge \frac{\sqrt[3]{3}n^2}{3\cdot(nx-y)}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3398 If $m \ge 0, x, y, z > 0$ and $x + y + z = s > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{(a^2+ab+b^2)^{m+1}}{(s^2-x^2)^m} + \frac{(b^2+bc+a^2)^{m+1}}{(s^2-y^2)^m} + \frac{(c^2+ca+a^2)^{m+1}}{(s^2-z^2)^m} \ge \frac{2^{2-m}\cdot 3^{2m+1}\left(\sqrt{3}\right)^{m+1}}{s^{2m}} \cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3399 In any triangle ABC with the usual notations the following inequality holds:

$$\frac{a^3}{h_a} + \frac{b^3}{h_b} + \frac{c^3}{h_c} \ge 8F$$

where $F =$ area ABC

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3400 Let be $x, y > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{x^2a^4 + y^2b^4}{m_c^2} + \frac{xb^4 + yc^4}{m_a^2} + \frac{xc^4 + ya^4}{m_b^2} \ge \frac{8\sqrt{3}}{3}(x+y)^2 F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3401 Find:

$$\Omega = \int_0^\infty \frac{x\left|\ln(x)\right|\arctan(x)}{\left(x^2+1\right)\left(x+1\right)^2}dx$$

Proposed by Vasile Mircea Popa-Romania

U.3402 *Prove that*:

$$\int_0^1 \frac{-K(k) + \frac{2\left(K(k)-E(k)\right)}{k^2}}{k}dk = 1 - \frac{\pi}{4}$$

$$where,\ \ E(k) = \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,d\theta \quad And \quad K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2\theta}}d\theta$$

Proposed by Shobhit Jain – India

U.3403 In any acute triangle ABC holds:

$$\frac{a}{p-a}\sqrt{\sin\frac{A}{2}}+\frac{b}{p-b}\sqrt{\sin\frac{B}{2}}+\frac{c}{p-c}\sqrt{\sin\frac{C}{2}}>2\sqrt[4]{8}$$

Proposed by Vasile Mircea Popa-Romania

U.3404 Find: $\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{5^{\frac{k}{n}}}{n+\frac{1}{k}}$

Proposed by Vasile Mircea Popa-Romania

U.3405 Find:

$$\Omega=\int_0^\infty\frac{x\ln^2(x)\arctan(x)}{(x+1)^4}dx$$

Proposed by Vasile Mircea Popa-Romania

U.3406 In any acute triangle ABC holds:

$$r_a\sqrt{\cos A}+r_b\sqrt{\cos B}+r_c\sqrt{\cos C}\le\frac{9\sqrt{2}}{4}R$$

Proposed by Vasile Mircea Popa-Romania

U.3407 Prove the below closed form

$$\int_0^{\frac{\pi}{2}}\ln\left(\sqrt{\sin x}+\sqrt{\cos x}\right)dx=-\frac{G}{2}+\frac{\pi}{4}\ln\left(2+\frac{3}{\sqrt{2}}\right)$$

Where, G is the Catalan's constant

Proposed by Ankush Kumar Parcha-India

U.3408 Prove the below closed form

$$\int_0^1\int_0^1\frac{\ln(1-x^2y^2)-xy\ln\left(\frac{1-xy}{1+xy}\right)}{1-x^2y^2}dx\,dy=\frac{\pi^2}{2}\ln(2)-\frac{21}{8}\zeta(3)$$

Where, $\zeta(3)$ is an Apery's constant

Proposed by Ankush Kumar Parcha -India

U.3409 Prove the below closed form

$$\int_0^1 \int_0^1 \frac{\ln^2\left(\frac{x}{y}\right) + \ln^2\left(\frac{y}{x}\right)}{\frac{x}{y} + \frac{y}{x}} dx\, dy = \frac{3}{8}\zeta(3)$$

Where $\zeta(3)$ is an Apery's constant

Proposed by Ankush Kumar Parcha -India

U.3410 Prove the below closed form

$$\int\int_{\left[0,\frac{\pi}{2}\right]^2} \frac{dx\, dy}{\sin^2(x) + \cos^2(y)} = \frac{\pi^2}{2\sqrt{2}}$$

