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3901. If in ΔABC we have:

$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{1}{5}$$

then:

$$3\sin C=2\sin A+2\sin B$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\tan\frac{A}{2}\tan\frac{B}{2}=\frac{1}{5}\Rightarrow\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\cdot\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\frac{1}{5}\Rightarrow$$

$$\Rightarrow\frac{s-c}{s}=\frac{1}{5}\Rightarrow 5s-5c=s\Rightarrow 4s=5c\Rightarrow$$

$$\Rightarrow 2a+2b+2c=5c\Rightarrow 3c=2a+2b\Rightarrow$$

$$\Rightarrow 3\cdot 2R\sin C=2\cdot 2R\sin A+2\cdot 2R\sin B\Rightarrow 3\sin C=2\sin A+2\sin B$$

3902. Prove that in ΔABC the following relationship holds:

$$\frac{a}{r_b}+\frac{b}{r_c}+\frac{c}{r_a}\ge 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{a}{r_b}+\frac{b}{r_c}+\frac{c}{r_a}\overset{AM-GM}{\ge}3\left(\frac{abc}{r_ar_br_c}\right)^{\frac{1}{3}}=$$

$$=3\left(\frac{4RF}{Fp}\right)^{\frac{1}{3}}=3\left(\frac{4R}{p}\right)^{\frac{1}{3}}\overset{Mitrinovic}{\ge}3\left(\frac{4}{p}\cdot\frac{2p}{3\sqrt{3}}\right)^{\frac{1}{3}}=\frac{6}{\sqrt{3}}=2\sqrt{3}$$

$$Equality\ holds\ for\ a=b=c.$$

3903. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\left(s+p_a\sqrt{\frac{w_a}{g_a}}\right)\left(w_a+w_b+m_c-n_a\sqrt{3}\right)}{h_a}\le 2\sqrt{3}(4R+r)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a \overset{?}{\geq} p_a^2 + \frac{(b-c)^2}{18} \overset{\substack{\text{Fustei} \\ \text{and} \\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a) + \frac{(b-c)^2}{4}\right)\left(s(s-a) + \frac{s(b-c)^2}{a}\right) \overset{?}{\geq} \left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+ \frac{(b-c)^4}{324} + \frac{(b-c)^2}{9}.\left(s(s-a) + \frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a} + \frac{1}{4}\right) + \frac{s(b-c)^4}{4a} \overset{?}{\geq} \frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4} +$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2} + \frac{(b-c)^4}{324} + s(s-a).\frac{(b-c)^2}{9} + \frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a} + \frac{1}{4} - \frac{2s(3s+a)(b-c)^2}{(2s+a)^2} - \frac{1}{9}\right) +$$

$$\left(\frac{s}{4a} - \frac{s^2(3s+a)^2}{(2s+a)^4} - \frac{1}{324} - \frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2 \overset{?}{\geq} 0 \; (\because (b-c)^2 \geq 0)$$

$$\Leftrightarrow \frac{s(s-a)(144s^3 - 52s^2a - 16sa^2 + 5a^3)}{36a(2s+a)^2} +$$

$$\frac{1296s^5 - 772s^4a - 608s^3a^2 + 48s^2a^3 + 37sa^4 - a^5}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2 + 92sa + 76a^2) + 81a^3\right)}{36a(2s+a)^2} +$$

$$\frac{(s-a)\left((s-a)(1296s^3 + 1820s^2a + 1736sa^2 + 1700a^3) + 1701a^4\right)(b-c)^2}{324a(2s+a)^4} \overset{?}{\geq} 0$$

$$\rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow (1) \text{ and}$$

$$n_a g_a \overset{\text{Bogdan Fustei}}{\geq} m_a w_a \rightarrow (2) \therefore (1) \bullet (2) \Rightarrow (m_a n_a)(n_a g_a) \geq p_a^2 . m_a w_a$$

$$\Rightarrow n_a \geq \sqrt{\frac{w_a}{g_a}}.p_a \Rightarrow \left(s + p_a.\sqrt{\frac{w_a}{g_a}}\right)\left(w_a + w_b + m_c - n_a.\sqrt{3}\right) \overset{\text{Lessel-Pelling}}{\leq}$$

$$\left(s + p_a.\sqrt{\frac{w_a}{g_a}}\right)\left(s.\sqrt{3} - n_a.\sqrt{3}\right) \leq (s + n_a)\left(s.\sqrt{3} - n_a.\sqrt{3}\right)$$

$$\left(\because 2h_a r_a \overset{\text{Bogdan Fustei}}{=} s^2 - n_a^2 \Rightarrow s - n_a > 0\right)$$

$$\Rightarrow \frac{\left(s + p_a.\sqrt{\frac{w_a}{g_a}}\right)\left(w_a + w_b + m_c - n_a.\sqrt{3}\right)}{h_a} \leq \frac{\sqrt{3}.(s^2 - n_a^2)}{h_a} \overset{\text{Bogdan Fustei}}{=} \frac{\sqrt{3}.2h_a r_a}{h_a}$$

$$\therefore \frac{\left(s+p_a.\sqrt{\frac{w_a}{g_a}}\right)(w_a+w_b+m_c-n_a.\sqrt{3})}{h_a} \le 2\sqrt{3}.r_a \text{ and analogs}$$

$$\Rightarrow \sum_{cyc}\frac{\left(s+p_a.\sqrt{\frac{w_a}{g_a}}\right)(w_a+w_b+m_c-n_a.\sqrt{3})}{h_a} \le 2\sqrt{3}.\sum_{cyc} r_a = 2\sqrt{3}.(4R+r)$$

$\forall \Delta ABC,'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$

3904. In any Δ ABC the following relationship holds :

$$\sqrt{h_a h_b}+\sqrt{h_b h_c}+\sqrt{h_c h_a} \le \frac{8}{9}.R+\frac{65}{9}.r$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{h_b h_c} = \frac{1}{2R}.\sum_{cyc}\sqrt{ca.ab}$$

$$= \frac{1}{2R}.\sum_{cyc}\left(\frac{a}{\sqrt{(s-b)(s-c)}}.\sqrt{bc(s-b)(s-c)}\right)$$

$$\overset{CBS}{\le} \frac{1}{2R}.\sqrt{\sum_{cyc}\frac{a^2(s-a)}{r^2 s}}.\sqrt{r^2 s^2.\sum_{cyc}\frac{bc}{s(s-a)}}$$

$$= \frac{1}{2R}.\sqrt{\frac{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{s}}.\sqrt{s^2.\sum_{cyc}\sec^2\frac{A}{2}}$$

$$= \frac{1}{2R}.\sqrt{4Rr+4r^2}.\sqrt{s^2.\frac{(4R+r)^2+s^2}{s^2}} \overset{?}{\le} \frac{8R+65r}{9}$$

$$\Leftrightarrow (16R^2+130Rr)^2 \underset{(*)}{\overset{?}{\ge}} 81(4Rr+4r^2)((4R+r)^2+s^2)$$

$$\text{Now, LHS of } (*) \overset{Blundon-Gerretsen}{\le} 81(4Rr+4r^2)\left((4R+r)^2+\frac{R(4R+r)^2}{4R-2r}\right)$$

$$\overset{?}{\le} (16R^2+130Rr)^2$$

$$\Leftrightarrow 256t^5-2448t^4+7692t^3-8207t^2+1053t+162 \underset{(**)}{\overset{?}{\ge}} 0 \left(t=\frac{R}{r}\right)$$

Case 1 $t \ge \frac{16689}{4309}$ and then : LHS of $(**)$ =

$$(t-2)\left(\left(\frac{64t^2-4t+25}{4}\right)(4t-15)^2+\frac{1632t-5949}{4}\right)>0$$

$$\left(\because t\geq\frac{16689}{4309}>\frac{5949}{1632}>2\right)\Rightarrow(**)\text{ is true}$$

$$\boxed{Case\ 2}\ t<\frac{16689}{4309}\ \text{and then : LHS of }(**)=$$

$$(t-2)\left(\left(\frac{641t^2+88t(t-2)+39(t^2-4)}{27}\right)(3t-11)^2+\frac{16689-4309t}{27}\right)\geq 0$$

$$\left(\because 2\overset{\text{Euler}}{\leq}t<\frac{16689}{4309}\right)\Rightarrow(**)\text{ is true}\ \therefore\ \text{combining both cases},(**)\Rightarrow(*)$$

$$\text{is true }\forall\,\Delta\,ABC\text{ and so},\sqrt{h_ah_b}+\sqrt{h_bh_c}+\sqrt{h_ch_a}\leq\frac{8}{9}.R+\frac{65}{9}.r\ \forall\,ABC,$$

$$''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

3905. In any Δ ABC the following relationship holds :

$$\sqrt{m_am_b}+\sqrt{m_bm_c}+\sqrt{m_cm_a}\leq\sqrt{\frac{74(a^2b^2+b^2c^2+c^2a^2)+7(a^4+b^4+c^4)}{12(a^2+b^2+c^2)}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\sqrt{bc}=\sum_{cyc}\sqrt{\frac{bc}{b+c}.(b+c)}\overset{CBS}{\leq}\sqrt{\sum_{cyc}\frac{bc}{b+c}}.\sqrt{\sum_{cyc}(b+c)}$$

$$=\sqrt{\frac{4s}{2s(s^2+2Rr+r^2)}.\sum_{cyc}\left(bc\left(a^2+\sum_{cyc}ab\right)\right)}=\sqrt{\frac{2(8Rrs^2+(s^2+4Rr+r^2)^2)}{s^2+2Rr+r^2}}$$

$$\overset{?}{\leq}2.\sqrt{\frac{7\left(\sum_{cyc}a^2\right)^2+60\sum_{cyc}a^2b^2}{36\sum_{cyc}a^2}}$$

$$=\sqrt{\frac{7(s^2-4Rr-r^2)^2+15((s^2+4Rr+r^2)^2-16Rrs^2)}{18(s^2-4Rr-r^2)}}$$

$$\Leftrightarrow 13s^6-(240Rr-29r^2)s^4+r^2(432R^2+176Rr+47r^2)s^2+$$

$$r^3(1280R^3+1136R^2r^2+328Rr^2+31r^3)\underset{(*)}{\overset{?}{\geq}}0\text{ and }\because 13(s^2-16Rr+5r^2)^3$$

$$\overset{\text{Gerretsen}}{\geq}0\ \therefore\text{ in order to prove }(*),\text{it suffices to prove : LHS of }(*)\overset{?}{\geq}$$

$$13(s^2-16Rr+5r^2)^3 \Leftrightarrow (192R-83r)s^4 - r(4776R^2-3208Rr+464r^2)s^2 +$$
$$r^2(27264R^3-24392R^2r^2+7964Rr^2-797r^3) \overset{?}{\underset{(**)}{\geq}} 0 \text{ and } \because$$
$$(192R-83r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } (**),$$
$$\text{it suffices to prove : LHS of } (**) \overset{?}{\geq} (192R-83r)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow (228R^2-228Rr+61r^2)s^2 \overset{?}{\underset{(***)}{\geq}} r(3648R^3-4596R^2r^2+1686Rr^2-213r^3)$$
$$\text{Now, } (R-r)(s^2-16Rr+5r^2) \overset{\text{Rouche}}{\geq} (R-r)\left(\begin{array}{c}2R^2-6Rr+4r^2-\\2(R-2r)\sqrt{R^2-2Rr}\end{array}\right)$$
$$=(R-2r)\left(\left(R-r-\sqrt{R^2-2Rr}\right)^2+r^2\right) \geq r^2(R-2r)\ \left(\because R-2r \overset{\text{Euler}}{\geq} 0\right)$$
$$\Rightarrow s^2 \geq 16Rr-5r^2+\frac{r^2(R-2r)}{R-r} \therefore \text{LHS of } (***) \geq$$
$$(228R^2-228Rr+61r^2)\left(16Rr-5r^2+\frac{r^2(R-2r)}{R-r}\right) \overset{?}{\geq} \text{RHS of } (***)$$
$$\Leftrightarrow 36t^3-62t^2-5t-30 \overset{?}{\geq} 0\ \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(36t^2+10t+15) \overset{?}{\geq} 0$$
$$\rightarrow \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$
$$\therefore \sum_{\text{cyc}} \sqrt{bc} \leq 2.\sqrt{\frac{7\left(\sum_{\text{cyc}} a^2\right)^2+60\sum_{\text{cyc}} a^2b^2}{36\sum_{\text{cyc}} a^2}} \text{ and implementing it on}$$
$$a \text{ triangle with sides } \frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}, \text{we arrive at :}$$
$$\frac{2}{3}.\sum_{\text{cyc}} \sqrt{m_bm_c} \leq 2.\sqrt{\frac{7.\frac{16}{81}.\left(\sum_{\text{cyc}} m_a^2\right)^2+60.\frac{16}{81}.\sum_{\text{cyc}} m_a^2m_b^2}{36.\frac{4}{9}.\sum_{\text{cyc}} m_a^2}}$$
$$=2.\sqrt{\frac{7.\frac{16}{81}.\left(\frac{3}{4}.\sum_{\text{cyc}} a^2\right)^2+60.\frac{16}{81}.\frac{9}{16}.\sum_{\text{cyc}} a^2b^2}{36.\frac{4}{9}.\frac{3}{4}.\sum_{\text{cyc}} a^2}} = 2.\sqrt{\frac{74\sum_{\text{cyc}} a^2b^2+7\sum_{\text{cyc}} a^4}{108\sum_{\text{cyc}} a^2}}$$
$$\Rightarrow \sum_{\text{cyc}} \sqrt{m_bm_c} \leq \sqrt{\frac{74\sum_{\text{cyc}} a^2b^2+7\sum_{\text{cyc}} a^4}{\frac{108}{9}.\sum_{\text{cyc}} a^2}} \text{ and so,}$$
$$\sqrt{m_am_b}+\sqrt{m_bm_c}+\sqrt{m_cm_a} \leq \sqrt{\frac{74(a^2b^2+b^2c^2+c^2a^2)+7(a^4+b^4+c^4)}{12(a^2+b^2+c^2)}}$$
$$\forall\, ABC, '' = '' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3906. In any Δ ABC the following relationship holds :

$$(m_a+m_b)(m_b+m_c)(m_c+m_a)\le\sqrt{\frac{27}{32}\cdot\left(3\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}b^2c^2\right)+5a^2b^2c^2\right)}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3}{32}\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}b^2c^2\right)+\frac{10}{27}\cdot m_a^2m_b^2m_c^2\overset{?}{\ge}\frac{1}{64}\cdot(a+b)^2(b+c)^2(c+a)^2$$

$$\Leftrightarrow\frac{3}{32}\left(2(s^2-4Rr-r^2)\right)\left((s^2+4Rr+r^2)^2-16Rrs^2\right)+$$
$$\frac{10}{27}\cdot\frac{s^6-s^4(12Rr-33r^2)-s^2(60R^2r^2+120Rr^3+33r^4)-r^3(4R+r)^3}{16}$$
$$\overset{?}{\ge}\frac{1}{64}\cdot 4s^2(s^2+4Rr+r^2)^2$$

