A Dual Technique for Constructing Inequalities

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Abstract

In this paper we present a simple algebraic technique for constructing inequalities from identities generated by Viete's relations. The main idea is to choose suitable auxiliary quantities as roots of a polynomial, derive an identity by summing the equations satisfied by these roots, and then combine the resulting identity with classical inequalities. Three applications are given.

1 Introduction

Many nontrivial inequalities arise from well-chosen algebraic identities. Once such an identity is available, classical tools such as the Cauchy–Schwarz inequality, Holder's inequality, Schur's inequality, or AM-GM can often turn it into a useful inequality.

The aim of this note is to describe a simple method based on Viete's relations. The method has two complementary steps. First, one constructs a polynomial whose roots are suitable expressions in the variables. Secondly, one evaluates the polynomial at its own roots and sums the resulting relations. This produces an identity which can then be estimated.

For this reason, the method may be viewed as a dual technique: it transforms variables into identities and then transforms identities back into inequalities.

2 The general construction

Let $r_1, r_2, \ldots, r_n$ be real numbers and consider the monic polynomial

$$P(x) = \prod_{i=1}^{n} (x - r_i).$$

By Viete's relations,

$$P(x) = x^n - S_1 x^{n-1} + S_2 x^{n-2} - \cdots + (-1)^n S_n,$$

where

$$S_k = \sum_{1 \le i_1 < \cdots < i_k \le n} r_{i_1} r_{i_2} \cdots r_{i_k}.$$

Since $P(r_i) = 0$ for every i, summing these identities gives

$$\sum_{i=1}^{n} r_i^n = S_1 \sum_{i=1}^{n} r_i^{n-1} - S_2 \sum_{i=1}^{n} r_i^{n-2} + \cdots + (-1)^{n+1} n S_n.$$

This is the basic identity used throughout the paper.

3 First application

Problem 1. *Let $a, b, c > 0$ such that*

$$a^2 + b^2 + c^2 = abc.$$

For every integer m, denote

$$s_m = a^m + b^m + c^m.$$

Prove that

$$\frac{3}{s_2} + \frac{s_4}{a^2b^2c^2} - \frac{1}{9} \geq s_{-2} \geq \frac{3}{s_2} + \frac{1}{3} - \frac{s_6}{a^3b^3c^3}.$$

Proof. Take

$$r_1 = \frac{a}{bc}, \qquad r_2 = \frac{b}{ca}, \qquad r_3 = \frac{c}{ab}.$$

Then

$$r_1 + r_2 + r_3 = \frac{a^2 + b^2 + c^2}{abc} = 1.$$

Also,

$$r_1r_2 + r_2r_3 + r_3r_1 = \frac{1}{c^2} + \frac{1}{a^2} + \frac{1}{b^2} = s_{-2},$$

and

$$r_1r_2r_3 = \frac{1}{abc}.$$

Therefore r_1, r_2, r_3 are roots of

$$x^3 - x^2 + s_{-2}x - \frac{1}{abc} = 0.$$

Substituting successively r_1, r_2, r_3 into this equation and summing, we get

$$\sum_{i=1}^{3} r_i^3 - \sum_{i=1}^{3} r_i^2 + s_{-2} \sum_{i=1}^{3} r_i - \frac{3}{abc} = 0.$$

Now

$$\sum_{i=1}^{3} r_i^3 = \frac{s_6}{a^3b^3c^3}, \qquad \sum_{i=1}^{3} r_i^2 = \frac{s_4}{a^2b^2c^2}, \qquad \sum_{i=1}^{3} r_i = 1.$$

Since $s_2 = abc$, we obtain

$$\frac{s_6}{a^3b^3c^3} + s_{-2} = \frac{3}{s_2} + \frac{s_4}{a^2b^2c^2}. \tag{1}$$