Proposed by Ankush Kumar Parcha -India

U.3411 Prove the below closed form

$$\int_0^1 \sinh^{-1}(x) \cosh^{-1}(x)\, dx = \frac{\pi}{2}\left(1 - \frac{4\sqrt{2\pi}}{\Gamma^2\left(\frac{1}{4}\right)}\right) i$$

Proposed by Ankush Kumar Parcha -India

U.3412 Prove the below closed form

$$\int\int_{\left[0,\frac{\pi}{2}\right]^2} \ln\left(\frac{\sin(x)}{\cos(y)} + \frac{\cos(x)}{\sin(y)}\right) dx\, dy = \frac{7\zeta(3)}{8} + \frac{\pi^2}{4}\ln 2$$

Where, $\zeta(3)$ is the Apery's constant

Proposed by Ankush Kumar Parcha -India

U.3413 If $\Omega \coloneqq \int_0^\infty \int_0^\infty \int_0^\infty \frac{\sin(x+y+z)\cdot\cos(x+y+z)}{\sqrt{x\sqrt{y\sqrt{z}}}} dx\, dy\, dz$

Then, show that: $\Omega = -\frac{\sqrt[4]{15+11\sqrt{2}-2\sqrt{116+82\sqrt{2}}}}{4\sqrt[4]{8}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{8}\right)}\pi^{\frac{5}{2}}$

Proposed by Ankush Kumar Parcha -India

U.3414 Prove the below closed form

$$\int_0^1 \left(\frac{1-x}{1+x}\right) \ln(x) \ln\left(\frac{1+x}{1-x}\right) dx = \left(\frac{x^2+4}{2}\right) \ln(2) - \frac{\pi^2}{12} - \frac{7}{12}\zeta(3)$$

Proposed by Ankush Kumar Parcha -India

U.3415 Prove the below closed form

$$\int\int_{\left[0,\frac{\pi}{2}\right]^2} \ln(\sin^2(x) + \cos^2(y))\, dx\, dy = \pi G - \frac{\pi^2}{2}\ln 2$$

Proposed by Ankush Kumar Parcha-India

U.3416 Prove the below closed form

$$\int_{\mathbb{R}} \left(\frac{\sin(x)}{x}\right)\left(\frac{\sin(2x)}{x}\right)\left(\frac{\sin(3x)}{x}\right) dx = \tau$$

Where, τ is the ratio between the circumference and radius of a circle

Proposed by Ankush Kumar Parcha -India

U.3417 If $\Omega \coloneqq \int_0^{\frac{1}{2}} \frac{\ln(x)}{x} \ln\left(\frac{1+x}{1-x}\right) dx$ Then, show that:

$$\Omega = Li_3\left(-\frac{1}{2}\right) + \ln 2\, Li_2\left(-\frac{1}{2}\right) - \frac{7}{8}\zeta(3) + \frac{\ln^3(2)}{3}$$

Where $Li_s(z)$ is the polylogarithm of Jonquiere's function, $\zeta(3)$ is the Apery's constant

Proposed by Ankush Kumar Parcha-India

U.3418 Prove the below closed form

$$\int\int_{\left[0,\frac{\pi}{4}\right]^2} \sqrt{\frac{\csc(2x)\cdot\csc(2y)}{\cos^2(x)\cos^2(y) - \sin^2(x)\sin^2(y)}}\, dx\, dy = \frac{\pi\varpi}{4}$$

Where, ϖ is the lemniscate constant.

Proposed by Ankush Kumar Parcha -India

U.3419 Prove the below closed form

$$\int_{\mathbb{R}} \frac{\ln^2(x)\sin^3(x)}{x} dx = \frac{\pi}{4}\gamma^2 - \frac{\pi\gamma}{4}\ln(3) + \frac{\pi^3}{48} - \frac{\pi}{8}\ln^2(3)$$

Where, γ is the Euler – Mascheroni constant.