$\left(\text{Reference : Solution to Inequality in Triangle by Dang Ngoc Minh} - 124;\ \text{published at www.ssmrmh.ro}\right)$

$$\Leftrightarrow 64s^6-(1200Rr-357r^2)s^4+r^2(3180R^2-660Rr-438r^2)s^2-$$
$$91r^3(4R+r)^3\ \underset{(*)}{\overset{?}{\ge}}\ 0\ \text{and}\ \because 64(s^2-16Rr+5r^2)^3\overset{\text{Gerretsen}}{\ge}0$$

$$\therefore\text{ in order to prove }(*),\text{ it suffices to prove : LHS of }(*)\overset{?}{\ge}64(s^2-16Rr+5r^2)^3$$
$$\Leftrightarrow(208R-67r)s^4-r(5108R^2-3340Rr+582r^2)s^2+$$
$$r^2(28480R^3-27792R^2r+8412Rr^2-899r^3)\ \underset{(**)}{\overset{?}{\ge}}\ 0\ \text{and}\ \because$$

$$(208R-67r)(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\ge}0\ \therefore\text{ in order to prove }(**),\text{ it suffices}$$
$$\text{to prove : LHS of }(**)\overset{?}{\ge}(208R-67r)(s^2-16Rr+5r^2)^2$$
$$\Leftrightarrow(387R^2-221Rr+22r^2)s^2\ \underset{(***)}{\overset{?}{\ge}}\ r(6192R^3-5660R^2r^2+1877Rr^2-194r^3)$$

$$\text{Finally, LHS of }(***)\overset{\text{Gerretsen}}{\ge}(387R^2-221Rr+22r^2)(16Rr-5r^2)\overset{?}{\ge}$$
$$\text{RHS of }(***)\Leftrightarrow 21(R-2r)(9R-2r)\overset{?}{\ge}0\to\text{true}\because R\overset{\text{Euler}}{\ge}2r\Rightarrow(***)\Rightarrow(**)\Rightarrow$$
$$(*)\text{ is true}\therefore\sqrt{\frac{3}{32}\cdot\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}b^2c^2\right)+\frac{10}{27}\cdot m_a^2m_b^2m_c^2}\ge\frac{1}{8}(a+b)(b+c)(c+a)$$

$$\text{and implementing it on a triangle with sides } \frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3} \text{ whose medians}$$

$$\text{as a consequence of trivial calculations} = \frac{a}{2}, \frac{b}{2}, \frac{c}{2} \text{ respectively, we get :}$$

$$\sqrt{\frac{3}{32}\cdot\left(\frac{4}{9}\cdot\sum_{cyc} m_a^2\right)\left(\frac{16}{81}\cdot\sum_{cyc} m_b^2 m_c^2\right)+\frac{10}{27}\cdot\frac{1}{64}\cdot a^2b^2c^2} \geq \frac{1}{8}\cdot\frac{8}{27}\prod_{cyc}(m_b+m_c)$$

$$\Rightarrow \sqrt{\frac{3}{32}\cdot\left(\frac{4}{9}\cdot\frac{3}{4}\cdot\sum_{cyc} a^2\right)\left(\frac{16}{81}\cdot\frac{9}{16}\cdot\sum_{cyc} b^2c^2\right)+\frac{10}{27}\cdot\frac{1}{64}\cdot a^2b^2c^2} \geq \frac{1}{27}\prod_{cyc}(m_b+m_c)$$

$$\Rightarrow \sqrt{\frac{81}{32}\cdot\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} b^2c^2\right)+\frac{135}{32}\cdot a^2b^2c^2} \geq (m_a+m_b)(m_b+m_c)(m_c+m_a)$$

$$\therefore (m_a+m_b)(m_b+m_c)(m_c+m_a) \leq \sqrt{\frac{27}{32}\cdot\left(3\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} b^2c^2\right)+5a^2b^2c^2\right)}$$

$$\forall\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3907. In any Δ ABC the following relationship holds :

$$|b-c| \leq \min\left\{s.\sqrt{\frac{R-2r}{R}}, \sqrt{\frac{128}{135}(2s^2-27Rr)}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s_0 = \text{semiperimeter (for own convenience)}, s = \sin\frac{A}{2}, c = \cos\frac{B-C}{2}$$

$$\text{and we first prove : } |b-c| \overset{?}{\leq} s_0.\sqrt{\frac{R-2r}{R}} \Leftrightarrow \frac{16R^2.s^2(1-c^2)}{16R^2.(1-s^2)\frac{(c+s)^2}{4}} \overset{?}{\leq} 1-4sc+4s^2$$

$$\Leftrightarrow 4s^3(c^3-s^3)+4cs^4(c-s)-(4c^3s-4s^4)-s^2(s^2+c^2-2cs)+$$

$$(c^2+2cs-3s^2) \overset{?}{\geq} 0 \Leftrightarrow 4s^3(c^2+s^2+cs)+4cs^4-4s(c^2+s^2+cs)-$$

$$s^2(c-s)+c+3s \overset{?}{\geq} 0 \left(\because c > s \text{ as } \cos\frac{B-C}{2} = \frac{b+c}{a}.\sin\frac{A}{2} > \sin\frac{A}{2}\right)$$

$$\Leftrightarrow (4s-4s^3)c^2-(8s^4-5s^2+1)c+3s^3-3s-4s^5 \overset{?}{\underset{(*)}{\leq}} 0 \text{ and } \because \text{ discriminant}$$

$$\delta = (8s^4-5s^2+1)^2-4(4s-4s^3)(3s^3-3s-4s^5)$$
$$= (32s^2(s^2-1)^2+(3s^2+1)^2) > 0 \text{ and } 4s-4s^3 > 0 \left(as\ 0 < \sin\frac{A}{2} < 1\right)$$

$\therefore$ in order to prove $(*)$, it suffices to prove :

$$2(4s-4s^3)c \overset{?}{\underset{(\bullet)}{\leq}} 8s^4-5s^2+1+\sqrt{\delta} \text{ AND } 2(4s-4s^3)c \overset{?}{\underset{(\bullet\bullet)}{\geq}} 8s^4-5s^2+1-\sqrt{\delta}$$

$$\text{Now, } 2(4s-4s^3)c-(8s^4-5s^2+1) \overset{0<c\leq 1}{\leq} 2(4s-4s^3)-(8s^4-5s^2+1)$$

and if it's ≤ 0, it's trivially $< \sqrt{\delta}$ and when it's > 0, then in order to prove $(\bullet)$, it suffices to prove : $\delta \overset{?}{\geq} \left(2(4s-4s^3)-(8s^4-5s^2+1)\right)^2$

$$\Leftrightarrow 16(1-s)(s+1)^4(2s-1)^2 \overset{?}{\geq} 0 \to \text{true} \Rightarrow (\bullet) \text{ is true}$$

Again, since $2(4s-4s^3)c > 0$ and $8s^4-5s^2+1 > 0 \therefore$ in order to prove $(\bullet\bullet)$, it suffices to prove : $\delta \overset{?}{\geq} (8s^4-5s^2+1)^2$

$$\Leftrightarrow 16s^2(1-s^2)((2s^2-1)^2+s^2+2) \overset{?}{\geq} 0 \to \text{true (strict inequality) and so,}$$

$$(\bullet) \text{ and } (\bullet\bullet) \text{ are both true} \Rightarrow (*) \text{ is true} \therefore |b-c| \leq s_0.\sqrt{\frac{R-2r}{R}}$$

$$\text{Again, } |b-c| \overset{?}{\leq} \sqrt{\frac{128}{135}\left(2s_0^2-27Rr\right)}$$

$$\Leftrightarrow \frac{16R^2.s^2(1-c^2)}{16R^2.(1-s^2)\frac{(c+s)^2}{4}} \overset{?}{\leq} \frac{256}{135}-\frac{128}{5}.\frac{4R^2.s.\frac{c-s}{2}}{16R^2.(1-s^2)\frac{(c+s)^2}{4}}$$

$$\Leftrightarrow (71s^2+64)c^2-(128s^3+304s)c+361s^2-64s^4 \overset{?}{\underset{(**)}{\geq}} 0 \text{ and } \because$$

$$\text{discriminant } \delta_0 = (128s^3+304s)^2-4(71s^2+64)(361s^2-64s^4)$$

$$= 108s^4(320s^2-77) \text{ which is clearly } \leq 0 \text{ for } s \in \left(0, \sqrt{\frac{77}{320}}\right]$$

$$\therefore \text{ we now focus on } s \in \left(\sqrt{\frac{77}{320}}, 1\right) \text{ and then, in order to prove } (**),$$

it suffices to prove : $2(71s^2+64)c \overset{?}{\leq} 128s^3+304s-\sqrt{108s^4(320s^2-77)}$ and $\because c \leq 1 \therefore$ it suffices to prove :

$$\sqrt{108s^4(320s^2-77)} \overset{?}{\leq} 128s^3+304s-2(71s^2+64)$$

$$\Leftrightarrow 4(64s^3-71s^2+152s-64)^2 \overset{?}{\geq} 108s^4(320s^2-77)$$

$$\left(\because 64s^3-71s^2+152s-64>0\ \forall s\in\left(\sqrt{\frac{77}{320}},1\right)\right)$$

$$\Leftrightarrow 64(1-s)(s+4)(2s-1)^2(71s^2+64)\overset{?}{\geq} 0 \to \text{true} \because s\in\left(\sqrt{\frac{77}{320}},1\right)$$

$$\Rightarrow (**) \text{ is true } \therefore |b-c|\leq\sqrt{\frac{128}{135}(2s_0^2-27Rr)} \text{ and so, whenever } "s" \text{ is}$$

$$\text{the semiperimeter, } |b-c|\leq s.\sqrt{\frac{R-2r}{R}}, \sqrt{\frac{128}{135}(2s^2-27Rr)}\ \therefore |b-c|\leq$$

$$\min\left\{s.\sqrt{\frac{R-2r}{R}}, \sqrt{\frac{128}{135}(2s^2-27Rr)}\right\}\ \forall\,\Delta\,ABC, "=" \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3908. In ΔABC the following relationship holds:

$$8\leq\prod\frac{m_b^2+m_c^2}{m_a^2}\leq\frac{8}{729}\left(\frac{2R^2}{r^2}+1\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\prod\frac{m_b^2+m_c^2}{m_a^2}=\frac{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}{m_a^2m_b^2m_c^2}=$$

$$=\frac{(\sum m_a^2)(\sum m_a^2m_b^2)-m_a^2m_b^2m_c^2}{m_a^2m_b^2m_c^2}=\sum m_a^2\frac{(\sum m_a^2m_b^2)}{m_a^2m_b^2m_c^2}-1=\sum m_a^2\sum\frac{1}{m_a^2}-1\leq$$

$$\overset{m_a\geq\sqrt{s(s-a)}}{\leq}\frac{3}{4}\sum a^2\sum\frac{1}{s(s-a)}-1\overset{Leibniz}{\leq}\frac{3}{4}.9R^2\left(\frac{4R+r}{s^2r}\right)-1\overset{s^2\geq 3r(4R+r)}{\leq}$$

$$\leq\frac{27}{4}R^2.\frac{1}{3r^2}-1=\frac{9}{4}\left(\frac{R}{r}\right)^2-1$$

We need to show:

$$\frac{9}{4}\left(\frac{R}{r}\right)^2-1\le\frac{8}{729}\left(\frac{2R^2}{r^2}+1\right)^3 \ \ or\ \ \frac{9}{4}x^2-1\overset{\frac{R}{r}=x\ge 2}{\le}\frac{8}{729}(2x^2+1)^3$$

$$\frac{9}{4}y-1\overset{x^2=y}{\le}\frac{8}{729}(2y+1)^3\ or\ 6561y-2916\le 32(2y+1)^3$$

$$6561y-2916\le 32(8y^3+12y^2+6y+1)$$

$$256y^3+384y^2-6369y+2948\ge 0$$

$$(y-4)(256x^4+1408x^2-737)\ge 0\ this\ is\ true\ as\ y=x^2=\left(\frac{R}{r}\right)^2\ge 2^2=4$$

$$\prod\frac{m_b^2+m_c^2}{m_a^2}=\frac{(m_a^2+m_b^2)(m_b^2+m_c^2)(m_c^2+m_a^2)}{m_a^2m_b^2m_c^2}\overset{Cesaro}{\ge}\frac{8m_a^2m_b^2m_c^2}{m_a^2m_b^2m_c^2}=8$$

Equality holds for an equilateral triangle.

3909. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{b+c}\cdot\sum_{cyc}\frac{b+c}{\sqrt{r_br_c}}\ge 9$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{m_a}{b+c}\cdot\sum_{cyc}\frac{b+c}{\sqrt{r_br_c}}\overset{CSB}{\ge}\left(\sum_{cyc}\sqrt{\frac{m_a}{b+c}}\cdot\sqrt{\frac{b+c}{\sqrt{r_br_c}}}\right)^2=$$

$$=\left(\sum_{cyc}\sqrt{\frac{m_a}{\sqrt{r_br_c}}}\right)^2\overset{AM-GM}{\ge}\left(3\left(\left(\frac{m_am_bm_c}{r_ar_br_c}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^2\overset{m_a\ge\sqrt{s(s-a)}}{\ge}$$

$$\ge 9\left(\frac{s\sqrt{s(s-a)(s-b)(s-c)}}{sF}\right)^{\frac{1}{3}}=9\left(\frac{sF}{sF}\right)^{\frac{1}{3}}=9$$

Equality holds for $a=b=c$.

3910. In ΔABC the following relationship holds:

$$8\le\prod\frac{s_b+s_c}{s_a}\le\frac{8}{729}\left(\frac{4R}{r}+1\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

Lemma:

$$\frac{R}{r} \ge \frac{m_b}{h_c} + \frac{m_c}{h_b}$$

Reference: " New Triangle inequalities with Brocard's Angle " by Bogdan Fustei, Mohamed Amine Ben Ajiba: www.ssmrmh.ro

$$\frac{s_b}{s_c} + \frac{s_c}{s_b} \overset{\substack{s_b \le m_b, s_c \ge h_c \\ s_c \le m_c, s_b \ge h_b}}{\le} \frac{m_b}{h_c} + \frac{m_c}{h_b} \overset{lemma}{\le} \frac{R}{r} \; (1)$$

$$\prod \frac{s_b + s_c}{s_a} = \frac{(s_b + s_c)(s_c + s_a)(s_a + s_b)}{s_a s_b s_c} = \prod \frac{(s_b + s_c)}{\sqrt{s_b s_c}} = \prod \left(\sqrt{\frac{s_b}{s_c}} + \sqrt{\frac{s_c}{s_b}}\right) \le$$

$$\overset{CBS}{\le} \prod \sqrt{2\left(\frac{s_b}{s_c} + \frac{s_c}{s_b}\right)} \overset{(1)}{\le} \prod \sqrt{\frac{2R}{r}} = \left(\sqrt{\left(\frac{2R}{r}\right)}\right)^3 \overset{\frac{R}{r} = x \ge 2}{=} \left(\sqrt{2x}\right)^3$$

We need to show:

$$\left(\sqrt{2x}\right)^3 \le \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3 \overset{\frac{R}{r} = x}{=} \frac{8}{729}(4x+1)^3 \; or, \left(\frac{2}{9}(4x+1)\right)^3 - \left(\sqrt{2x}\right)^3 \ge 0$$

$$or, \left(\frac{2}{9}(4x+1) - \sqrt{2x}\right)\left(\left(\frac{2(4x+1)}{9}\right)^2 + \frac{2(4x+1)}{9}\sqrt{2x} + \left(\sqrt{2x}\right)^2\right) \overset{\substack{a^3 - b^3 = \\ (a-b)(a^2+ab+b^2)}}{\ge} 0$$

$$\left(\begin{array}{c} True \; as \; x = \frac{R}{r} \ge 2 \; then \\ \left(\left(\frac{2(4x+1)}{9}\right)^2 + \frac{2(4x+1)}{9}\sqrt{2x} + \left(\sqrt{2x}\right)^2\right) > 0 \; and \\ \left(\frac{2}{9}(4x+1) - \sqrt{2x}\right) \ge \left(\frac{2}{9}(4 \times 2 + 1) - \sqrt{2 \times 2}\right) = 2 - 2 = 0 \end{array}\right)$$

$$\prod \frac{s_b + s_c}{s_a} = \frac{(s_b + s_c)(s_c + s_a)(s_a + s_b)}{s_a s_b s_c} \overset{Cesaro}{\ge} \frac{8 s_a s_b s_c}{s_a s_b s_c} = 8$$

Equality holds for an equilateral triangle.