By the power mean inequality,

$$\frac{s_6}{3} \geq \left(\frac{s_2}{3}\right)^3.$$

Hence

$$\frac{s_6}{s_2^3} \geq \frac{1}{9}.$$

Since $s_2 = abc$, this gives

$$\frac{s_6}{a^3b^3c^3} \geq \frac{1}{9}. \tag{2}$$

Similarly,

$$\frac{s_4}{3} \geq \left(\frac{s_2}{3}\right)^2,$$

so

$$\frac{s_4}{s_2^2} \geq \frac{1}{3}.$$

Again using $s_2 = abc$, we get

$$\frac{s_4}{a^2b^2c^2} \geq \frac{1}{3}. \tag{3}$$

From (1) and (2),

$$s_{-2} = \frac{3}{s_2} + \frac{s_4}{a^2b^2c^2} - \frac{s_6}{a^3b^3c^3} \leq \frac{3}{s_2} + \frac{s_4}{a^2b^2c^2} - \frac{1}{9}.$$

From (1) and (3),

$$s_{-2} \geq \frac{3}{s_2} + \frac{1}{3} - \frac{s_6}{a^3b^3c^3}.$$

The proof is complete. □

4 Second application

Lemma 1. *Let* $a, b, c, d \geq 0$ *and*

$$a + b + c + d = 1.$$

Define

$$q = \sum_{1\leq i<j\leq 4} a_i a_j, \qquad r = \sum_{1\leq i<j<k\leq 4} a_i a_j a_k,$$

where $a_1 = a$*,* $a_2 = b$*,* $a_3 = c$*, and* $a_4 = d$*. Then*

$$r + \frac{1}{5}\sum_{i=1}^{4} a_i^3 \geq \frac{8}{45}q. \tag{4}$$

Proof. By Newton's identity,

$$\sum_{i=1}^{4} a_i^3 = 1 - 3q + 3r.$$

Thus (4) is equivalent to

$$9 - 35q + 72r \geq 0. \tag{5}$$

We prove (5).

Consider

$$F(a, b, c, d) = 9 - 35q + 72r$$

on the compact simplex

$$a, b, c, d \geq 0, \qquad a + b + c + d = 1.$$

Therefore F attains its minimum.

First assume that the minimum is attained at an interior point. Using the method of Lagrange multipliers, for every pair i, j we get

$$(a_i - a_j)(35 - 72(1 - a_i - a_j)) = 0. \tag{6}$$

Indeed,

$$\frac{\partial q}{\partial a_i} = 1 - a_i,$$

and

$$\frac{\partial r}{\partial a_i} = \sum_{\substack{1\leq u<v\leq 4 \\ u,v\neq i}} a_u a_v.$$

Subtracting the equations corresponding to i and j gives (6).

If all variables are equal, then

$$a = b = c = d = \frac{1}{4}.$$

In this case,

$$q = \frac{3}{8}, \qquad r = \frac{1}{16},$$

and therefore

$$F = \frac{3}{8} > 0.$$

If not all variables are equal, then by (6), any two unequal variables must satisfy

$$a_i + a_j = \frac{37}{72}.$$

Thus the interior critical point has, up to permutation, the form

$$\left(\frac{13}{48}, \frac{35}{144}, \frac{35}{144}, \frac{35}{144}\right).$$

For this point, a direct calculation gives

$$F = \frac{3887}{10368} > 0.$$

Hence F cannot be negative at an interior minimum.

It remains to examine the boundary. Suppose, without loss of generality, that $d = 0$. Then $a, b, c \geq 0$, $a + b + c = 1$, and

$$F = 9 - 35(ab + bc + ca) + 72abc.$$

Let

$$Q = ab + bc + ca, \qquad R = abc.$$

We write

$$9 - 35Q + 72R = (9 + 81R - 36Q) + (Q - 9R). \tag{7}$$

By Schur's inequality, for $a + b + c = 1$,

$$1 + 9abc \geq 4(ab + bc + ca).$$

Multiplying by 9, we obtain

$$9 + 81R - 36Q \geq 0. \tag{8}$$

Moreover,

$$Q \geq 3(abc)^{2/3} = 3R^{2/3}.$$

Since $R \leq 1/27$, it follows that

$$3R^{2/3} \geq 9R.$$

Thus

$$Q - 9R \geq 0. \tag{9}$$

From (7), (8), and (9), we get

$$F \geq 0$$

on the boundary.