Proposed by Ankush Kumar Parcha-India

U.3420 Prove the below closed form

$$\sum_{n\in\mathbb{N}_0} \frac{(-1)^{\frac{n(n+1)}{2}}}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(\frac{n}{2}+1\right)} = \frac{\cos(2) - \sin(2)}{\sqrt{\pi}}$$

Proposed by Ankush Kumar Parcha-India

U.3421 Prove the below closed form

$$\int_0^{2\pi} \frac{(1+\pi\sin(x))\cdot(1+\pi\cos(x))}{(1+e\sin(x))\cdot(1+e\cos(x))}\,dx = 2\pi\left(\frac{\pi}{e}\right)^2$$

Proposed by Ankush Kumar Parcha-India

U.3422 Prove the below closed form

$$\int\int\int_{[0,1]^3} \sum_{x,y,z} \ln\left(\frac{\frac{x}{x+y}}{\frac{y+z}{z}}\right) dx\,dy\,dz = \ln\left(\frac{e^3}{4096}\right)$$

Proposed by Ankush Kumar Parcha -India

U.3423 Prove the below closed form

$$\sum_{n\in\mathbb{Z}_0^+} \frac{2^{-2n}}{\left(n+\frac{1}{2}\right)\left(n+\frac{3}{2}\right)\left(n+\frac{5}{2}\right)}\binom{2n}{n} = \frac{3\pi}{6}$$

Proposed by Ankush Kumar Parcha -India

U.3424 Prove the below closed form

$$\int_0^1 \frac{(1-x)(2-x)(3-x)}{(1+x^2)\ln x}\,dx = \ln\left(\frac{\pi^5}{2048}\right)$$

Proposed by Ankush Kumar Parcha-India

U.3425 If $\Omega \coloneqq \int_0^1 \ln\left(\frac{1+x(\sqrt{2}-1)}{1-x(\sqrt{2}-1)}\right)\left(\frac{1+x}{x}\right)dx$

Then, show that: $\Omega = \frac{\pi^2}{8} - \frac{\ln^2(1+\sqrt{2})}{2} - \sqrt{2}\ln(1+\sqrt{2}) + (1+\sqrt{2})\ln(2)$

Proposed by Ankush Kumar Parcha-India

U.3426 Prove the below closed form

$$\int_0^{2\pi} \frac{dx}{\sin^4(x)+\sin^2(x)+1} = 2\left(\frac{2^{\frac{1}{2}}}{3^{\frac{3}{4}}}\right)\pi$$

Proposed by Ankush Kumar Parcha -India

U.3427 Prove the below closed form

$$\int_0^{\frac{\pi}{2}} x\ln\left(1+\frac{1}{\csc(x)}+\frac{1}{\sec(x)}\right)dx = \frac{\pi}{2}G - \frac{\pi^2}{16}\ln(2)$$

Where, G is the Catalan's constant.

Proposed by Ankush Kumar Parcha -India

U.3428 Prove the below closed form

$$\int_{\mathbb{R}}\left(\frac{1}{x}-\frac{\sin(x)}{x^2}\right)^2 dx=\frac{\pi}{3}$$

Proposed by Ankush Kumar Parcha-India

U.3429 Prove the below closed form

$$\int\int_{\mathbb{R}_+^2}\left(\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right)+\cos^{-1}\left(\frac{1-y^2}{1+y^2}\right)\frac{dx\,dy}{(1+x^2)(x^2+y^2)(1+y^2)}=\pi G-\frac{\pi^3}{8}+\frac{3\pi^2}{4}\ln(2)$$

Where, G is the Catalan's constant

Proposed by Ankush Kumar Parcha-India

U.3430 Prove the below closed form

$$\sum_{n\in\mathbb{Z}}\frac{n^2+1}{n^4+1}=\frac{\pi\sqrt{2}\sinh(\pi\sqrt{2})}{\cosh(\pi\sqrt{2})-\cos(\pi\sqrt{2})}$$

Proposed by Ankush Kumar Parcha -India

U.3431 Prove the below closed form

$$\sum_{n\in\mathbb{Z}}\frac{1}{\Gamma(n-1)+\Gamma(n)+\Gamma(n+1)}=\frac{\delta}{e}+\gamma+e-1$$