3911. In ΔABC the following relationship holds:

$$\frac{\tan(A) + \tan(B)}{\tan^2(C)} + \frac{\tan(B) + \tan(C)}{\tan^2(A)} + \frac{\tan(A) + \tan(C)}{\tan^2(B)} \ge 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\tan(A)+\tan(B)}{tan^2(C)}=\sum_{cyc}\frac{\frac{\sin(A+B)}{\cos(A).\cos(B)}}{tan^2(C)}=\sum_{cyc}\frac{cos^2(C).\sin(C)}{sin^2(C).\cos(A).\cos(B)}=$$

$$=\sum_{cyc}\frac{cos^2(C)}{\sin(C).\cos(A).\cos(B)}\overset{AM-GM}{\geq}$$

$$\geq 3\left(\frac{cos^2(A).cos^2(B).cos^2(C)}{sin(A).\sin(B).\sin(C).cos^2(A).cos^2(B).cos^2(C)}\right)^{\frac{1}{3}}=$$

$$=3\left(\frac{1}{\prod_{cyc}\sin(A)}\right)^{\frac{1}{3}}\geq 3\left(\frac{1}{\frac{3\sqrt{3}}{8}}\right)^{\frac{1}{3}}=3\left(\frac{8}{\sqrt{27}}\right)^{\frac{1}{3}}=2\sqrt{3}$$

Equality holds for $a=b=c$.

3912. In ΔABC the following relationship holds:

$$\frac{ab}{r_c^2}+\frac{bc}{r_a^2}+\frac{ac}{r_b^2}\geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{ab}{r_c^2}+\frac{bc}{r_a^2}+\frac{ac}{r_b^2}\overset{AM-GM}{\geq}3\left(\left(\frac{abc}{r_ar_br_c}\right)^2\right)^{\frac{1}{3}}=$$

$$=3\left(\left(\frac{4RF}{Fp}\right)^2\right)^{\frac{1}{3}}=3\left(\frac{16R^2}{p^2}\right)^{\frac{1}{3}}\overset{Mitrinovic}{\geq}3\left(\frac{16}{p^2}\cdot\frac{4p^2}{27}\right)^{\frac{1}{3}}=4$$

Equality holds for $a=b=c$.

3913. In any Δ ABC the following relationship holds :

$$\mathrm{h}_a^3+\mathrm{h}_\mathrm{b}^3+\mathrm{h}_\mathrm{c}^3\leq \mathrm{r}_a^3+\mathrm{r}_\mathrm{b}^3+\mathrm{r}_\mathrm{c}^3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\mathrm{r}_a^3\overset{?}{\geq}\sum_{cyc}\mathrm{h}_a^3$$

$$\Leftrightarrow(4\mathrm{R}+\mathrm{r})^3-12\mathrm{Rs}^2\overset{?}{\geq}\frac{(\mathrm{s}^2+4\mathrm{Rr}+\mathrm{r}^2)^3-24\mathrm{Rrs}^2(\mathrm{s}^2+2\mathrm{Rr}+\mathrm{r}^2)}{8\mathrm{R}^3}$$

$$\Leftrightarrow -s^6+(12Rr-3r^2)s^4-(96R^4+3r^4)s^2+$$
$$512R^6+384R^5r+96R^4r^2-56R^3r^3-48R^2r^4-12Rr^5-r^6 \underset{①}{\overset{?}{\geq}} 0$$

$$\text{Now, since}: P=-s^4(s^2-4R^2-4Rr-3r^2) \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{ in order to prove } ①, \text{it suffices to prove}: \text{LHS of } ① \overset{?}{\geq} P$$
$$\Leftrightarrow -(4R^2-8Rr+6r^2)s^4-(96R^4+3r^4)s^2+$$
$$512R^6+384R^5r+96R^4r^2-56R^3r^3-48R^2r^4-12Rr^5-r^6 \underset{②}{\overset{?}{\geq}} 0$$

$$\text{Again, since}: Q=-(4R^2-8Rr+6r^2)(s^2-4R^2-4Rr-3r^2)s^2 \overset{\text{Gerretsen}}{\geq} 0$$

$$\therefore \text{ in order to prove } ②, \text{it suffices to prove}: \text{LHS of } ② \overset{?}{\geq} Q$$
$$\Leftrightarrow 512R^6+384R^5r+96R^4r^2-56R^3r^3-48R^2r^4-12Rr^5-r^6$$
$$\underset{③}{\overset{?}{\geq}} (112R^4-16R^2r^2+4Rr^3+21r^4)s^2$$

$$\text{Finally}, (112R^4-16R^2r^2+4Rr^3+21r^4)s^2 \overset{\text{Gerretsen}}{\leq}$$
$$(112R^4-16R^2r^2+4Rr^3+21r^4)(4R^2+4Rr+3r^2) \overset{?}{\leq} \text{LHS of } ③$$
$$\Leftrightarrow 8t^6-24t^4-3t^3-18t^2-12t-8 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$
$$\Leftrightarrow (t-2)\left(8t^5+16t^4+8t^3+13t^2+8t+4\right) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$
$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true} \therefore h_a^3+h_b^3+h_c^3 \leq r_a^3+r_b^3+r_c^3 \ \forall \Delta ABC,$$
$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3914. In ΔABC the following relationship holds:

$$\frac{r_a^2}{h_b}+\frac{r_b^2}{h_c}+\frac{r_c^2}{h_a} \geq 9r$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{r_a^2}{h_b}+\frac{r_b^2}{h_c}+\frac{r_c^2}{h_a} \overset{\text{Bergstrom}}{\geq} \frac{(r_a+r_b+r_c)^2}{h_a+h_b+h_c} = \frac{(r_a+r_b+r_c)^2}{\frac{ac}{2R}+\frac{ab}{2R}+\frac{bc}{2R}} = \frac{2R(r_a+r_b+r_c)^2}{ab+bc+ac} \geq$$

$$\geq \frac{3.2R(4R+r)^2}{(a+b+c)^2} = \frac{6R(4R+r)^2}{4s^2} \overset{\text{Mitrinovici}}{\geq} \frac{6.\frac{2s}{3\sqrt{3}}(4R+r)^2}{4s^2} =$$

$$= \frac{(4R+r)(4R+r)}{s\sqrt{3}} \overset{\text{Doucet}}{\geq} 4R+r \overset{\text{Euler}}{\geq} 9r$$

Equality holds for $a=b=c$.

3915. *If in ΔABC $a \ge b$ and $x \in (0,1)$ then prove that :*

$$\frac{3c(b+c-a)}{x^2+x+1}+\frac{2(a-b)(a-b-c)}{x+1}+(a-b)c>0$$

Proposed by Pavlos Trifon-Greece

Solution by Tapas Das-India

$$x \in (0,1) \text{ then } 1<x+1<2 \text{ and } x^2+x+1<1^2+1+1=3 \text{ so}$$
$$\frac{1}{x+1}<1\ \&\ \frac{1}{x^2+x+1}>\frac{1}{3}$$

$$3c(b+c-a)>0 \text{ as in any } \Delta ABC\ b+c>a$$

$$2(a-b)(a-b-c)=-2(a-b)(b+c-a)<0$$
$$\text{as } a \ge b\ \&\ \text{in any } \Delta ABC\ b+c>a \text{ and } (a-b)c>0 \text{ as } a\ge b$$

$$\frac{3c(b+c-a)}{x^2+x+1}+\frac{2(a-b)(a-b-c)}{x+1}+(a-b)c>0$$

$$\frac{1}{3}3c(b+c-a)+2(a-b)(a-b-c)+(a-b)c>0$$

$$c(b+c-a)+2(a-b)(a-b-c)+(a-b)c>0$$

$$c\big(c-(a-b)\big)+2(a-b)\big((a-b)-c\big)+(a-b)c>0$$

$$c^2-c(a-b)+2(a-b)^2-2c(a-b)+(a-b)c>0$$

$$c^2-2c(a-b)+2(a-b)^2>0$$

$$\big((a-b)-c\big)^2+(a-b)^2>0 \text{ true}$$

3916. In ΔABC the following relationship holds:

$$\frac{16R}{r}-\frac{(\sin A+\sin B+\sin C)^4}{4\sin^2 A\sin^2 B\sin^2 C}\le 5$$

Proposed by Samir Cabiyev-Azerbaijan

Solution by Daniel Sitaru-Romania

$$\frac{16R}{r}-\frac{(sinA+sinB+sinC)^4}{4sin^2Asin^2Bsin^2C}=\frac{16R}{r}-\frac{\left(\frac{a+b+c}{2R}\right)^4}{4\left(\frac{abc}{8R^3}\right)^2}=$$

$$=\frac{16R}{r}-\frac{\left(\frac{2s}{2R}\right)^4}{4\left(\frac{4RF}{8R^3}\right)^2}=\frac{16R}{r}-\left(\frac{s}{R}\right)^4\cdot\frac{64R^6}{4\cdot 16R^2r^2s^2}=$$

$$=\frac{16R}{r}-\frac{s^4}{R^4}\cdot\frac{R^4}{r^2s^2}=\frac{16R}{r}-\frac{s^2}{r^2}\overset{GERRETSEN}{\leq}\frac{16R}{r}-\frac{16Rr-5r^2}{r^2}=$$

$$=\frac{16R}{r}-\frac{16R}{r}+5=5$$

Equality holds for $A=B=C$.

3917. *Let ΔDEF be the orthic triangle of ΔABC.*
I, I_a, I_b, I_c – incenter and excenters of ΔABC. Prove that:

$$\sum\frac{EF}{AI\cdot AI_a}\leq\frac{3R}{8\sqrt{3}r^2}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Georgia

$$Lemmas:\ EF=a\cdot\cos A,\ AI_a=\frac{r}{\sin\left(\frac{A}{2}\right)}\ and\ AI=\frac{r}{\sin\left(\frac{A}{2}\right)}$$

$$Schur\ \dot{I}nequality:\ a^n(a-b)(a-c)+b^n(b-a)(b-c)+c^n(c-a)(c-b)\geq 0$$

$$For\ n=3$$

$$a^5+b^5+c^5+abc(a^2+b^2+c^2)\geq a^4b+ab^4+b^4c+c^4b+c^4a+a^4c$$

$$a^4b+ab^4+b^4c+c^4b+c^4a+a^4c=ab(a^3+b^3)+bc(b^3+c^3)+ac(a^3+c^3)\geq$$
$$\geq a^2b^2(a+b)+b^2c^2(b+c)+a^2c^2(a+c)=$$
$$=a^3b^2+a^2b^3+b^3c^2+b^2c^3+c^3a^2+c^2a^3\Rightarrow$$

$$a^5+b^5+c^5+abc(a^2+b^2+c^2)\geq a^3b^2+a^2b^3+b^3c^2+c^3b^2+c^3a^2+a^3c^2$$

$$LHS=\sum\frac{EF}{AI\cdot AI_a}=\sum\frac{a\cos A}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}}=\sum\frac{a\cos A\cdot\sin^2\left(\frac{A}{2}\right)}{rr_a}=\sum\frac{a\cos A\cdot\sin^2\left(\frac{A}{2}\right)}{rs\tan\left(\frac{A}{2}\right)}=$$

$$=\sum\frac{a\cos A\sin A}{2rs}=\sum\frac{a^2\cos A}{4Rrs}=\frac{1}{4Rrs}\sum a^2\cos A=$$

$$=\frac{1}{4Rrs}\sum\frac{a^2b^2+a^2c^2-a^4}{2bc}=\frac{1}{4Rrs}\cdot\frac{\sum(a^3b^2+a^2b^3)-\sum a^5}{2abc}\le$$

$$\le\frac{1}{4Rrs}\cdot\frac{abc(a^2+b^2+c^2)}{2abc}=\frac{\sum a^2}{8Rrs}\le\frac{9R^2}{8Rrs}=\frac{9R}{8rs}\le\frac{9R}{24\sqrt{3}r^2}=\frac{3R}{8\sqrt{3}r^2}$$

Equality holds for $a=b=c$.

3918. *Let Ω be the Brocard point of ΔABC.*
O_a, O_b, O_c circumcenters of $\Delta B\Omega C, \Delta A\Omega C, \Delta A\Omega B$. Prove that:

$$\sum\frac{OO_a}{a\cdot AH}\ge\frac{2\sqrt{3}r}{R^2}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

Lemma: $AH=2R\cdot\cos A$

Let $OO_{a\cap BC}=M$, $BM=MC$. $OM=x$ *and* $O_aM=y$.

$$\cot A=\frac{2x}{a},\ \cot C=\frac{2y}{a}.\ OO_a=x+y=\frac{a(\cot A+\cot C)}{2}=$$

$$=\frac{a\sin(A+C)}{2\sin A\sin C}=\frac{a\sin B}{2\sin A\sin C}=\frac{a\cdot\frac{b}{2R}}{2\cdot\frac{a}{2R}\cdot\frac{c}{2R}}=\frac{b}{c}R$$

$$LHS=\sum\frac{OO_a}{a\cdot AH}=\sum\frac{b}{2ac\cdot\cos A}=\sum\frac{b^2}{2abc\cos A}\ge\frac{\sum a^2}{2abc\sum\cos A}\ge$$

$$\geq \frac{4s^2}{2abc\cdot\frac{3}{2}} = \frac{4s^2}{3\cdot 4Rrs} = \frac{s}{3Rr} \geq \frac{3\sqrt{3}r}{3R\cdot\frac{R}{2}} = \frac{2\sqrt{3}r}{R^2}$$

Equality holds for $a = b = c$.

