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# A FEW LIMITS OF INTEGRALS

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**Abstract:** In the next paper we will prove a theorem and we will solve three problems from the well-known math magazine Crux Mathematicorum, Vol. 52, No. 2, February, 2026.

**Theorem:** Let $I=[a,b]\subset R$, and the sequences $(a_n)_{n\geq 1}$, $(b_n)_{n\geq 1}$, and $(c_n)_{n\geq 1}$:
(i) $a_n\leq b_n$, $c_n>0$, for any $n\in N^*$; (ii) $a_n, b_n, c_n x\in I, n\in N^*$ for any $x\in I, n\in N^*$;
(iii) $(a_n c_n)_{n\geq 1}$ and $(b_n c_n)_{n\geq 1}$ have finite limits and $\lim\limits_{n\to\infty} a_n c_n=\lim\limits_{n\to\infty} b_n c_n=l\in I$;
(iv) there exists $\alpha\in R$ such that $\lim\limits_{n\to\infty} n^{\alpha}(b_n-a_n)=L\in R$.
If the function $f:I\to R$ is continuous on $I$, then:

$$\lim_{n\to\infty} n^{\alpha}\int_{a_n}^{b_n} f(c_n x)dx=L\cdot f(l).$$

**Proof:** For a $n\in N^*$, let the function $g_n:I\to R$ defined by $g_n(x)=f(c_n x)$, $x\in I$. Since $g_n$ it is a continuous function on $I$, according to the mean theorem (MVT) of integral calculus we have that there exists $\xi_n\in[a_n,b_n]$ such that

$$\int_{a_n}^{b_n} g_n(x)dx=(b_n-a_n)g_n(\xi_n),\ (*),$$

i.e. there exists $a_n\leq\xi_n\leq b_n$ such that

$$\int_{a_n}^{b_n} f(c_n x)dx=(b_n-a_n)f(c_n\xi_n),\ n\in N^*,\ (**).$$

Since $c_n>0$, from (*) we have $a_n c_n\leq c_n\xi_n\leq b_n c_n$. Taking into account (iii), according to the squeeze theorem it follows that
$\lim\limits_{n\to\infty} f_n(c_n\xi_n)=f(l)$, (***).
Taking into account (**) and (***), we get the conclusion of the proposition.
The idea of the proof in the proposition above can be used in solving problems with limits of definite integrals.

**Problem 1:**

Find the limit $\lim_{n\to\infty}\sqrt[n]{n!}\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx$, where $(s_n)_{n\geq1}, s_n=\sum_{k=1}^{n}\frac{1}{k^2}$ and $f:(0,\infty)\to(0,\infty)$ is a continuous function on $(0,\infty)$.

**Solution:** Using $\lim_{n\to\infty} s_n=\frac{\pi^2}{6}$, by (MVT) there exists $\xi_n\in\left(s_n,\frac{\pi^2}{6}\right)$ such that

$$\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx=\left(\tfrac{\pi^2}{6}-s_n\right)f(\xi_n),\forall n\in N^*,\ (1).\text{ So,}$$

$$\sqrt[n]{n!}\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx=\frac{\sqrt[n]{n!}}{n}\cdot f(\xi_n)\cdot n\left(\frac{\pi^2}{6}-s_n\right)=\frac{\sqrt[n]{n!}}{n}\cdot f(\xi_n)\cdot\frac{\frac{\pi^2}{6}-s_n}{\frac{1}{n}}$$

and since

$$\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e},\quad \lim_{n\to\infty}\sqrt[n]{n!}\int_{s_n}^{\frac{\pi^2}{6}} f(x)dx=\frac{1}{e}\cdot f\left(\lim_{n\to\infty}\xi_n\right)\cdot\lim_{n\to\infty}\frac{\frac{\pi^2}{6}-s_n}{\frac{1}{n}}\underset{\left(\frac{0}{0}\right)}{\overset{Stolz}{=}}$$

$$=\frac{1}{e}f\left(\frac{\pi^2}{6}\right)\lim_{n\to\infty}\frac{-s_{n+1}+s_n}{\frac{1}{n+1}-\frac{1}{n}}=\frac{f\left(\frac{\pi^2}{6}\right)}{e}\lim_{n\to\infty}\frac{s_{n+1}-s_n}{\frac{1}{n(n+1)}}=$$

$$=\frac{f\left(\frac{\pi^2}{6}\right)}{e}\lim_{n\to\infty}\frac{\frac{1}{(n+1)^2}}{\frac{1}{n(n+1)}}=\frac{f\left(\frac{\pi^2}{6}\right)}{e}\lim_{n\to\infty}\frac{n}{n+1}=\frac{f\left(\frac{\pi^2}{6}\right)}{e}.$$

**Problem 2:**

Find the limit $\lim_{n\to\infty}\sqrt[n]{n!}\int_{\gamma}^{\gamma_n} f(x)dx$, where $(\gamma_n)_{n\geq1}$, $\gamma_n=-\ln n+\sum_{k=1}^{n}\frac{1}{k}$, $\lim_{n\to\infty}\gamma_n=\gamma$ and $f:(0,\infty)\to(0,\infty)$ is a continuous function on $(0,\infty)$.