Where $\Gamma(z), \Re\{z\}>0$ is the gamma function, δ is the Gompertz constant or Euler-Gompertz constant, γ is Euler's constant or Euler – Mascheroni constant

Proposed by Ankush Kumar Parcha-India

U.3432 Prove the below closed form

$$\int_{\mathbb{R}^+}\ln\left(1+\frac{1}{x^2}+\frac{1}{x^4}\right)dx=\pi\sqrt{3}$$

Proposed by Ankush Kumar Parcha -India

U.3433 Let be $(a_n)_{n\geq1}, a_n=n!$ and $(b_n)_{n\geq1}, b_n>0, \forall n\in\mathbb{N}^*$ with $\lim\limits_{n\to\infty}\frac{n\cdot b_{n+1}}{b_n}=b>0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^2}-\sqrt[n]{a_n^2}\right)\cdot\sqrt[n]{b_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

U.3434 Let $s, t \geq 0$ and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}, a_n, b_n > 0, \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n \cdot n^s} = a > 0$,

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n\, n^t} = b > 0. \text{ Find: } \lim_{n\to\infty} \frac{\sqrt[n]{a_n \cdot b_n}}{n^{s+t}}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3435 Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{a_{n+1}}{n \cdot a_n} = a > 0, \lim_{n\to\infty} \frac{n \cdot b_{n+1}}{b_n} = b > 0$$

Find: $\lim_{n\to\infty} \sqrt[n]{a_n \cdot b_n}$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3436 Let $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{n \cdot a_{n+1}}{a_n} = a > 0. \text{ Find: } \lim_{n\to\infty} \sqrt[n]{n! \cdot a_n}$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3437 Let be $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that

$$\lim_{n\to\infty} \frac{n \cdot a_{n+1}}{a_n} = a > 0. \text{ Find: } \lim_{n\to\infty} \sqrt[n]{(2n-1)!! \cdot a_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

U.3438 If $t \geq 0$ find:

$$\lim_{n\to\infty} \left(\frac{\left(\sqrt[n+1]{(2n+1)!!}\right)^{t+1}}{(n+1)^t} - \frac{\left(\sqrt[n]{(2n-1)!!}\right)^{t+1}}{n^t} \right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3439 Let be $\gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}, \lim_{n\to\infty} \gamma_n = \gamma =$ Euler – Mascheroni constant. Find:

$$\lim_{n\to\infty} (\gamma_n - \gamma) \cdot \sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3440 Let be $(a_n)_{n\geq 1}, a_n = n!$ and $(b_n)_{n\geq 1}, b_n > 0, \forall n \in \mathbb{N}^*$ such that

$$\lim_{n\to\infty} \frac{n \cdot b_{n+1}}{b_n} = b > 0. \text{ Find: } \lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}^2} - \sqrt[n]{a_n^2} \right) \sqrt[n]{b_n}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

U.3441 Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{(2n+1)!!\,(n+1)}\right)-\sqrt[n]{(2n-1)!!\,n}$$

Proposed by D.M. Bătinețu – Giurgiu Daniel Sitaru-Romania

U.3442 Let be $(a_n)_{n\geq1}, a_n>0, \forall n\in\mathbb{N}^*$ with $\lim_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n}=a>0$ and $(b_n)_{n\geq1}, b_n=n!$. Find:

$$\lim_{n\to\infty}\sqrt[n]{a_n}\cdot\left(\frac{(n+1)^3}{\sqrt[n+1]{b_{n+1}}}-\frac{n^3}{\sqrt[n]{b_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3443 If $(a_n)_{n\geq1}, a_n>0, \forall n\in\mathbb{N}^*$ such that $\lim_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n}=a>0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\left((n+1)!\right)^2\cdot a_{n+1}}-\sqrt[n]{(n!)^2\cdot a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3444 If $(a_n)_{n\geq1}, a_n>0, \forall n\in\mathbb{N}^*$ with $\lim_{n\to\infty}\frac{n\cdot a_{n+1}}{a_n}=a>0$ and $(b_n)_{n\geq1}, b_n=(2n-1)!!$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}\cdot b_{n+1}^2}-\sqrt[n]{a_n\cdot b_n^2}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3445 If $s,t>0, s+t=1$ and $(a_n)_{n\geq1},(b_n)_{n\geq1}, a_n,b_n>0, \forall n\geq1$ with