3919. In any acute triangle ABC the following relationship holds :

$$bc\sqrt{\cot A} + ca\sqrt{\cot B} + ab\sqrt{\cot C} > \frac{8R^2}{3}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \sqrt{\cot A} = x, \sqrt{\cot B} = y, \sqrt{\cot C} = z \therefore \sum_{cyc} \sqrt{\cot A} \overset{?}{>} 2 \Leftrightarrow \sum_{cyc} x > 2$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)^2 > 4 = 4.\sqrt{\sum_{cyc} x^2y^2}\left(\because \sum_{cyc} \cot A\cot B = \sum_{cyc} x^2y^2 = 1\right)$$

$$\Leftrightarrow \left(\sum_{cyc} x\right)^4 \underset{(*)}{\overset{?}{>}} 16\sum_{cyc} x^2y^2$$

Assigning $y + z = X, z + x = Y, x + y = Z \Rightarrow X + Y - Z = 2z > 0, Y + Z - X = 2x > 0$ and $Z + X - Y = 2y > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y \Rightarrow X, Y, Z$ *form* sides of *a* triangle with semiperimeter, circumradius and inradius $= s, R, r$ (say)

$$\Rightarrow 2\sum_{cyc} x = \sum_{cyc} X = 2s \Rightarrow \sum_{cyc} x = s \to (i) \Rightarrow x = s - X, y = s - Y, z = s - Z \text{ and so}$$

$$\sum_{cyc} xy = \sum_{cyc} (s - X)(s - Y) \Rightarrow \sum_{cyc} xy = 4Rr + r^2 \to (ii) \text{ and } \sum_{cyc} x^2y^2 =$$

$$\left(\sum_{cyc} xy\right)^2 - 2xyz\left(\sum_{cyc} x\right) \overset{\text{via (i) and (ii)}}{=} (4Rr + r^2)^2 - 2\left(\prod_{cyc}(s - X)\right).s$$

$$= (4Rr + r^2)^2 - 2r^2s.s \Rightarrow \sum_{cyc} x^2y^2 = r^2((4R + r)^2 - 2s^2) \to (iii) \therefore \text{via (i) \& (iii)},$$

$$(*) \Leftrightarrow s^4 > 16r^2((4R + r)^2 - 2s^2) \Leftrightarrow s^4 + 32r^2s^2 \overset{(**)}{>} 16r^2(4R + r)^2$$

$$\text{Now, LHS of } (**) \overset{\text{Gerretsen}}{\geq} (16Rr + 27r^2)s^2 \overset{\text{Gerretsen}}{\geq} (16Rr + 27r^2)(16Rr - 5r^2)$$

$$\overset{?}{>} 16r^2(4R + r)^2 \Leftrightarrow 76r(R - 2r) + 148Rr + r^2 \overset{?}{>} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore \boxed{\sum_{cyc}\sqrt{\cot A} \overset{(\bullet)}{>} 2} \text{ and WLOG we may assume } a \ge b \ge c$$

$$\text{and then : } bc \le ca \le ab \text{ and } \sqrt{\cot A} \le \sqrt{\cot B} \le \sqrt{\cot C}$$

$$\therefore bc.\sqrt{\cot A} + ca.\sqrt{\cot B} + ab.\sqrt{\cot C} \overset{\text{Chebyshev}}{\ge} \frac{1}{3}\left(\sum_{cyc} bc\right)\left(\sum_{cyc}\sqrt{\cot A}\right)$$

$$\overset{\text{via }(\bullet)}{>} \frac{2(s^2+4Rr+r^2)}{3} > \frac{2(4R^2+4Rr+r^2)}{3}$$

$$\left(\begin{array}{c}\because \text{ABC is acute} \Rightarrow \prod_{cyc}\cos A > 0 \Rightarrow \frac{s^2-(2R+r)^2}{4R^2} > 0 \\ \Rightarrow s > 2R+r > 2R \Rightarrow s^2 > 4R^2\end{array}\right) > \frac{8R^2}{3} \text{ and so,}$$

$$bc.\sqrt{\cot A} + ca.\sqrt{\cot B} + ab.\sqrt{\cot C} > \frac{8R^2}{3}\ \forall \text{ acute } \Delta\ ABC\ (QED)$$

3920. In ΔABC the following relationship holds:

$$\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} \ge \frac{729r^3}{\left(m_a\sqrt{r_a}+m_b\sqrt{r_b}+m_c\sqrt{r_c}\right)^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} =$$

$$= \frac{m_ar_br_c + m_br_ar_c + m_cr_br_a}{r_ar_br_c} \ge \frac{729r^3}{\left(m_a\sqrt{r_a}+m_b\sqrt{r_b}+m_c\sqrt{r_c}\right)^2}$$

Let's prove that :

$$\frac{(m_ar_br_c + m_br_ar_c + m_cr_br_a)\left(m_a\sqrt{r_a}+m_b\sqrt{r_b}+m_c\sqrt{r_c}\right)^2}{r_ar_br_c} \ge 729r^3$$

$$\frac{(m_ar_br_c + m_br_ar_c + m_cr_br_a)\left(m_a\sqrt{r_a}+m_b\sqrt{r_b}+m_c\sqrt{r_c}\right)^2}{r_ar_br_c} \overset{AM-GM}{\ge}$$

$$\ge \frac{3\left((m_ar_br_c)(r_ar_br_c)^2\right)^{\frac{1}{3}}\left(3\left((m_ar_br_c)(r_ar_br_c)^{\frac{1}{2}}\right)^{\frac{1}{3}}\right)^2}{r_ar_br_c} =$$

$$= \frac{27\left((m_ar_br_c)(r_ar_br_c)\right)}{r_ar_br_c} = 27(m_ar_br_c) \ge 27h_ah_bh_c \ge 27.27r^3 = 729r^3$$

Equality holds for $a = b = c$.

3921. In ΔABC the following relationship holds:

$$\frac{1}{\sin^5\frac{A}{2}}+\frac{1}{\sin^5\frac{B}{2}}+\frac{1}{\sin^5\frac{C}{2}}\geq 96$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{1}{\sin^5\frac{A}{2}}+\frac{1}{\sin^5\frac{B}{2}}+\frac{1}{\sin^5\frac{C}{2}}=\frac{1^6}{\sin^5\frac{A}{2}}+\frac{1^6}{\sin^5\frac{B}{2}}+\frac{1^6}{\sin^5\frac{C}{2}}\overset{RADON}{\geq}$$

$$=\frac{(1+1+1)^6}{\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^5}\overset{JENSEN}{\geq}\frac{3^6}{\left(3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)^5}=$$

$$=\frac{3^6}{\left(3\sin\frac{A+B+C}{6}\right)^5}=\frac{3}{\sin^5\frac{\pi}{6}}=\frac{3}{\frac{1}{32}}=96$$

Equality holds for $A=B=C$.

3922. In ΔABC the following relationship holds:

$$\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}\geq\frac{1}{3\sqrt{3}R^5}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Daniel Sitaru-Romania

$$\frac{1}{a^5}+\frac{1}{b^5}+\frac{1}{c^5}=\frac{1^6}{a^5}+\frac{1^6}{b^5}+\frac{1^6}{c^5}\overset{RADON}{\geq}$$

$$\geq\frac{(1+1+1)^6}{(a+b+c)^5}=\frac{3^6}{(2s)^5}\overset{MITRINOVICI}{\geq}\frac{3^6}{\left(2\cdot\frac{3\sqrt{3}}{2}R\right)^5}=\frac{3^6}{3^5\cdot 9\sqrt{3}R^5}=\frac{3}{9\sqrt{3}R^5}=\frac{1}{3\sqrt{3}R^5}$$

Equality holds for $a=b=c$.

3923. ***Find:***

$$\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{9\pi}{7}\right)+\sin^4\left(\frac{10\pi}{7}\right)$$

Proposed by Deivy Garcia-Peru

Solution by Amin Hajiyev-Azerbaijan

$$S=\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{9\pi}{7}\right)+\sin^4\left(\frac{10\pi}{7}\right)=\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\pi+\frac{2\pi}{7}\right)+\sin^4\left(\pi+\frac{3\pi}{7}\right)$$

$$=\sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{2\pi}{7}\right)+\sin^4\left(\frac{3\pi}{7}\right)$$

$$\textit{De Moivre's formula}\ \cos(n\theta)+i\sin(n\theta)=(\cos(\theta)+i\sin(\theta))^n$$

$$\cos(7\theta)+i\sin(7\theta)=(\cos(\theta)+i\sin(\theta))^7$$

$$\sin(7\theta)=\Im\{(\cos(\theta)+i\sin(\theta))^7\}$$

$$\textit{Binomial theorem}\ \ (a+b)^n=\sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$

$$\sin(7\theta)=7\cos^6(\theta)\sin(\theta)-\frac{7}{3}\cos^4(\theta)\sin^3(\theta)+\frac{7}{5}\cos^2(\theta)\sin^5(\theta)-\sin^7(\theta)=$$

$$=7\sin(\theta)-56\sin^3(\theta)+112\sin^5(\theta)-64\sin^7(\theta)$$

$$\theta=\frac{\pi k}{7}\quad k=\{1,2,3,\dots,6\}\quad k\neq 7$$

$$\sin(7\theta)=0\to\ 7\sin(\theta)-56\sin^3(\theta)+112\sin^5(\theta)-64\sin^7(\theta)=0$$

$$7-56\sin^2(\theta)+112\sin^4(\theta)-64\sin^6(\theta)=0$$

$$\textit{Substitution}\ \sin^2(\theta)=x\quad x_1=\sin^2\left(\frac{\pi}{7}\right),x_2=\sin^2\left(\frac{2\pi}{7}\right),x_3=\sin^2\left(\frac{3\pi}{7}\right)$$

$$64x^3-112x^2+56x-7=0$$

$$\textit{Vieta's formulas}\Rightarrow\begin{cases}x_1+x_2+x_3=\dfrac{112}{64}=\dfrac{7}{4}\\ x_1x_2+x_2x_3+x_1x_3=\dfrac{56}{64}=\dfrac{7}{8}\end{cases}$$

$$S=x_1^2+x_2^2+x_3^2=(x_1+x_2+x_3)^2-2(x_1x_2+x_2x_3+x_1x_3)=$$

$$=\frac{49}{16}-2\cdot\frac{7}{8}=\frac{21}{16}$$

$$\textit{Therefore}\ \ \sin^4\left(\frac{\pi}{7}\right)+\sin^4\left(\frac{9\pi}{7}\right)+\sin^4\left(\frac{10\pi}{7}\right)=\frac{21}{16}$$

3924. **In any Δ ABC the following relationship holds :**

$$2\sqrt{3}.\sum_{cyc}\frac{h_a}{w_a+w_b+m_c-n_a.\sqrt{3}}\geq\frac{s}{r}+\sum_{cyc}\frac{n_a}{r_a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$2\sqrt{3}\sum_{cyc}\frac{h_a}{w_a+w_b+m_c-n_a.\sqrt{3}} \overset{Lessel-Pelling}{\geq} 2\sqrt{3}.\sum_{cyc}\frac{h_a}{s.\sqrt{3}-n_a.\sqrt{3}}$$

$$=\frac{2\sqrt{3}}{\sqrt{3}}.\sum_{cyc}\frac{h_a(n_a+s)}{s^2-n_a^2} \overset{Bogdan\ Fustei}{=} 2\sum_{cyc}\frac{h_a(s+n_a)}{2h_a r_a} = s\sum_{cyc}\frac{1}{r_a}+\sum_{cyc}\frac{n_a}{r_a}$$

$$=\frac{s}{r}+\sum_{cyc}\frac{n_a}{r_a} \therefore 2\sqrt{3}.\sum_{cyc}\frac{h_a}{w_a+w_b+m_c-n_a.\sqrt{3}} \geq \frac{s}{r}+\sum_{cyc}\frac{n_a}{r_a}\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3925. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{(n_a+s)(w_a+w_b+m_c-n_a.\sqrt{3})}{n_a} \leq 2r\sqrt{3}\sum_{cyc}\frac{h_a}{n_a-\sqrt{4r^2+(b-c)^2}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{(n_a+s)(w_a+w_b+m_c-n_a.\sqrt{3})}{n_a} \overset{Lessel-Pelling}{\leq}$$

$$\leq \sum_{cyc}\frac{(n_a+s)(s.\sqrt{3}-n_a.\sqrt{3})}{n_a} = \sum_{cyc}\frac{\sqrt{3}.(s^2-n_a^2)}{n_a} \overset{Bogdan\ Fustei}{=} \sum_{cyc}\frac{\sqrt{3}.(2h_a r_a)}{n_a}$$

$$\therefore \sum_{cyc}\frac{(n_a+s)(w_a+w_b+m_c-n_a.\sqrt{3})}{n_a} \leq 2\sqrt{3}.\sum_{cyc}\frac{h_a r_a}{n_a} \to (1) \text{ and}$$

$$2r.\sqrt{3}.\sum_{cyc}\frac{h_a}{n_a-\sqrt{4r^2+(b-c)^2}} \overset{Bogdan\ Fustei}{=} 2r.\sqrt{3}.\sum_{cyc}\frac{h_a}{n_a-\frac{an_a}{s}} =$$

$$= 2\sqrt{3}.\sum_{cyc}\left(\left(\frac{rs}{s-a}\right)\left(\frac{h_a}{n_a}\right)\right) =$$

$$= 2\sqrt{3}.\sum_{cyc}\frac{h_a r_a}{n_a} \overset{via\ (1)}{\geq} \sum_{cyc}\frac{(n_a+s)(w_a+w_b+m_c-n_a.\sqrt{3})}{n_a} \text{ and so}$$

$$\sum_{cyc}\frac{(n_a+s)(w_a+w_b+m_c-n_a.\sqrt{3})}{n_a} \leq 2r.\sqrt{3}.\sum_{cyc}\frac{h_a}{n_a-\sqrt{4r^2+(b-c)^2}}\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3926. In any Δ ABC holds :

$$\frac{|b-c|}{a} \le \min\left\{\frac{\sqrt{s^2-27r^2}}{s}, \sqrt{\frac{2(2s^2-27Rr)}{27r^2}}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Let s_0 = semiperimeter (for own convenience), $s = \sin\frac{A}{2}, c = \cos\frac{B-C}{2}$

Case 1 $c \ge 2s$ and now, $\frac{|b-c|}{a} \overset{?}{\le} \sqrt{\frac{2(2s_0^2-27Rr)}{27r^2}}$

$$\Leftrightarrow \frac{1-c^2}{1-s^2} \overset{?}{\le} \frac{4}{27} \cdot \frac{(1-s^2)\frac{(c+s)^2}{4}}{s^2\frac{(c-s)^2}{4}} - \frac{2R}{4Rs\left(\frac{c-s}{2}\right)}$$

$$\Leftrightarrow (27c^4s^2 - 54c^3s^3 + 31c^2s^4 + 8cs^5 + 4s^6) - (35c^2s^2 - 65cs^3 + 62s^4) + (4c^2 - 19cs + 31s^2) \underset{(*)}{\overset{?}{\ge}} 0 \Leftrightarrow s^6(27t^4 - 54t^3 + 31t^2 + 8t + 4) -$$

$$s^4(35t^2 - 65t + 62) + s^2(4t^2 - 19t + 31) \overset{?}{\ge} 0 \ \left(t = \frac{c}{s}\right)$$

$$\Leftrightarrow s^4(27t^2(t-1)^2 + 4(t+1)^2) - s^2(35t^2 - 65t + 62) + 4t^2 - 19t + 31 \underset{(**)}{\overset{?}{\ge}} 0$$

Now, LHS of $(**)$ is a quadratic polynomial in $''s^2''$ with discriminant $\delta =$
$(35t^2 - 65t + 62)^2 - 4(27t^4 - 54t^3 + 31t^2 + 8t + 4)(4t^2 - 19t + 31)$
$= -27(t-1)^2(t-2)^2\big((t-2)(16t+20)+9\big) \le 0 \ (\because t \ge 2) \Rightarrow (**) \Rightarrow (*)$ is true

Case 2(a) $c \le 2s$ and $s \le \frac{1}{2}$ and then, to prove $(**)$, it suffices to prove :

$2s^2(27t^4 - 54t^3 + 31t^2 + 8t + 4) \overset{?}{\le} 35t^2 - 65t + 62 - \sqrt{\delta}$ & $\because s^2 \le \frac{1}{4}$, it suffices

to prove : $2(35t^2 - 65t + 62) - (27t^4 - 54t^3 + 31t^2 + 8t + 4) \overset{?}{\ge} 2\sqrt{\delta}$

$$\Leftrightarrow 3(2-t)(9t^3 - 13t + 20) \underset{①}{\overset{?}{\ge}} 2\sqrt{\delta} \text{ and now, } 9t^3 - 13t + 20 =$$