Consequently,

$$9 - 35q + 72r \geq 0$$

on the whole simplex, and the lemma is proved. □

Problem 2. *Let* $a, b, c, d > 0$ *and*

$$a + b + c + d = 1.$$

Prove that

$$\sum_{i=1}^{4} a_i^4 - \frac{4}{5}\sum_{i=1}^{4} a_i^3 + 4abcd \geq 2\left(\sum_{1\leq i<j\leq 4} a_i a_j\right)^2 - \frac{37}{45}\sum_{1\leq i<j\leq 4} a_i a_j,$$

where $a_1 = a$, $a_2 = b$, $a_3 = c$, *and* $a_4 = d$.

Proof. Let

$$q = \sum_{1\leq i<j\leq 4} a_i a_j, \qquad r = \sum_{1\leq i<j<k\leq 4} a_i a_j a_k.$$

The numbers a, b, c, d are roots of

$$x^4 - x^3 + qx^2 - rx + abcd = 0.$$

Substituting a, b, c, d into this polynomial and summing, we obtain

$$\sum_{i=1}^{4} a_i^4 - \sum_{i=1}^{4} a_i^3 + q\sum_{i=1}^{4} a_i^2 - r\sum_{i=1}^{4} a_i + 4abcd = 0.$$

Since $\sum_{i=1}^{4} a_i = 1$, this gives

$$\sum_{i=1}^{4} a_i^4 + q\sum_{i=1}^{4} a_i^2 + 4abcd = \sum_{i=1}^{4} a_i^3 + r. \tag{10}$$

But

$$\sum_{i=1}^{4} a_i^2 = 1 - 2q.$$

Therefore (10) becomes

$$\sum_{i=1}^{4} a_i^4 + 4abcd = \sum_{i=1}^{4} a_i^3 + r - q + 2q^2.$$

Hence

$$\sum_{i=1}^{4} a_i^4 - \frac{4}{5}\sum_{i=1}^{4} a_i^3 + 4abcd = 2q^2 - q + r + \frac{1}{5}\sum_{i=1}^{4} a_i^3.$$

By Lemma 1,

$$r + \frac{1}{5}\sum_{i=1}^{4} a_i^3 \geq \frac{8}{45}q.$$

Consequently,

$$\sum_{i=1}^{4} a_i^4 - \frac{4}{5}\sum_{i=1}^{4} a_i^3 + 4abcd \geq 2q^2 - q + \frac{8}{45}q = 2q^2 - \frac{37}{45}q.$$

This is the desired inequality. □

Remark 1. *The constant* $\frac{37}{45}$ *is sharp. Equality is attained in the limiting case*

$$(a, b, c, d) = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0\right),$$

and its permutations.

5 Third application

Problem 3. *Let $a, b, c > 0$ such that*

$$a^3 + b^3 + c^3 = 3.$$

Define

$$f(x, y) = \left(x - \frac{1}{y}\right)^2 + \left(\frac{1}{y} - \frac{y}{x}\right)^2 + \left(\frac{y}{x} - x\right)^2.$$

Furthermore, define

$$u_a = \sqrt{\left(b^{-2} + c^{-2} + 1\right)\left(b^2 + c^2 + 1\right)},$$

and define u_b, u_c cyclically. Prove that

$$\sum_{\text{cyc}} u_a f(b, c) + 6 \geq 2 \sum_{\text{cyc}} \frac{b^3}{a^3}.$$

Proof. By the Cauchy–Schwarz inequality,

$$u_a = \sqrt{\left(b^{-2} + c^{-2} + 1\right)\left(b^2 + c^2 + 1\right)} \geq b + \frac{1}{c} + \frac{c}{b}. \tag{11}$$