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^s}=a>0, \lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^t}=b>0$. Find: $\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}b_{n+1}}-\sqrt[n]{a_n\cdot b_n}\right)$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3446 Let $(a_n)_{n\geq1}, a_n>0, \forall n\in\mathbb{N}^*$ with $\lim_{n\to\infty}\frac{\sqrt[n]{a_n}}{\sqrt[n]{(2n-1)!!}}=a>0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3447 If $t\geq0$, find:

$$\lim_{n\to\infty}\left(\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{t+1}}{(n+1)^t}-\frac{\left(\sqrt[n]{n!}\right)^{t+1}}{n^t}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.3448 Let be $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ sequences of real strictly positive numbers such that

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n}=a>0, \lim_{n\to\infty}\frac{b_{n+1}\cdot n}{b_n}=b>0$$

Find: $\lim_{n\to\infty}\left(\frac{(n+1)^3}{\sqrt[n+1]{a_{n+1}}}\cdot\sqrt[n+1]{b_{n+1}}-\frac{n^3}{\sqrt[n]{a_n}}\cdot\sqrt[n]{b_n}\right)$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3449 Let be $(a_n)_{n\ge1}, a_n>0, \forall n\in\mathbb{N}^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0$. Find:

$$\lim_{n\to\infty} n\cdot\sqrt[n]{a_n}\cdot\int_0^1 x^3\cdot(1-x^2)^n\,dx$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3450 Let $(a_n)_{n\ge1}, (b_n)_{n\ge1}$ sequences of real strictly positive numbers such that

$\lim_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0,\ \lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0$. Find:

$$\lim_{n\to\infty}\frac{1}{\sqrt[n]{a_n}}\cdot\sum_{k=2}^{n}\frac{k}{\sqrt[k]{b_k}}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

ROMANIAN MATHEMATICAL MAGAZINE-R.M.M.-SUMMER 2026

PROBLEMS FOR JUNIORS

JP.601 If $a,b,c>0$ and $\lambda\ge 0$ then:

$$(\lambda+1)\sum\frac{a^2}{b+\lambda c}+2\sqrt{3(ab+bc+ca)}\ge 3(a+b+c)$$

Proposed by Marin Chirciu – Romania

JP.602 If $a,b,c>0$ and $\lambda\ge 0$ then:

$$2\sum\frac{a^3}{b+\lambda c}+\frac{\lambda+1}{2}\sum bc\ge 2\sum a^2$$

Proposed by Marin Chirciu – Romania

JP.603 If $a,b,c>0$ then:

$$2\left(a^a+b^b+c^c\right)\ge a^b+a^c+b^a+b^c+c^a+c^b$$

Proposed by Daniel Sitaru – Romania

JP.604 If $a,b>0; n\in\mathbb{N}^*$ then:

$$\frac{a^{2n}}{b^{2n}}+a^{2n}b^{4n}+\frac{1}{a^{4n}b^{2n}}\ge\frac{b}{a}+a^2b+\frac{1}{ab^2}$$

Proposed by Daniel Sitaru – Romania

JP.605 Find if it exists $x, y \in \mathbb{R}^*$ such that:

$$4^{x+y} + \sqrt{2026^x + 2026^y - 2} = 2^{x+y+1} - 1$$

Proposed by Adrian Gobej – Romania

JP.606 Solve the following equation:

$$2016^{2-x^2} + 2016^{2017x^2 - 4\sqrt{x} - 2013} = 2017$$

Proposed by Adrian Gobej – Romania

JP.607 Solve for reals:

$$\sqrt[4]{8x-7} + \sqrt[4]{3-2x^4} = 2$$

Proposed by Marin Chirciu – Romania

JP.608 Let be the triangle ABC, AA_1, BB_1, CC_1 internal bisectors and A_2, B_2, C_2 contact points to the bisectors with the circumcircle of the triangle. Prove that:

$$A_1A_2 \cdot B_2C_2 + B_1B_2 \cdot A_2C_2 + C_1C_2 \cdot A_2B_2 \geq Rs$$

where p represent the semiperimeter and R the circumradii of triangle ABC.