$\frac{(45t+63)(10t-7)^2 + 115t + 6913}{500} > 0$ and $c \le 2s \Rightarrow 2 - t \ge 0$ and so, ① $\Leftrightarrow$

$$9(2-t)^2(9t^3 - 13t + 20)^2 \overset{?}{\ge} -108(t-1)^2(2-t)^2(16t^2 - 12t - 31)$$

$(\delta$ evaluated earlier$) \Leftrightarrow (9t^3 - 13t + 20)^2 + 12(t-1)^2(16t^2 - 12t - 31) \overset{?}{\ge} 0$

$$\Leftrightarrow \frac{((t-1)^2(1296t^2+3888t+6780)+6960t-5135)(2t-1)^2+14412t+147}{64} \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t = \frac{c}{s} = \frac{\frac{b+c}{a}\sin\frac{A}{2}}{\sin\frac{A}{2}} > 1 \Rightarrow (1) \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

Case 2(b) $c \leq 2s \wedge s \geq \frac{1}{2}$; Let : $c - s = u \Rightarrow c = s + u$ & then : $(*) \Leftrightarrow 27s^2u^4 + 54s^3u^3 + (31s^4 - 35s^2 + 4)u^2 + (16s^5 - 5s^3 - 11s)u + 16s^6 - 32s^4 + 16s^2 \underset{(\bullet)}{\overset{?}{\geq}} 0$

$\because 27s^2(u-1+s)^4 \geq 0 \therefore$ to prove $(\bullet)$, it suffices to prove : LHS of $(\bullet) \overset{?}{\geq} 27s^2(u-1+s)^4 \Leftrightarrow 54s^2(2-s)u^3 - (131s^4 - 324s^3 + 197s^2 - 4)u^2 - (92s^5 - 324s^4 + 329s^3 - 108s^2 + 11s)u + 108s^5 - 11s^6 - 194s^4 + 108s^3 - 11s^2 \underset{(\bullet\bullet)}{\overset{?}{\geq}} 0$; $\because 54s^2(2-s)u(u-1+s)^2 \geq 0 \therefore$ to prove $(\bullet\bullet)$, suffices to prove :

LHS of $(\bullet\bullet) \overset{?}{\geq} 54s^2(2-s)u(u-1+s)^2 \overset{\text{factoring out }(1-s)}{\Leftrightarrow} (1+s)(23s^2+4)u^2 + (38s^4 - 70s^3 - 11s^2 - 11s)u + 11s^5 - 97s^4 + 97s^3 - 11s^2 \underset{(\bullet\bullet\bullet)}{\overset{?}{\geq}} 0$

Discriminant δ_0 of LHS of $(\bullet\bullet\bullet) = (38s^4 - 70s^3 - 11s^2 - 11s)^2 - 4(1+s)(23s^2+4)(11s^5 - 97s^4 + 97s^3 - 11s^2)$

$= 27(4s^4 + 28s^3 + 63s^2 + 2s + 11)(2s-1)^2 \geq 0$ & $\because u > 0 \therefore$ to prove $(\bullet\bullet\bullet)$, it suffices to prove : $2(1+s)(23s^2+4)u \overset{?}{\leq} -(38s^4 - 70s^3 - 11s^2 - 11s) - \sqrt{\delta_0}$

& $\because u = c - s \leq 1 - s \therefore$ suffices to prove : $\sqrt{\delta_0} \overset{?}{\leq} -(38s^4 - 70s^3 - 11s^2 - 11s) - 2(1+s)(23s^2+4)(1-s) = (2s-1)(4s^3 + 37s^2 + 5s + 8) \Leftrightarrow$

$27(4s^4 + 28s^3 + 63s^2 + 2s + 11)(2s-1)^2 \overset{?}{\leq} (2s-1)^2(4s^3 + 37s^2 + 5s + 8)^2$

$\left(\because s \geq \frac{1}{2}\right) \Leftrightarrow 4(1-s)(s+4)(23s^2+4)(s+1)^2(2s-1)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \Rightarrow (\bullet\bullet\bullet)$true

$\therefore$ combining *all* cases, $\frac{|b-c|}{a} \leq \sqrt{\frac{2(2s_0^2 - 27Rr)}{27r^2}}$ & **finally**, $\frac{|b-c|}{a} \overset{?}{\leq} \frac{\sqrt{s_0^2 - 27r^2}}{s_0}$

$$\Leftrightarrow \frac{27s^2(c-s)^2}{(1-s^2)(c+s)^2} \overset{?}{\leq} 1 - \frac{1-c^2}{1-s^2} \overset{\text{factoring out }(c-s)}{\Leftrightarrow} (c+s)^3 \overset{?}{\geq} 27s^2(c-s)$$

$$\Leftrightarrow (c+7s)(c-2s)^2 \overset{?}{\geq} 0 \rightarrow \textbf{true}; \text{so}, \frac{|b-c|}{a} \leq \min\left\{\frac{\sqrt{s^2-27r^2}}{s}, \sqrt{\frac{2(2s^2-27Rr)}{27r^2}}\right\}$$

$\forall \Delta ABC$, where $s \equiv$ semiperimeter, " = " iff $\cos\frac{B-C}{2} = 2\sin\frac{A}{2}$ (QED)

3927. In any Δ ABC the following relationship holds :

$$w_a \le \frac{3\sqrt{3}(b^2+c^2+6bc)}{32s}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{3\sqrt{3}.(b^2+c^2+6bc)}{32s} = \frac{3\sqrt{3}.((b+c)^2+4bc)}{32s} \overset{AM-GM}{\ge} \frac{12\sqrt{3}.(b+c).\sqrt{bc}}{32s}$$

$$\overset{?}{\ge} \frac{2\sqrt{bc}}{b+c}.\sqrt{s(s-a)} \Leftrightarrow 27(b+c)^4 \overset{?}{\ge} 256s^3(s-a) \Leftrightarrow 27(2s-a)^4 \overset{?}{\ge} 256s^3(s-a)$$

$$\Leftrightarrow 176t^4 - 608t^3 + 648t^2 - 216t + 27 \overset{?}{\ge} 0 \ \left(t = \frac{s}{a}\right)$$

$$\Leftrightarrow (2t-3)^2\big((44t+24)(t-1)+27\big) \overset{?}{\ge} 0 \to \text{true} \because t > 1$$

$$\therefore \frac{3\sqrt{3}.(b^2+c^2+6bc)}{32s} \ge \frac{2\sqrt{bc}}{b+c}.\sqrt{s(s-a)} = w_a \Rightarrow w_a \le \frac{3\sqrt{3}.(b^2+c^2+6bc)}{32s}$$

$\forall\, \Delta$ ABC, " = " iff b = c *and* b + c = 2*a* ⇒ " = " iff Δ ABC is equilateral (QED)

3928. In ΔABC the following relationship holds:

$$9 \le \sum \cot^2\frac{A}{2} \le \left(\frac{2R}{r} - 1\right)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \cot^2\frac{A}{2} = \sum \frac{s^2}{r_a^2} = s^2\left(\left(\sum\frac{1}{r_a}\right)^2 - 2\sum\frac{1}{r_ar_b}\right) =$$

$$= s^2\left(\frac{1}{r^2} - \frac{2(4R+r)}{s^2r}\right) = \frac{s^2-2r^2-8Rr}{r^2}$$

$$\sum \cot^2\frac{A}{2} = \frac{s^2-2r^2-8Rr}{r^2} \overset{Gerretsen}{\le} \frac{(4R^2+4Rr+3r^2)-2r^2-8Rr}{r^2} =$$

$$= \frac{(2R-1)^2}{r^2} = \left(\frac{2R}{r} - 1\right)^2$$

$$\sum \cot^2\frac{A}{2} = \frac{s^2 - 2r^2 - 8Rr}{r^2} \overset{Gerretsen}{\geq} \frac{(16Rr - 5r^2) - 2r^2 - 8Rr}{r^2} =$$

$$= \frac{8Rr - 7r^2}{r^2} = \frac{8R}{r} - 7 \overset{Euler}{\geq} 8 \times 2 - 7 = 9$$

Equality holds for an equilateral triangle.

3929.

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\left(\frac{\cos A}{1+\sin A}\right)^n \geq 3\left(2-\sqrt{3}\right)^n, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cos A}{1+\sin A} = \sum_{cyc}\frac{\cos^2\frac{A}{2} - \sin^2\frac{A}{2}}{\cos^2\frac{A}{2} + \sin^2\frac{A}{2} + 2\cos\frac{A}{2}.\sin\frac{A}{2}}$$

$$= \sum_{cyc}\frac{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)\left(\cos\frac{A}{2}-\sin\frac{A}{2}\right)}{\left(\cos\frac{A}{2}+\sin\frac{A}{2}\right)^2} = \sum_{cyc}\frac{\cos\frac{A}{2}-\sin\frac{A}{2}}{\cos\frac{A}{2}+\sin\frac{A}{2}} = \sum_{cyc}\frac{1-\tan\frac{A}{2}}{1+\tan\frac{A}{2}}$$

$$= \sum_{cyc}\frac{s-r_a}{s+r_a} = \sum_{cyc}\frac{-s-r_a+2s}{s+r_a} = -3 + \sum_{cyc}\frac{2s(s+r_b)(s+r_c)}{(s+r_a)(s+r_b)(s+r_c)}$$

$$= -3 + \sum_{cyc}\frac{2s(s^2+s(r_b+r_c)+r_br_c)}{s^3+s^2(4R+r)+s.s^2+rs^2} = -3 + \frac{2s(3s^2+2s(4R+r)+s^2)}{2s^3+s^2(4R+r)+rs^2}$$

$$= -3 + \frac{4s(2s^2+s(4R+r)+rs-rs)}{s(2s^2+s(4R+r)+rs)} = 1 - \frac{4r}{2s+4R+2r} = 1 - \frac{2r}{s+2R+r}$$

$$\overset{?}{\geq} 3\left(2-\sqrt{3}\right) \Rightarrow 3\sqrt{3}-5 \overset{?}{\geq} \frac{2r}{s+2R+r} \Leftrightarrow \left(3\sqrt{3}-5\right)(s+2R+r) \underset{(*)}{\overset{?}{\geq}} 2r$$

$$\text{Now, since } s \overset{Mitrinovic}{\geq} 3\sqrt{3}r \text{ and } R \overset{Euler}{\geq} 2r \text{ and } \because 3\sqrt{3}-5 > 0$$

$$\therefore \text{LHS of } (*) \geq \left(3\sqrt{3}-5\right)\left(3\sqrt{3}+5\right)r = (27-25)r = 2r \Rightarrow (*) \text{ is true}$$

$$\therefore \sum_{cyc}\frac{\cos A}{1+\sin A} \geq 3\left(2-\sqrt{3}\right) \text{ and so, } \sum_{cyc}\left(\frac{\cos A}{1+\sin A}\right)^n \overset{Holder}{\geq} \frac{1}{3^{n-1}}.\left(\sum_{cyc}\frac{\cos A}{1+\sin A}\right)^n$$

$$\geq \frac{1}{3^{n-1}}.\left(3\left(2-\sqrt{3}\right)\right)^n = 3\left(2-\sqrt{3}\right)^n \therefore \sum_{cyc}\left(\frac{\cos A}{1+\sin A}\right)^n \geq 3\left(2-\sqrt{3}\right)^n$$

$\forall$ n $\in \mathbb{N}$ and $\forall$ Δ ABC, " = " iff Δ ABC is equilateral (QED)

3930. In ΔABC the following relationship holds:

$$9\sum \tan^2\frac{A}{2}\le \frac{R}{2r}\sum \cot^2\frac{A}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \cot^2\frac{A}{2}=\sum\frac{s^2}{r_a^2}=s^2\left(\left(\sum\frac{1}{r_a}\right)^2-2\sum\frac{1}{r_ar_b}\right)=$$

$$=s^2\left(\frac{1}{r^2}-\frac{2(4R+r)}{s^2r}\right)=\frac{s^2-2r^2-8Rr}{r^2}$$

$$\sum \cot^2\frac{A}{2}=\frac{s^2-2r^2-8Rr}{r^2}\overset{Gerretsen}{\ge}\frac{(16Rr-5r^2)-2r^2-8Rr}{r^2}=$$

$$=\frac{8Rr-7r^2}{r^2}=\frac{8R}{r}-7$$

$$\sum \tan^2\frac{A}{2}=\frac{(4R+r)^2}{s^2}-2\overset{Doucet}{\le}\frac{(4R+r)^2}{3r(4R+r)}-2=\frac{4R-5r}{3r}$$

$$We\ need\ to\ show:$$

$$9\sum \tan^2\frac{A}{2}\le\frac{R}{2r}\sum cot^2\frac{A}{2}\ or,9\times\frac{4R-5r}{3r}\le\left(\frac{8R}{r}-7\right)\frac{R}{2r}$$

$$\frac{R}{r}=x\ge 2$$

$$12x-15\overset{Euler}{\le}(8x-7)\times\frac{x}{2}\ or\ 8x^2-7x\ge 24x-30$$

$$8x^2-31x+30\ge 0\ or\ (x-2)(8x-15)\ge 0\ true\ as\ x\ge 2$$

Equality holds for an equilateral triangle.