**Solution:** Using $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$, (1), by (MVT) there exists $\xi_n\in(\gamma,\gamma_n)$ such that

$$\int_\gamma^{\gamma_n} f(x)dx=(\gamma_n-\gamma)f(\xi_n),\ \forall n\in N^*, \text{ so}$$

$$\sqrt[n]{n!}\int_\gamma^{\gamma_n} f(x)dx=\sqrt[n]{n!}(\gamma_n-\gamma)f(\xi_n)=\frac{\sqrt[n]{n!}}{n}\cdot n(\gamma_n-\gamma)f(\xi_n),\ (2).$$

From (1) and (2)

$$\lim_{n\to\infty}\sqrt[n]{n!}\int_\gamma^{\gamma_n} f(x)dx=\frac{1}{e}f\left(\lim_{n\to\infty}\xi_n\right)\lim_{n\to\infty}n(\gamma_n-\gamma)=$$

$$=\frac{1}{e}f(\gamma)\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}}\underset{\left(\frac{0}{0}\right)}{\overset{Cesaoro-Stols}{=}}\frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}}=\frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{-\gamma_{n+1}+\gamma_n}{\frac{1}{n(n+1)}}=$$

$$=\frac{f(\gamma)}{e}\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}-\frac{1}{n+1}}{\frac{1}{n^2}}\overset{\left(\frac{1}{n}=x\right)}{=}\frac{f(\gamma)}{e}\lim_{\substack{x\to0\\x>0}}\frac{\ln(1+x)-\frac{x}{1+x}}{x^2}=$$

$$=\frac{f(\gamma)}{e}\lim_{\substack{x\to0\\x>0}}\frac{(x+1)\ln(1+x)-x}{x^2(1+x)}=\frac{f(\gamma)}{e}\lim_{\substack{x\to0\\x>0}}\frac{(x+1)\ln(1+x)-x}{x^2}=$$

$$=\frac{f(\gamma)}{e}\lim_{\substack{x\to0\\x>0}}\frac{\ln(1+x)+1-1}{2x}=\frac{f(\gamma)}{2e}\lim_{\substack{x\to0\\x>0}}\ln(1+x)^{\frac{1}{x}}=\frac{f(\gamma)}{2e}\ln e=\frac{f(\gamma)}{2e}.$$

**Problem 3:**

Find the limit $\lim_{n\to\infty}\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}}\int_\gamma^{\gamma_n} f(x)dx$, where $(\gamma_n)_{n\ge1}$, $\gamma_n=-\ln n+\sum_{k=1}^n\frac{1}{k}$ with $\lim_{n\to\infty}\gamma_n=\gamma$, and $f:(0,\infty)\to(0,\infty)$ is a continuous function on $(0,\infty)$.

**Solution:** $$\lim_{n\to\infty}\frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}{n^n}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}\cdot\sqrt[n+1]{(2n+1)!!}}{(n+1)^{n+1}}\cdot\frac{n^n}{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!}}=$$

$$=\lim_{n\to\infty}\frac{\sqrt[n+1]{(2n+1)!!}}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{1}{e}\lim_{n\to\infty}\sqrt[n]{\frac{(2n-1)!!}{n^n}}=\frac{1}{e}\lim_{n\to\infty}\frac{(2n+1)!!}{(n+1)^{n+1}}\cdot\frac{n^n}{(2n-1)!!}=$$

$$=\frac{1}{e}\lim_{n\to\infty}\frac{2n+1}{n+1}\left(\frac{n}{n+1}\right)^n=\frac{2}{e^2}, (1).$$

By (MVT) $\exists\xi_n\in(\gamma,\gamma_n)$ such that

$$\int_{\gamma}^{\gamma_n} f(x)dx=(\gamma_n-\gamma)f(\xi_n), \forall n\in N^*, \text{so}$$

$$\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}}\int_{\gamma}^{\gamma_n} f(x)dx=\frac{\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}}{n}\cdot n(\gamma_n-\gamma)f(\xi_n), (2).$$

From (1) and (2) yields that

$$\lim_{n\to\infty}\sqrt[n]{\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\sqrt[n]{(2n-1)!!}}\int_{\gamma}^{\gamma_n} f(x)dx=$$

$$=\frac{2}{e^2}f(\gamma)\lim_{n\to\infty}\frac{\gamma_n-\gamma}{\frac{1}{n}}\underset{\left(\frac{0}{0}\right)}{\overset{Cesaoro-Stols}{=}}\frac{2f(\gamma)}{e^2}\lim_{n\to\infty}\frac{\gamma_{n+1}-\gamma_n}{\frac{1}{n+1}-\frac{1}{n}}=\frac{2f(\gamma)}{e^2}\lim_{n\to\infty}\frac{-\gamma_{n+1}+\gamma_n}{\frac{1}{n(n+1)}}=$$

$$=\frac{2f(\gamma)}{e^2}\lim_{n\to\infty}\frac{\ln\frac{n+1}{n}-\frac{1}{n+1}}{\frac{1}{n^2}}\overset{\left(\frac{1}{n}=x\right)}{=}\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{\ln(1+x)-\frac{x}{1+x}}{x^2}=$$

$$=\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{(x+1)\ln(1+x)-x}{x^2(1+x)}=\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{(x+1)\ln(1+x)-x}{x^2}=$$

$$=\frac{2f(\gamma)}{e^2}\lim_{\substack{x\to0\\x>0}}\frac{\ln(1+x)+1-1}{2x}=\frac{2f(\gamma)}{2e^2}\lim_{\substack{x\to0\\x>0}}\ln(1+x)^{\frac{1}{x}}=\frac{f(\gamma)}{e^2}\ln e=\frac{f(\gamma)}{e^2}.$$

## References

**[1]. Neculai Stanciu,** *Evaluating Limits of Integrals Using The Mean Value Theorem*, Crux Mathematicorum, Vol. 52(2), February, 2026, 83-87.