Proposed by Marian Ursărescu, Florică Anastase – Romania

JP.609 In acute triangle ABC, A', B', C' are symmetric points of the points A, B, C to the sides BC, AC, and AB respectively. Prove that:

$$\frac{\sigma[A'B'C']}{\sigma[ABC]} = 4\left(\frac{r}{R}\right)^2 + 8 \cdot \frac{r}{R} - 1$$

where $\sigma[ABC]$ represent area of ΔABC.

Proposed by Marian Ursărescu, Florică Anastase – Romania

JP.610 Solve for real numbers:

$$\log_{2\sqrt{8+2\sqrt{15}}}(x^2+x+2)=\log_{\sqrt{4+\sqrt{15}}}(x^2+x+1)$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

JP.611 Find the angle between the real plans:

$$P_1: 2x+y+3z-1=0$$

$$P_2: 3x-2y+z+1=0$$

Proposed by Daniel Sitaru – Romania

JP.612 Find the angle between the line d and the real plan P.

$$d:\begin{cases}x=2+3t\\ y=3-2t\\ z=1+5t\end{cases}; t\in\mathbb{R}; P: 2x+y+z-5=0$$

Proposed by Daniel Sitaru – Romania

JP.613 Find the angle between the lines:

$$d_1:\begin{cases}x=3-2t\\ y=1+t\\ z=-1+3t\end{cases}; d_2:\begin{cases}x=2+4t\\ y=-1+2t\\ z=4-t\end{cases}; t\in\mathbb{R}$$

Proposed by Daniel Sitaru – Romania

JP.614 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c}+\frac{b^4+c^4}{h_a}+\frac{c^4+a^4}{h_b}\geq 288r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

JP.615 In ΔABC the following relationship holds:

$$\frac{a^4+b^4}{h_c^2}+\frac{b^4+c^4}{h_a^2}+\frac{c^4+a^4}{h_b^2}\geq 96r^2$$

Proposed by Nguyen Hung Cuong – Vietnam

PROBLEMS FOR SENIORS

SP.601 What is the largest positive value of the power k such that

$$a^k + b^k + c^k \le 3$$

for all nonnegative real numbers a, b, c, d with $a \le b \le c \le d$ and

$$ab + bc + cd + da = 4?$$

Proposed by Vasile Cîrtoaje – Romania

SP.602 Let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that $a_1 + a_2 + \cdots + a_n = n$. Prove that:

$$\sum_{i=1}^{n} \sqrt{\frac{n - a_i}{n - 1 + a_i}} \ge \sqrt{n(n-1)}$$

Proposed by Vasile Cîrtoaje – Romania

SP.603 Prove that $\frac{5}{3}$ is the largest positive value of the power k such that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} \ge a^k + b^k + c^k + d^k$$

for all positive real numbers a, b, c, d with at most one of them smaller than 1 and

$$ab + ac + ad + bc + bd + cd = 6.$$

Proposed by Vasile Cîrtoaje – Romania

SP.604 Let be $A(2,1,0)$; $B(1,2,1)$; $C(3,3,3)$; $D(0,0,4)$. Find the distance from the point D to the real plan (ABC).

Proposed by Daniel Sitaru – Romania

SP.605 Let be $A(1,2,1)$; $B(2,1,3)$; $C(4,4,4)$.Find the distance from the point A to the line BC.

Proposed by Daniel Sitaru – Romania

SP.606 Let be $A(2,1,0)$; $B(0,1,2)$; $C(3,0,1)$; $D(4,4,4)$. Find the volume of the tetrahedron $ABCD$.