3931. In any Δ ABC the following relationship holds:

$$\sum_{cyc}\frac{h_b\cdot\sqrt{h_c}}{h_a^2}\ge\sqrt{\frac{3}{r}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{h_b\cdot\sqrt{h_c}}{h_a^2}=\sum_{cyc}\frac{\left(\frac{1}{h_a}\right)^2}{\frac{1}{h_b}\cdot\sqrt{\frac{1}{h_c}}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\frac{1}{h_a}\right)^2}{\sum_{cyc}\left(\frac{1}{h_b}\cdot\sqrt{\frac{1}{h_c}}\right)}\overset{\text{CBS}}{\geq}\frac{\frac{1}{r^2}}{\sqrt{\sum_{cyc}\frac{1}{h_bh_c}}\cdot\sqrt{\sum_{cyc}\frac{1}{h_b}}}$$

$$=\frac{\sqrt{r}}{r^2\cdot\sqrt{\sum_{cyc}h_a}}\cdot\sqrt{\frac{2r^2s^2}{R}}=\frac{\sqrt{r}}{r^2\cdot\sqrt{\frac{\sum_{cyc}ab}{2R}}}\cdot\sqrt{\frac{2r^2s^2}{R}}\geq\frac{\sqrt{3r}}{r^2\cdot\sqrt{\frac{\left(\sum_{cyc}a\right)^2}{2R}}}\cdot\sqrt{\frac{2r^2s^2}{R}}$$

$$=\frac{\sqrt{3r}}{r^2\cdot\sqrt{\frac{4s^2}{2R}}}\cdot\sqrt{\frac{2r^2s^2}{R}}=\sqrt{\frac{3}{r}}\therefore\sum_{cyc}\frac{h_b\cdot\sqrt{h_c}}{h_a^2}\geq\sqrt{\frac{3}{r}}\ \forall\ \Delta\ ABC,$$

$$''=''\ \text{iff}\ \Delta\ ABC\text{ is equilateral (QED)}$$

3932. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{r_b\cdot\sqrt{r_c}}{r_a^2}\geq\sqrt{\frac{3}{r}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{r_b\cdot\sqrt{r_c}}{r_a^2}=\sum_{cyc}\frac{\left(\frac{1}{r_a}\right)^2}{\frac{1}{r_b}\cdot\sqrt{\frac{1}{r_c}}}\overset{\text{Bergstrom}}{\geq}\frac{\left(\sum_{cyc}\frac{1}{r_a}\right)^2}{\sum_{cyc}\left(\frac{1}{r_b}\cdot\sqrt{\frac{1}{r_c}}\right)}\overset{\text{CBS}}{\geq}\frac{\frac{1}{r^2}}{\sqrt{\sum_{cyc}\frac{1}{r_br_c}}\cdot\sqrt{\sum_{cyc}\frac{1}{r_b}}}$$

$$=\frac{\sqrt{r}}{r^2\cdot\sqrt{\sum_{cyc}r_a}}\cdot\sqrt{rs^2}=\frac{s}{r.\sqrt{4R+r}}\geq\frac{\sqrt{3r(4R+r)}}{r.\sqrt{4R+r}}$$

$$\left(\because s^2\overset{\text{Gerretsen}}{\geq}12Rr+3r^2+4r(R-2r)\overset{\text{Euler}}{\geq}12Rr+3r^2\right)=\sqrt{\frac{3}{r}}$$

$$\therefore\sum_{cyc}\frac{r_b\cdot\sqrt{r_c}}{r_a^2}\geq\sqrt{\frac{3}{r}}\ \forall\ \Delta\ ABC,''=''\ \text{iff}\ \Delta\ ABC\text{ is equilateral (QED)}$$

3933. In ΔABC the following relationship holds:

$$1 \le \sum \tan^2\frac{A}{2} \le \frac{R}{r} - 1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum \tan^2\frac{A}{2} = \left(\frac{4R+r}{s}\right)^2 - 2$$

$$We\ need\ to\ show:$$

$$\left(\frac{4R+r}{s}\right)^2 - 2 \le \frac{R}{r} - 1\ or\ \left(\frac{4R+r}{s}\right)^2 \le \frac{R}{r} + 1$$

$$(4R+r)^2 \le s^2\left(\frac{R}{r}+1\right)\ or\ (4R+r)^2 \overset{Gerretsen}{\le} (16Rr - 5r^2)\left(\frac{R+r}{r}\right)$$

$$16R^2 + 8Rr + r^2 \le 16R^2 + 11Rr - 5r^2\ or\ 3Rr \ge 6r^2\ or\ R \ge 2r\ true\ by\ Euler$$

$$\sum \tan^2\frac{A}{2} = \left(\frac{4R+r}{s}\right)^2 - 2 \overset{Doucèt}{\ge} 3 - 2 = 1$$

Equality holds for an equilateral triangle.

3934.

In any Δ ABC the following relationship holds :

$$\frac{a^3}{h_a^3} + \frac{b^3}{h_b^3} + \frac{c^3}{h_c^3} \le \frac{a^3}{r_a^3} + \frac{b^3}{r_b^3} + \frac{c^3}{r_c^3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^3}{h_a^3} \overset{?}{\ge} \sum_{cyc}\frac{a^3}{r_a^3} \Leftrightarrow \sum_{cyc}\frac{a^6}{8r^3s^3} \overset{?}{\ge} \sum_{cyc}\frac{a^3(s-a)^3}{r^3s^3}$$

$$\Leftrightarrow \sum_{cyc} a^6 \overset{?}{\ge} 8\sum_{cyc}\left(a^3(s^3 - 3s^2a + 3sa^2 - a^3)\right)$$

$$\Leftrightarrow 9\left(\left(\sum_{cyc} a^3\right)^2 - 2\sum_{cyc} a^3b^3\right) \overset{?}{\ge} 8s^3\left(\sum_{cyc} a^3\right) - 24s^2\left(2\sum_{cyc} a^2b^2 - 16r^2s^2\right) +$$

$$24s\left(\left(\sum_{cyc}a^2\right)\left(\sum_{cyc}a^3\right)-2s\sum_{cyc}a^2b^2+abc\sum_{cyc}ab\right)$$

$$\Leftrightarrow 9\left(4s^2(s^2-6Rr-3r^2)^2-2\left((s^2+4Rr+r^2)^3-24Rrs^2(s^2+2Rr+r^2)\right)\right)\overset{?}{\ge}$$

$$16s^4(s^2-6Rr-3r^2)-48s^2((s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2)+$$

$$24s\begin{pmatrix}4s(s^2-4Rr-r^2)(s^2-6Rr-3r^2)-2s((s^2+4Rr+r^2)^2-16Rrs^2)\\+8Rrs(s^2+4Rr+r^2)\end{pmatrix}$$

$$\Leftrightarrow s^6-(12Rr+15r^2)s^4-r^2(72R^2+120Rr+39r^2)s^2-9r^3(4R+r)^3 \boxed{\underset{①}{\overset{?}{\ge}}} 0 \text{ and}$$

$$\because (s^2-16Rr+5r^2)^3 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ in order to prove ①, it suffices to prove :}$$

$$\text{LHS of ①} \overset{?}{\ge} (s^2-16Rr+5r^2)^3 \Leftrightarrow (18R-15r)s^4-$$

$$r(348R^2-300Rr+18r^2)s^2-r^3(2136R^2-546Rr+67r^2) \boxed{\underset{②}{\overset{?}{\ge}}} 0$$

$$\text{Again, since } (18R-15r)(s^2-16Rr+5r^2)^2 \overset{\text{Gerretsen}}{\ge} 0 \therefore \text{ in order to prove ②,}$$

$$\text{it suffices to prove : LHS of ②} \overset{?}{\ge} (18R-15r)(s^2-16Rr+5r^2)^2$$

$$\Leftrightarrow (57R^2-90Rr+33r^2)s^2 \boxed{\underset{③}{\overset{?}{\ge}}} r(712R^3-1146R^2r+576Rr^2-77r^3)$$

$$\text{Finally, } (57R^2-90Rr+33r^2)s^2 \overset{\text{Gerretsen}}{\ge} (57R^2-90Rr+33r^2)(16Rr-5r^2)$$

$$\overset{?}{\ge} r(712R^3-1146R^2r+576Rr^2-77r^3) \Leftrightarrow 200t^3-579t^2+402t-88 \overset{?}{\ge} 0$$

$$\left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(200t^2-179t+44) \overset{?}{\ge} 0 \to \text{true} \because t \overset{\text{Euler}}{\ge} 2$$

$$\Rightarrow ③ \Rightarrow ② \Rightarrow ① \text{ is true } \forall \Delta ABC \therefore \frac{a^3}{h_a^3}+\frac{b^3}{h_b^3}+\frac{c^3}{h_c^3} \le \frac{a^3}{r_a^3}+\frac{b^3}{r_b^3}+\frac{c^3}{r_c^3}\ \forall \Delta ABC,$$

$$''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3935. In any Δ ABC the following relationship holds :

$$\frac{a^2}{h_a^3}+\frac{b^2}{h_b^3}+\frac{c^2}{h_c^3} \le \frac{a^2}{r_a^3}+\frac{b^2}{r_b^3}+\frac{c^2}{r_c^3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{h_a^3} \overset{?}{\ge} \sum_{cyc}\frac{a^2}{r_a^3} \Leftrightarrow \sum_{cyc}\frac{a^5}{8r^3s^3} \overset{?}{\ge} \sum_{cyc}\frac{a^2(s-a)^3}{r^3s^3}$$

$$\Leftrightarrow \sum_{cyc} a^5 \overset{?}{\geq} 8\sum_{cyc}\left(a^2(s^3 - 3s^2a + 3sa^2 - a^3)\right)$$

$$\Leftrightarrow 9\left(\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a^3\right) - 2s\sum_{cyc} a^2b^2 + abc\sum_{cyc} ab\right) \overset{?}{\geq}$$

$$8s^3\left(\sum_{cyc} a^2\right) - 24s^2\left(\sum_{cyc} a^3\right) + 24s\left(2\sum_{cyc} a^2b^2 - 16r^2s^2\right)$$

$$\Leftrightarrow 9\left(\begin{matrix}4s(s^2 - 4Rr - r^2)(s^2 - 6Rr - 3r^2) - 2s((s^2 + 4Rr + r^2)^2 - 16Rrs^2)\\ +8Rrs(s^2 + 4Rr + r^2)\end{matrix}\right) \overset{?}{\geq}$$

$$16s^3(s^2 - 4Rr - r^2) - 48s^3(s^2 - 6Rr - 3r^2) + $$
$$48s((s^2 + 4Rr + r^2)^2 - 16Rrs^2 - 8r^2s^2)$$

$$\Leftrightarrow s^4 - (10Rr + 10r^2)s^2 - r^2(24R^2 - 78Rr - 21r^2) \boxed{\underset{①}{\overset{?}{\geq}}} 0 \text{ and } \because$$

$$(s^2 - 16Rr + 5r^2)^2 \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{ in order to prove } ①, \text{it suffices to prove :}$$

$$\text{LHS of } ① \overset{?}{\geq} (s^2 - 16Rr + 5r^2)^2 \Leftrightarrow (11R - 10r)s^2 \boxed{\underset{②}{\overset{?}{\geq}}} r(140R^2 - 119Rr + 2r^2)$$

$$\text{Again, } (11R - 10r)s^2 \overset{\text{Gerretsen}}{\geq} (11R - 10r)(16Rr - 5r^2) \overset{?}{\geq}$$
$$r(140R^2 - 119Rr + 2r^2) \Leftrightarrow 12r(R - 2r)(3R - 2r) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow ② \Rightarrow ① \text{ is true } \forall\, \Delta\, ABC \therefore \frac{a^2}{h_a^3} + \frac{b^2}{h_b^3} + \frac{c^2}{h_c^3} \leq \frac{a^2}{r_a^3} + \frac{b^2}{r_b^3} + \frac{c^2}{r_c^3}\ \forall\, \Delta\, ABC,$$
$$'' = ''\text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3936. ***If in ΔABC, $A = \frac{\pi}{7}, B = \frac{2\pi}{7}, C = \frac{4\pi}{7}$ then prove that:***

$$R^4\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right) > \frac{27}{49}$$

Proposed by Tapas Das-India

Solution by Daniel Sitaru-Romania

$$R^4\left(\frac{1}{a^4} + \frac{1}{b^4} + \frac{1}{c^4}\right) = R^4\left(\frac{\left(\frac{1}{a^2}\right)^2}{1} + \frac{\left(\frac{1}{b^2}\right)^2}{1} + \frac{\left(\frac{1}{c^2}\right)^2}{1}\right) \overset{RADON}{\geq}$$

$$\geq R^4 \cdot \frac{\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)^2}{1 + 1 + 1} = R^4 \cdot \frac{\left(\frac{2}{R^2}\right)^2}{3} = \frac{4}{3} > \frac{27}{49}$$

THE HEPTAGONAL TRIANGLE REVISITED

DANIEL SITARU, CLAUDIA NĂNUŢI - ROMANIA

ABSTRACT. In this paper are proved the characteristic metric relationships in the heptagonal triangle.

We call heptagonal triangle the obtuse scalene triangle whose vertices coincide with the first, second and fourth vertices of a regular heptagon (from an arbitrary starting vertex).
Thus its sides coincide with one side and the adjacent shorter and longer diagonals of the regular heptagon.

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In conclusion, in a heptagonal triangle with sides $a < b < c; \mu(A) = \frac{\pi}{7};$ $\mu(B) = \frac{2\pi}{7}; \mu(C) = \frac{4\pi}{7}$ the following relationship holds:

$$a^2 = c^2 - bc; b^2 = a^2 + ac; c^2 = b^2 + ba$$

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c};$$

$$h_a = h_b + h_c; h_a^2 + h_b^2 + h_c^2 = \frac{a^2 + b^2 + c^2}{2}$$

$$F = \frac{\sqrt{7}}{4} R^2;$$

$$a^2 + b^2 + c^2 = 7R^2; \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{2}{R^2}$$

□

REFERENCES

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3937. In any Δ ABC the following relationship holds :

$$\frac{R}{r} \sum_{cyc} \frac{1 + \tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)^2} \geq 5$$

Proposed by Tapas Das-India

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1+\tan^2\frac{B}{2}\tan^2\frac{C}{2}}{\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)^2}=\sum_{cyc}\frac{s^4+s^2(s-a)^2}{s^2\left(4R\cos^2\frac{A}{2}\right)^2}=\sum_{cyc}\frac{s^2+(s-a)^2}{16R^2.\frac{a^2s^2(s-a)^2}{16R^2r^2s^2}}$$

$$=r^2\sum_{cyc}\frac{s^2+(s-a)^2}{a^2(s-a)^2}=r^2\left(\sum_{cyc}\frac{(s-a)^2+a^2+2a(s-a)}{a^2(s-a)^2}+\sum_{cyc}\frac{1}{a^2}\right)$$

$$=r^2\left(\frac{2}{16R^2r^2s^2}\sum_{cyc}a^2b^2+\frac{1}{r^2s^2}\sum_{cyc}r_a^2+\frac{2}{4Rr}\sum_{cyc}\frac{bc}{s(s-a)}\right)$$

$$=r^2\left(\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{8R^2r^2s^2}+\frac{(4R+r)^2-2s^2}{r^2s^2}+\frac{2}{4Rr}.\frac{s^2+(4R+r)^2}{s^2}\right)$$

$$\left(\because\sum_{cyc}\frac{bc}{s(s-a)}=\sum_{cyc}\sec^2\frac{A}{2}\right)\therefore\frac{R}{r}\sum_{cyc}\frac{1+\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2}$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2+8R^2((4R+r)^2-2s^2)+4Rr(s^2+(4R+r)^2)}{8Rrs^2}\overset{?}{\geq}5$$

$$\Leftrightarrow s^4-(16R^2+44Rr-2r^2)s^2+128R^4+128R^3r+56R^2r^2+12Rr^3+r^4\underset{(*)}{\overset{?}{\geq}}0$$

$$\text{Now, since } P=s^2(s^2-16Rr+5r^2)\overset{\text{Gerretsen}}{\geq}0\therefore\text{ in order to prove }(*),$$

$$\text{it suffices to prove : LHS of }(*)\overset{?}{\geq}P$$

$$\Leftrightarrow 128R^4+128R^3r+56R^2r^2+12Rr^3+r^4\underset{(**)}{\overset{?}{\geq}}(16R^2+28Rr+3r^2)s^2$$

$$\text{Finally, }(16R^2+28Rr+3r^2)s^2\overset{\text{Gerretsen}}{\leq}(16R^2+28Rr+3r^2)(4R^2+4Rr+3r^2)$$

$$\overset{?}{\leq}\text{LHS of }(**)\Leftrightarrow 16t^4-12t^3-29t^2-21t-2\overset{?}{\geq}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(16t^3+20t^2+11t+1)\overset{?}{\geq}0\to\text{true}\because t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{R}{r}\sum_{cyc}\frac{1+\tan^2\frac{A}{2}\tan^2\frac{B}{2}}{\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)^2}\geq 5\ \forall\ \Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