Indeed, this follows by applying Cauchy–Schwarz to the vectors

$$\left(\frac{1}{b}, \frac{1}{c}, 1\right) \quad \text{and} \quad (c, 1, b).$$

Their scalar product is

$$\frac{c}{b} + \frac{1}{c} + b.$$

Now consider the three numbers

$$m, \qquad \frac{1}{n}, \qquad \frac{n}{m}.$$

Their product is 1, their sum is

$$m + \frac{1}{n} + \frac{n}{m},$$

and the sum of their pairwise products is

$$\frac{m}{n} + \frac{1}{m} + n.$$

Therefore these numbers are roots of

$$t^3 - \left(m + \frac{1}{n} + \frac{n}{m}\right)t^2 + \left(\frac{m}{n} + \frac{1}{m} + n\right)t - 1 = 0.$$

Substituting the three roots into this polynomial and summing gives

$$m^3 + \frac{1}{n^3} + \frac{n^3}{m^3} = \left(m + \frac{1}{n} + \frac{n}{m}\right)\left(m^2 + \frac{1}{n^2} + \frac{n^2}{m^2}\right) - \left(\frac{m}{n} + \frac{1}{m} + n\right)\left(m + \frac{1}{n} + \frac{n}{m}\right) + 3.$$

A direct expansion gives

$$f(m, n) = 2\left[m^2 + \frac{1}{n^2} + \frac{n^2}{m^2} - \left(\frac{m}{n} + \frac{1}{m} + n\right)\right].$$

Hence

$$m^3 + \frac{1}{n^3} + \frac{n^3}{m^3} = \left(m + \frac{1}{n} + \frac{n}{m}\right)\frac{f(m,n)}{2} + 3. \tag{12}$$

Applying (12) with $(m,n) = (b,c)$, we get

$$\left(b + \frac{1}{c} + \frac{c}{b}\right) f(b,c) = 2\left(b^3 + \frac{1}{c^3} + \frac{c^3}{b^3}\right) - 6.$$

Since $f(b,c) \geq 0$, inequality (11) gives

$$u_a f(b,c) \geq 2\left(b^3 + \frac{1}{c^3} + \frac{c^3}{b^3}\right) - 6.$$

Summing cyclically,

$$\sum_{\text{cyc}} u_a f(b,c) \geq 2\sum_{\text{cyc}} b^3 + 2\sum_{\text{cyc}} \frac{1}{c^3} + 2\sum_{\text{cyc}} \frac{c^3}{b^3} - 18.$$

Since

$$a^3 + b^3 + c^3 = 3,$$

we have

$$\sum_{\text{cyc}} b^3 = 3.$$

Also, by Titu Andreescu's lemma,

$$\sum_{\text{cyc}} \frac{1}{a^3} \geq \frac{(1+1+1)^2}{a^3 + b^3 + c^3} = 3.$$

Therefore

$$\sum_{\text{cyc}} u_a f(b,c) \geq 2\sum_{\text{cyc}} \frac{c^3}{b^3} - 6.$$

Renaming the cyclic indices gives

$$\sum_{\text{cyc}} u_a f(b,c) + 6 \geq 2\sum_{\text{cyc}} \frac{b^3}{a^3}.$$

The proof is complete. □

6 Conclusion

The three examples above illustrate how Viete's relations can be used as a source of useful algebraic identities. Once the appropriate auxiliary roots are chosen, the resulting polynomial identity may be combined with classical inequalities to produce nontrivial estimates.

The strength of the method lies in the choice of the roots. This choice creates a bridge between symmetric algebraic identities and inequality estimates. In this sense, the technique is dual: it passes from variables to identities, and then from identities back into inequalities.

References

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