Proposed by Daniel Sitaru – Romania

SP.607 Let be the function $f:[0,1]\to\mathbb{R}$ integrable such that $f(1)=1$ and

$$\int_x^y f(t)\,dt=\frac{1}{2}\big(yf(y)-xf(x)\big),\forall x,y\in[0,1]$$

Find:

$$I=\int_0^{\frac{\pi}{4}} f(x)\cdot\tan^2 x\,dx$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

SP.608 If $a,b,c\in(0,1)$ and $x,y,z>0$ such that $a=(bc)^x$, $b=(ca)^y$, $c=(ab)^z$ and $xyz=1$ then holds:

$$\sqrt[n]{\sum_{cyc} a^n(y+z+2)^{2n-1}}\geq 6\cdot\sqrt[3]{abc}, n\in\mathbb{N}^*, n\geq 2$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

SP.609 If $a,b,c>0$ then:

$$\sqrt{a^2b^2+c^2}+\sqrt{c^2b^2+a^2}+\sqrt{c^2a^2+b^2}\geq\sqrt{(a+b+c)^2+(ab+bc+ca)^2}$$

Proposed by Daniel Sitaru – Romania

SP.610 If $a, b, c > 0$ then:

$$\sqrt{a^2+1}+\sqrt{b^2+1}+\sqrt{c^2+1} \geq \sqrt{(a+b+c)^2+9}$$

Proposed by Daniel Sitaru – Romania

SP.611 If $x, y, z > 0$ then:

$$(x^5+y^5+z^5)(x^6+y^6+z^6)(x^2+y^2+z^2)^5 \geq (x^3+y^3+z^3)^7$$

Proposed by Daniel Sitaru – Romania

SP.612 Let be $A(1,2,3); B(4,1,1); C(0,3,2); D(2,1,3)$. Find the volume of the parallelepiped builded on the vectors $\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}$.

Proposed by Daniel Sitaru – Romania

SP.613 Let be $A(2,4,1); B(4,2,5)$. Find the mediator plan of the segment $[AB]$.

Proposed by Daniel Sitaru – Romania

SP.614 Let be $A(1,3,2); B(3,1,1); C(4,2,0)$. Find the area of the parallelogram builded on $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Proposed by Daniel Sitaru – Romania

SP.615 Let be $A(1,2,3)$ and the real plan: $P: 2x+y+z-5=0$

Find the parametrical equations of the perpendicular line from A to the real plan P.

Proposed by Daniel Sitaru – Romania

UNDERGRADUATE PROBLEMS

UP.601 Let $\{L_n\}_{n\geq 0}$ be the Lucas' sequence defined by $L_0 = 2, L_1 = 1$ and for all $n \geq 2, L_n = L_{n-1} + L_{n-2}$. Determine the sum of the lengths of the intervals, disjoint two by two, formed by all $f(x) = 1$, where

$$f(x) = \frac{L_1^2}{x + L_1^2} + \frac{L_2^2}{x + L_2^2} + \cdots + \frac{L_n^2}{x + L_n^2}$$

Proposed by Jose Luiz Diaz Barrero-Spain

UP.602 Find all non-negative integers for which

$$\sum_{k=0}^{n} \frac{k}{k+1} \binom{n}{k}^2$$

is and integer number.

Proposed by Jose Luiz Diaz Barrero-Spain

UP.603 Find all positive real values of the constant k such that

$$9(a^2 + k)(b^2 + k)(c^2 + k) \leq (1 + k)^3(a + b + c)^2$$

for any nonnegative real numbers a, b, c with $ab + bc + ca = 3$.

Proposed by Vasile Cîrtoaje – Romania

UP.604 Prove that $\frac{3}{2}$ is the smallest positive value of the power k such that

$$\frac{1}{a^k} + \frac{1}{b^k} + \frac{1}{c^k} \geq a^2 + b^2 + c^2$$

for all positive real numbers a, b, c with at most one of them smaller than 1 and

$$ab + bc + ca = 3.$$

Proposed by Vasile Cîrtoaje – Romania

UP.605 We consider the function $u:\mathbb{R}\to\mathbb{R}$, periodic with period 2π. For the period $[0,2\pi]$ we have:

$u(x)=\cos(x)$ **if** $x\in\left[0,\frac{\pi}{2}\right)$; $u(x)=0$ **if** $\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right)$; $u(x)=\cos(x)$ **if** $x\in\left[\frac{3\pi}{2},2\pi\right)$.