3938. In any Δ ABC the following relationship holds :

$$s^2\geq 27r^2\cdot\frac{b^2-bc+c^2}{bc}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$s^2 \overset{?}{\geq} 27r^2.\frac{b^2-bc+c^2}{bc} \Leftrightarrow bc.s^3 \overset{?}{\geq} 27(s-a)(s-b)(s-c)(b^2-bc+c^2)$$

$$\Leftrightarrow (z+x)(x+y)(x+y+z)^3 \overset{?}{\geq} 27xyz((z+x)^2-(z+x)(x+y)+(x+y)^2)$$

$$\left(\begin{matrix} x=s-a, y=s-b, z=s-c \Rightarrow a=y+z, b=z+x, c=x+y \\ and\ s=x+y+z \end{matrix}\right)$$

$$\Leftrightarrow x^5+4(y+z)x^4+6x^3(y+z)^2-26x^3yz+4x^2\left((y+z)^3-3yz(y+z)\right)-$$

$$12x^2yz(y+z)+x((y+z)^2-2yz)^2+77xy^2z^2-20xyz(y+z)^2+yz(y+z)^3 \overset{?}{\geq} 0$$

$$\Leftrightarrow x^5+4mx.x^4+6x^3.m^2x^2-26x^3.nx^2+4x^2(m^3x^3-3nx^2.mx)-$$

$$12x^2.nx^2.mx+x(m^2x^2-2nx^2)^2+77x.n^2x^4-20x.nx^2.m^2x^2+nx^2.m^3x^3$$

$$\left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right) \Leftrightarrow 81n^2+(m^3-24m^2-24m-26)n+(m+1)^4 \overset{?}{\underset{(*)}{\geq}} 0$$

$and\ it's\ trivially\ true\ when: m^3-24m^2-24m-26>0\ and\ so,$
we now focus on the case when $: m^3-24m^2-24m-26 \leq 0$, that is,
$$when: m \leq m_0 \,\big|\, m_0^3-24m_0^2-24m_0-26=0$$
$(\because m^3-24m^2-24m-26$ has only one zero $: m_0 \approx 25)$

Now, LHS of $(*)$ is a quadratic expression in "n" with discriminant $\delta =$
$$(m^3-24m^2-24m-26)^2-324(m+1)^4 \boxed{=}$$
$$(m-2)^2(m-2)(m^3-24m^2-24m-26-18(m+1)^2)\ and \because$$
$$m^3-24m^2-24m-26 \leq 0 \therefore \delta \leq 0 \text{ if } m \geq 2\ (and\ m \leq m_0)$$
and in that case, LHS of $(*) \geq 0 \Rightarrow (*)$ is true and if $m<2$, then $: \delta > 0$
and then, in order to prove $(*)$, it suffices to prove :

$$162n \overset{?}{\leq} -(m^3-24m^2-24m-26)-\sqrt{\delta}\ and \because \frac{(y+z)^2}{x^2} \overset{AM-GM}{\geq} \frac{4yz}{x^2}$$

$\therefore m^2 \geq 4n$ and so, it suffices to prove :

$$162.\frac{m^2}{4} \overset{?}{\leq} -(m^3-24m^2-24m-26)-\sqrt{\delta}$$

$$\Leftrightarrow 2\sqrt{(m-2)^2(m-2)(m^3-24m^2-24m-26-18(m+1)^2)} \overset{?}{\leq}$$
$$(2-m)(2m^2+37m+26)$$

$$\Leftrightarrow (m-2)^2(2m^2+37m+26)^2 \overset{?}{\geq}$$
$$4(m-2)^2(m-2)(m^3-24m^2-24m-26-18(m+1)^2)\ (\because 2-m>0)$$

$$\Leftrightarrow 81(4m+1)(m^2-4)^2 \overset{?}{\geq} 0 \rightarrow true \Rightarrow (*) \text{ is true} \therefore s^2 \geq 27r^2.\frac{b^2-bc+c^2}{bc}$$

$\forall \Delta ABC$, " = " iff $y=z$ and $y+z=2x \Rightarrow$ " = " iff ΔABC is equilateral (QED)

3939. In any Δ ABC the following relationship holds :

$$m_a \le \frac{\sqrt{s^2-11r^2}}{4r}\cdot h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a \overset{?}{\le} \frac{\sqrt{s^2-11r^2}}{4r}.h_a \Leftrightarrow \frac{a(2m_a)}{4rs} \overset{?}{\le} \frac{\sqrt{s^2-11r^2}}{4r}$$

$$\Leftrightarrow a^2(2b^2+2c^2-a^2) \overset{?}{\le} s^2(s^2-11r^2)$$

$$\Leftrightarrow 16a^2(2b^2+2c^2-a^2) \overset{?}{\le} (a+b+c)^4 - 11\left(2\sum_{cyc} a^2b^2 - \sum_{cyc} a^4\right)$$

$$\Leftrightarrow 7a^4+3b^4+3c^4+\sum_{cyc} a^3b + \sum_{cyc} ab^3 - 12a^2(b^2+c^2) - 4b^2c^2 +$$

$$3abc\sum_{cyc} a \overset{?}{\ge} 0 \Leftrightarrow 7(y+z)^4+3(z+x)^4+3(x+y)^4+\sum_{cyc}(y+z)^3(z+x)+$$

$$\sum_{cyc}(y+z)(z+x)^3 - 12(y+z)^2((z+x)^2+(x+y)^2) - 4(z+x)^2(x+y)^2 +$$

$$6(y+z)(z+x)(x+y)(x+y+z) \overset{?}{\ge} 0 \left(\begin{gathered}\text{substituting :}\\ a=y+z, b=z+x, c=x+y\end{gathered}\right)$$

$$\Leftrightarrow x^4+4x^3(y+z)+2x^2(y+z)^2-11x^2yz-11xyz(y+z)+4yz(y+z)^2 \overset{?}{\ge} 0$$

$$\Leftrightarrow x^4+4x^3.mx+2x^2.m^2x^2-11x^2.nx^2-11x.nx^2.mx+4nx^2.m^2x^2 \overset{?}{\ge} 0$$

$$\left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right) \Leftrightarrow (4n+2)m^2-(11n-4)m+1-11n \underset{(*)}{\overset{?}{\ge}} 0$$

Now, LHS of $(*)$ is a quadratic expression in "m" with discriminant $\delta =$ $(11n-4)^2-4(4n+2)(1-11n) \boxed{=} 297n^2-16n+8>0$ ($\because \delta_0 = 256n^2-32(297n^2)<0$) & so, in order to prove $(*)$, it suffices to prove :

$$2(4n+2)m \overset{?}{\ge} 11n-4+\sqrt{\delta} \text{ and } \because \frac{(y+z)^2}{x^2} \overset{AM-GM}{\ge} \frac{4yz}{x^2} \therefore m^2 \ge 4n$$

$$\Rightarrow m \ge 2\sqrt{n} \text{ and so, it suffices to prove : } 2(4n+2)(2\sqrt{n}) \boxed{\overset{?}{\ge}} 11n-4+\sqrt{\delta}$$

$$\Leftrightarrow 4t(4t^2+2)+4-11t^2 \overset{?}{\ge} \sqrt{297t^4-16t^2+8}\ (\sqrt{n}=t)$$

$$\Leftrightarrow 16t^3-11t^2+8t+4 \overset{?}{\ge} \sqrt{297t^4-16t^2+8}$$

$$\Leftrightarrow (16t^3-11t^2+8t+4)^2 \overset{?}{\ge} 297t^4-16t^2+8$$

$$\left(\begin{gathered}\because \text{discriminant of } 16t^2-11t+8 = 121-512<0 \Rightarrow 16t^3-11t^2+8t>0\\ \Rightarrow 16t^3-11t^2+8t+4>0\end{gathered}\right)$$

$$\Leftrightarrow 8(2t+1)(8t+1)(2t^2+1)(t-1)^2 \boxed{\overset{?}{\ge}} 0 \to \text{true} \Rightarrow (*) \text{ is true}$$

$$\therefore m_a \le \frac{\sqrt{s^2-11r^2}}{4r}.h_a\ \forall\ \Delta\ ABC,\text{''}=\text{''}\ \text{iff } y=z \text{ and } yz=x^2$$
$$\Rightarrow \text{''}=\text{''}\ \text{iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3940.

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\frac{r}{h_a^2}+\frac{1}{h_b}}{\frac{1}{h_b}+\frac{1}{h_c}}\ge 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{3r}{h_a}=x,\frac{3r}{h_b}=y,\frac{3r}{h_c}=z \text{ and then : } \sum_{cyc}x=3\to\text{(i)}$$

$$\text{Now, }\sum_{cyc}\frac{\frac{r}{h_a^2}+\frac{1}{h_b}}{\frac{1}{h_b}+\frac{1}{h_c}}=\sum_{cyc}\frac{\frac{r^2}{h_a^2}+\frac{r}{h_b}}{\frac{r}{h_b}+\frac{r}{h_c}}=\sum_{cyc}\frac{\frac{x^2}{9}+\frac{y}{3}}{\frac{y}{3}+\frac{z}{3}}\overset{?}{\ge}2\Leftrightarrow\sum_{cyc}\frac{x^2}{y+z}+3\sum_{cyc}\frac{y}{y+z}\overset{?}{\ge}6$$

$$\Leftrightarrow\sum_{cyc}\frac{x^2}{y+z}+\sum_{cyc}\frac{y(x+y+z)}{y+z}\overset{?}{\ge}6\left(\because 3=\sum_{cyc}x \text{ via (i)}\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{x^2+xy}{y+z}+\sum_{cyc}y\overset{?}{\ge}6\Leftrightarrow\sum_{cyc}\frac{x^2+xy}{y+z}\overset{?}{\ge}3\left(\because\sum_{cyc}x=3\right)$$

$$\Leftrightarrow\sum_{cyc}\frac{x^2+xy}{y+z}\overset{?}{\ge}\sum_{cyc}x\Leftrightarrow\sum_{cyc}\left(\frac{x^2+xy}{y+z}-x\right)\overset{?}{\ge}0\Leftrightarrow\sum_{cyc}\frac{x^2-xz}{y+z}\overset{?}{\ge}0$$

$$\Leftrightarrow\frac{1}{(x+y)(y+z)(z+x)}.\sum_{cyc}\left((x^2-xz)\left(x^2+\sum_{cyc}xy\right)\right)\overset{?}{\ge}0$$

$$\Leftrightarrow\left(\sum_{cyc}xy\right)\left(\sum_{cyc}x^2-\sum_{cyc}xy\right)+\sum_{cyc}x^4-\sum_{cyc}xy^3\ \boxed{\overset{?}{\ge}(*)}\ 0$$

$$\text{Now, } x^4+y^4+y^4+y^4 \underbrace{\overset{AM-GM}{\geq}}_{①} 4xy^3,\ y^4+z^4+z^4+z^4 \underbrace{\overset{AM-GM}{\geq}}_{②} 4yz^3 \text{ and}$$

$$z^4+x^4+x^4+x^4 \underbrace{\overset{AM-GM}{\geq}}_{③} 4zx^3 \text{ and } ①+②+③ \Rightarrow \sum_{cyc} x^4 \geq \sum_{cyc} xy^3 \text{ and this}$$

$$\text{combined with } \sum_{cyc} x^2 \geq \sum_{cyc} xy \Rightarrow (*) \text{ is true } \therefore \sum_{cyc} \frac{\frac{r}{h_a^2}+\frac{1}{h_b}}{\frac{1}{h_b}+\frac{1}{h_c}} \geq 2\ \forall\ \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3941. In any ΔABC the following relationship holds :

$$\sum_{cyc} \frac{\frac{r}{r_a^2}+\frac{1}{r_b}}{\frac{1}{r_b}+\frac{1}{r_c}} \geq 2$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{3r}{r_a}=x, \frac{3r}{r_b}=y, \frac{3r}{r_c}=z \text{ and then : } \sum_{cyc} x = 3 \to (i)$$

$$\text{Now, } \sum_{cyc} \frac{\frac{r}{r_a^2}+\frac{1}{r_b}}{\frac{1}{r_b}+\frac{1}{r_c}} = \sum_{cyc} \frac{\frac{r^2}{r_a^2}+\frac{r}{r_b}}{\frac{r}{r_b}+\frac{r}{r_c}} = \sum_{cyc} \frac{\frac{x^2}{9}+\frac{y}{3}}{\frac{y}{3}+\frac{z}{3}} \overset{?}{\geq} 2 \Leftrightarrow \sum_{cyc} \frac{x^2}{y+z} + 3\sum_{cyc} \frac{y}{y+z} \overset{?}{\geq} 6$$

$$\Leftrightarrow \sum_{cyc} \frac{x^2}{y+z} + \sum_{cyc} \frac{y(x+y+z)}{y+z} \overset{?}{\geq} 6 \left(\because 3 = \sum_{cyc} x \text{ via (i)}\right)$$

$$\Leftrightarrow \sum_{cyc} \frac{x^2+xy}{y+z} + \sum_{cyc} y \overset{?}{\geq} 6 \Leftrightarrow \sum_{cyc} \frac{x^2+xy}{y+z} \overset{?}{\geq} 3 \left(\because \sum_{cyc} x = 3\right)$$

$$\Leftrightarrow \sum_{cyc} \frac{x^2+xy}{y+z} \overset{?}{\geq} \sum_{cyc} x \Leftrightarrow \sum_{cyc} \left(\frac{x^2+xy}{y+z} - x\right) \overset{?}{\geq} 0 \Leftrightarrow \sum_{cyc} \frac{x^2-xz}{y+z} \overset{?}{\geq} 0$$

$$\Leftrightarrow \frac{1}{(x+y)(y+z)(z+x)} \cdot \sum_{cyc} \left((x^2-xz)\left(x^2+\sum_{cyc} xy\right)\right) \overset{?}{\geq} 0$$

$$\Leftrightarrow\left(\sum_{cyc} xy\right)\left(\sum_{cyc} x^2-\sum_{cyc} xy\right)+\sum_{cyc} x^4-\sum_{cyc} xy^3 \overset{?}{\underset{(*)}{\geq}} 0$$

Now, $x^4+y^4+y^4+y^4 \underbrace{\overset{AM-GM}{\geq}}_{①} 4xy^3, y^4+z^4+z^4+z^4 \underbrace{\overset{AM-GM}{\geq}}_{②} 4yz^3$ and

$z^4+x^4+x^4+x^4 \underbrace{\overset{AM-GM}{\geq}}_{③} 4zx^3$ and $①+②+③ \Rightarrow \sum_{cyc} x^4 \geq \sum_{cyc} xy^3$ and this

combined with $\sum_{cyc} x^2 \geq \sum_{cyc} xy \Rightarrow (*)$ is true $\therefore \sum_{cyc} \frac{\frac{r}{r_a^2}+\frac{1}{r_b}}{\frac{1}{r_b}+\frac{1}{r_c}} \geq 2\ \forall\ \Delta\ ABC,$

$'' ='' $ iff Δ ABC is equilateral (QED)

3942. *I, I_a, I_b, I_c – incenter and excenters of ΔABC. Prove that:*

$$\sum \frac{F_{I_aBC}}{AI\cdot AI_a} \geq \frac{3\sqrt{3}}{4}$$

Proposed by Sarkhan Adgozalov-Azerbaijan

Solution by Qurban Muellim-Azerbaijan

Lemma: $\sum \sin(2A) = 4\prod \sin A$

$$LHS=\sum\frac{F_{I_aBC}}{AI\cdot AI_a}=\sum\frac{a\sin^2\left(\frac{A}{2}\right)}{2r}=\frac{a+b+c-(a\cos A+b\cos B+c\cos C)}{4r}=$$

$$=\frac{2s-R\sum\sin(2A)}{4r}=\frac{2s-4R\cdot\prod\sin A}{4r}=\frac{2s-4R\cdot\frac{abc}{8R^3}}{4r}=$$

$$=\frac{2s-4R\cdot\frac{4rsR}{8R^3}}{4r}=\frac{s}{2r}-\frac{s}{2R}\geq\frac{3\sqrt{3}r}{2r}-\frac{3\sqrt{3}R}{4R}=\frac{3\sqrt{3}}{4}$$

Equality holds for $a=b=c$.