Prove the equality:

$$\int_0^\infty \frac{u(x)}{1+x^2}dx=\frac{\pi}{4e}+\frac{e^2+1}{2e}\arctan\left(\frac{1}{e}\right)$$

Proposed by Vasile Mircea Popa – Romania

UP.606 If $f:\mathbb{R}\to(1,\infty)$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous functions, and $y_n=\sqrt[n]{n!\,F_n}$, $n\in\mathbb{N}^*-\{1\}$, where $(F_n)_{n\geq 0}$ is Fibonacci sequence, find:

$$\lim_{n\to\infty}\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.607 Calculate the integral:

$$\int_0^1 \frac{x\ln x}{(x+1)(x^2+1)}dx$$

Proposed by Vasile Mircea Popa – Romania

UP.608 If $f:\mathbb{R}\to(1,\infty)$ and $g:\mathbb{R}\to\mathbb{R}$ are continuous functions, and $(L_n)_{n\geq 0}$ is Lucas sequence, and $y_n=\sqrt[n]{(2n-1)!!\,L_n}$, $n\in\mathbb{N}^*-\{1\}$, find:

$$\lim_{n\to\infty}\int_{y_n}^{y_{n+1}}\frac{\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}{\left(f(y_{n+1}-x)\right)^{g(x-y_n)}+\left(f(x-y_n)\right)^{g(y_{n+1}-x)}}dx.$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.609 If $f:[a,b]\to(0,\infty)$ is continuous functions such that $f(a+b-x)+f(x)=c$, $\forall x\in[a,b]$ and $a+b=\frac{\pi}{2}$, find:

$$\int_a^b \frac{\sin^n x+f(x)+d}{\sin^n x+\cos^n x+c+2d}dx,$$

where $n\in\mathbb{N}^*$ and $d\ge 0$.

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.610 If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a continuous function and $\gamma_n=-\ln n+\sum_{k=1}^n\frac{1}{k}$, find

$$\lim_{n\to\infty} n\int_\gamma^{\gamma_n}\frac{f(x-\gamma)}{f(\gamma_n-x)+f(x-\gamma)}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.611 If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a continuous function and $(x_n)_{n\ge1}, x_n=\sum_{k=1}^n\frac{1}{k}$, find

$$\lim_{n\to\infty}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.612 If $f:\mathbb{R}_+^*\to\mathbb{R}_+^*$ is a continuous function and $(a_n)_{n\ge1}$ is defined by $a_1=a_2=1$, $a_{n+1}=\sum_{k=1}^n\frac{a_k}{k}, \forall n\ge 2$ and $(x_n)_{n\ge1}, x_n=\sum_{k=1}^n\frac{1}{a_k}$, find $\lim_{n\to\infty}\frac{1}{n}\int_{e^{x_n}}^{e^{x_{n+1}}}\frac{f(x-e^{x_n})}{f(e^{x_{n+1}}-x)+f(x-e^{x_n})}dx$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

UP.613 Let be $A(1,2,0)$; $B(2,0,1)$; $C(3,3,3)$. Find the area of ΔABC.

Proposed by Daniel Sitaru – Romania

UP.614 If $0 < a \le b$ then:

$$\int_a^b \int_a^b \frac{dxdy}{x^2 - xy + y^2} + 3\left(\ln\left(\frac{b}{a}\right)\right)^2 \ge 16 \int_a^b \int_a^b \frac{dxdy}{(x+y)^2}$$

Proposed by Daniel Sitaru – Romania

UP.615 Prove that $\frac{8}{5}$ is the smallest positive value of the constant k such that

$$\frac{1}{(a+b)^2 + k} + \frac{1}{(b+c)^2 + k} + \frac{1}{(c+a)^2 + k} \le \frac{3}{4+k}$$

for all side lengths a, b, c of a triangle with $ab + bc + ca = 3$.

Proposed by Vasile Cîrtoaje – Romania

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