3943.

In any Δ ABC the following relationship holds :

$$R\prod_{cyc}(s+n_a)\left(w_a+w_b+m_c-n_a.\sqrt{3}\right)\leq 48\sqrt{3}r(r_ar_br_c)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$R\prod_{cyc}(s+n_a)(w_a+w_b+m_c-n_a.\sqrt{3})\overset{Lessel-Pelling}{\leq}$$

$$R\prod_{cyc}(s+n_a)(s.\sqrt{3}-n_a.\sqrt{3})=R.3\sqrt{3}.\prod_{cyc}(s^2-n_a^2)\overset{Bogdan\ Fustei}{=}$$

$$8R.3\sqrt{3}.\prod_{cyc}(h_ar_a)=24\sqrt{3}.\frac{2r^2s^2}{R}.(r_ar_br_c)=48\sqrt{3}r.(r_ar_br_c)^2\ (\because rs^2=r_ar_br_c)$$

$$\text{and so, } R\prod_{cyc}(s+n_a)(w_a+w_b+m_c-n_a.\sqrt{3})\leq 48\sqrt{3}r(r_ar_br_c)^2\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3944. In any Δ ABC the following relationship holds :

$$n_a\leq\frac{3\sqrt{3}(5b^2-6bc+5c^2)}{16s}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$n_a^2\overset{?}{\leq}\frac{27(5b^2-6bc+5c^2)^2}{256s^2}\overset{Bogdan\ Fustei}{\Leftrightarrow}$$

$$\frac{as(s-a)+s(b-c)^2}{a}\overset{?}{\leq}\frac{27(5b^2-6bc+5c^2)^2}{256s^2}$$

$$\Leftrightarrow 27(y+z)\left(5(z+x)^2+5(x+y)^2-6(z+x)(x+y)\right)^2\overset{?}{\geq}$$

$$256(x+y+z)^3(x(y+z)+(y-z)^2)$$

$$\left(\text{denoting } x=s-a, y=s-b, z=s-c\Rightarrow a=y+z, b=z+x, c=x+y \text{ and } s=x+y+z\right)$$

$$\Leftrightarrow 176x^4(y+z)-160x^3(y+z)^2+1024x^3yz-24\left((y+z)^3-3yz(y+z)\right)-$$

$$456x^2yz(y+z)+456x^2yz(y+z)+56x((y+z)^2-2yz)^2-160xyz(y+z)^2-$$

$$224xy^2z^2+419(y+z)(((y+z)^2-2yz)^2+y^2z^2-yz(y+z)^2)-$$

$$1201yz\left((y+z)^3-3yz(y+z)\right)+1214y^2z^2(y+z)\overset{?}{\geq}0\Leftrightarrow 176x^4.mx-$$

$$160x^3.m^2x^2+1024x^3.nx^2-24x^2(m^3x^3-3nx^2.mx)-456x^2.nx^2.mx+$$

$$56x(m^2x^2-2nx^2)^2-160x.nx^2.m^2x^2-224x.n^2x^4+$$

$$419mx((m^2x^2-2nx^2)^2+n^2x^4-nx^2.m^2x^2)-1201.nx^2.(m^3x^3-3nx^2.mx)+$$

$$1214.n^2x^4.mx\overset{?}{\geq}0\ \left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right)$$

$$\Leftrightarrow 6912mn^2-(3296m^3+384m^2+384m-1024)n+$$

$$419m^5+56m^4-24m^3-160m^2+176m \overset{?}{\underset{(*)}{\geq}} 0$$

Now, since $56m^4-24m^3-160m^2+176m=$
$$\frac{m}{4}\left(\left(419m^2+475m+\frac{1385}{4}\right)(2m-1)^2+270m+\frac{1431}{4}\right)>0$$

$\therefore$ $(*)$ is trivially true when : $3296m^3+384m^2+384m-1024\leq 0$ and so we now focus on the case when : $3296m^3+384m^2+384m-1024>0$ and now, let us assume : $11m-4\leq 0$ and then : $3296m^3+384m^2+384m-1024$
$$=\frac{32}{121}\left(\left(1133m^2+544m+\frac{3628}{11}\right)(11m-4)-\frac{28080}{11}\right)<0, \text{but}:$$
$3296m^3+384m^2+384m-1024>0$ and so, our assumption is incorrect and hence : $11m-4>0 \to (1)$

Now, LHS of $(*)$ is a quadratic expression in "n" with discriminant $\delta=$
$$(3296m^3+384m^2+384m-1024)^2-$$
$$27648m(419m^5+56m^4-24m^3-160m^2+176m)$$
$$\equiv (m+1)^3(m-2)^2(4-11m)\leq 0\ (\because 4-11m<0 \text{ via } (1)) \Rightarrow (*) \text{ is true}$$

$$\text{and so}, n_a\leq \frac{3\sqrt{3}.(5b^2-6bc+5c^2)}{16s}\ \forall\Delta ABC,$$
$$"=" \text{ iff } m=2 \Rightarrow "=" \text{ iff } b+c=2a \text{ (QED)}$$

3945. In any Δ ABC the following relationship holds :

$$\frac{2(b^2+bc+c^2)}{3(b+c)}\cos\frac{A}{2}\leq p_a\leq\frac{3\sqrt{3}.(3b^2-bc+3c^2)}{20s}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$p_a^2\overset{?}{\leq}\frac{27(3b^2-bc+3c^2)^2}{400s^2}\overset{\text{Bogdan Fustei and Mohamed Amine Ben Ajiba}}{\Longleftrightarrow}$$
$$\frac{s(s-a)(2s+a)^2+s(3s+a)(b-c)^2}{(2s+a)^2}\overset{?}{\leq}\frac{27(3b^2-bc+3c^2)^2}{400s^2}$$
$$\Leftrightarrow 27\big(y+z+2(x+y+z)\big)^2\left(3(z+x)^2+3(x+y)^2-(z+x)(x+y)\right)^2\overset{?}{\geq}$$
$$400(x+y+z)^3\left(x\big(y+z+2(x+y+z)\big)^2+\big(y+z+3(x+y+z)\big)(y-z)^2\right)$$
$$\begin{pmatrix} x=s-a, y=s-b, z=s-c\Rightarrow \\ a=y+z, b=z+x, c=x+y \text{ and } s=x+y+z\end{pmatrix}\Leftrightarrow 1100x^6+3900x^5.mx+$$
$$4215x^4.m^2x^2--2760x^4.nx^2+1210x^3(m^3x^3-3nx^2.mx)-5810x^3.nx^2.mx$$
$$+57x^2((m^2x^2-2nx^2)^2-2n^2x^4)-10398x^2.nx^2.(m^2x^2-2nx^2)-$$

$$15618x^2.n^2x^4+606x.mx((m^2x^2-2nx^2)^2+n^2x^4-nx^2.m^2x^2)-3588x.nx^2(m^3x^3-3nx^2.mx)+2082x.n^2x^4.mx+587(m^2x^2-2nx^2)((m^2x^2-2nx^2)^2-3n^2x^4)-284nx^2((m^2x^2-2nx^2)^2-2n^2x^4)+5488n^2x^4.m^2x^2+1742n^3x^6\overset{?}{\geq}0$$

$$\left(m=\frac{y+z}{x},n=\frac{yz}{x^2}\right)\Leftrightarrow(11907m^2+15876m+5292)n^2-(3806m^4+6618m^3+10626m^2+9440m+2760)n+587m^6+606m^5+57m^4+1210m^3+4215m^2+3900m+1100\underset{(*)}{\overset{?}{\geq}}0$$

Now, LHS of $(*)$ is a quadratic expression in "n" with discriminant $\delta=(3806m^4+6618m^3+10626m^2+9440m+2760)^2-4(11907m^2+15876m+5292)\left(\begin{matrix}587m^6+606m^5+57m^4+1210m^3+\\4215m^2+3900m+1100\end{matrix}\right)$

$$=-6400(m+1)^3(2105m^5-3852m^4-5862m^3+5408m^2+8328m+2448)$$

and if : $P=2105m^5-3852m^4-5862m^3+5408m^2+8328m+2448\geq 0$, then : $\delta\leq 0\Rightarrow(*)$ is true and so, we now focus on the case when : $P<0$ $(\Rightarrow\delta>0)$ and then, in order to prove $(*)$, it suffices to prove :

$$2(11907m^2+15876m+5292)n\overset{?}{\leq}3806m^4+6618m^3+10626m^2+9440m+2760-\sqrt{\delta}$$

and $\because n\overset{AM-GM}{\leq}\frac{m^2}{4}\therefore$ it suffices to prove :

$$2(11907m^2+15876m+5292).\frac{m^2}{4}\overset{?}{\leq}3806m^4+6618m^3+10626m^2+9440m+2760-\sqrt{\delta}\Leftrightarrow 2\sqrt{\delta}\underset{(**)}{\overset{?}{\leq}}5(-859m^4-528m^3+3192m^2+3776m+1104)$$

and let us assume $Q=859m^4+528m^3-3192m^2-3776m-1104\geq 0$ whenever : $P<0$ and then : $P<0\leq\frac{m}{5}.Q\Rightarrow mQ>5P\Rightarrow T$

$$=1611m^5-3298m^4-4353m^3+5136m^2+7124m+2040\overset{①}{<}0$$

Case 1 $m\geq 1$ and then : $T=(m-1)(m-2)^2(1611m^2+4757m+6544)+6244m^2-26200m+28216>0\because\Delta$ of $(6244m^2-26200m+28216)=(26200)^2-4(6244)(28216)\ll 0\therefore T>0$

Case 2 $0<m<1$; then : $T=(m-1)(m-2)(4m-1)^2\left(\frac{3222m+4681}{32}\right)+\frac{13024m+3801m(1-m^2)+2578m(1-m)}{2}+\frac{15m+2m^3+m^2+55918}{32}>0$ $\therefore$ combining both cases, $T>0\ \forall\ m>0$ and this is a contradiction to ① and hence our assumption is incorrect $\therefore$ $Q<0$ and so, $(**)\Leftrightarrow$

$$-25600(m+1)^3(2105m^5-3852m^4-5862m^3+5408m^2+8328m+2448)\overset{?}{\leq}$$

$$25(-859m^4 - 528m^3 + 3192m^2 + 3776m + 1104)^2 \Leftrightarrow$$

$$33075(3m+2)^4(m-2)^2(27m^2+68m+44) \overset{?}{\geq} 0 \to \text{true} \Rightarrow (**) \Rightarrow (*) \text{ is true}$$

$$\therefore p_a \leq \frac{3\sqrt{3}.(3b^2 - bc + 3c^2)}{20s} \text{ and again, } \frac{2(b^2+bc+c^2)}{3(b+c)}\cos\frac{A}{2} \overset{?}{\leq} p_a \Leftrightarrow$$

$$\frac{s(s-a)(2s+a)^2 + s(3s+a)(b-c)^2}{(2s+a)^2} \overset{?}{\geq} \frac{4(b^2+bc+c^2)^2}{9(b+c)^2}\cdot\frac{s(s-a)}{bc}$$

$$\overset{\substack{\text{via earlier substitution}\\ \text{and following simplification}}}{\Leftrightarrow}$$

$$(y-z)^2\begin{pmatrix} 48x^5 + 120x^4(y+z) + 92x^3(y+z)^2 + 112x^3yz + \\ 24x^2(y+z)(y^2+yz+z^2) + 288x^2yz(y+z) + 180xyz(y+z)^2 + \\ 36yz(y+z)(y^2+yz+z^2) + 36y^2z^2(y+z) \end{pmatrix} \overset{?}{\geq} 0$$

$$\to \text{true} \therefore p_a \geq \frac{2(b^2+bc+c^2)}{3(b+c)}\cos\frac{A}{2} \text{ and so,}$$

$$\frac{2(b^2+bc+c^2)}{3(b+c)}\cos\frac{A}{2} \leq p_a \leq \frac{3\sqrt{3}.(3b^2-bc+3c^2)}{20s},$$

" =" for lower bound iff $b = c$ and

" =" for upper bound iff Δ ABC is equilateral (QED)

3946. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \leq \frac{w_b}{w_c} + \frac{w_c}{w_b} \leq \frac{r_b + r_c}{w_a}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt{\frac{b}{c}} + \sqrt{\frac{c}{b}} \overset{?}{\leq} \frac{w_b}{w_c} + \frac{w_c}{w_b} \Leftrightarrow \frac{b}{c} + \frac{c}{b} + 2 \overset{?}{\leq} \frac{w_b^2}{w_c^2} + \frac{w_c^2}{w_b^2} + 2$$

$$\Leftrightarrow \frac{(b+c)^2}{bc} \overset{?}{\leq} \frac{\left(sa - \frac{s(s-b)(c-a)^2}{(c+a)^2} - \frac{s(s-c)(a-b)^2}{(a+b)^2}\right)^2}{\frac{4ca}{(c+a)^2}.s(s-b).\frac{4ab}{(a+b)^2}.s(s-c)}$$

$$\Leftrightarrow \frac{\begin{pmatrix}(y+z)(2y+z+x)^2(2z+x+y)^2 - y(z-x)^2(2z+x+y)^2 - \\ z(x-y)^2(2y+z+x)^2\end{pmatrix}}{16(y+z)^2yz(2x+y+z)^2(2y+z+x)^2(2z+x+y)^2} \overset{?}{\geq}$$

$$\begin{pmatrix} x = s-a, y = s-b, z = s-c \Rightarrow \\ a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z \end{pmatrix} \overset{\text{Following simplification}}{\Leftrightarrow}$$

$$x^6(y-z)^2(y+z)^2 + 6x^5(y-z)^2(y+z)\left((y^2+yz+z^2)+yz\right) +$$

$$x^4(y-z)^2(15(y^2+yz+z^2)^2+31yz(y^2+yz+z^2)+16y^2z^2)+$$
$$x^3(y-z)^2(y+z)\left(20(y^2+yz+z^2)(y^2+z^2)+24y^2z^2+64yz(y^2+yz+z^2)\right)+$$
$$x^2(y-z)^2(y+z)^2(15(y^2+z^2)^2+65yz(y^2+z^2)+69y^2z^2)+$$
$$x(y-z)^2(y+z)\left(\begin{array}{c}6(y^2+z^2)(y^4+y^3z+y^2z^2+yz^3+z^4)+\\32yz(y^2+z^2)(y^2+yz+z^2)+52y^2z^2(y^2+yz+z^2)\end{array}\right)+$$
$$(y-z)^2\left(\begin{array}{c}6yz(y^2+yz+z^2)(y^4+y^3z+y^2z^2+yz^3+z^4)+\\13y^2z^2(y^2+yz+z^2)^2+\\(y+z)(y^2+z^2)(y^2+yz+z^2)(y^3+z^3)+y^3z^3(y^2+yz+z^2)\end{array}\right)\overset{?}{\geq}0$$

$$\to \text{true} \therefore \frac{w_b}{w_c}+\frac{w_c}{w_b}\geq\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}} \text{ and again, } \frac{w_b}{w_c}+\frac{w_c}{w_b}\overset{?}{\leq}\frac{r_b+r_c}{w_a}$$

$$\Leftrightarrow \frac{\left(sa-\frac{s(s-b)(c-a)^2}{(c+a)^2}-\frac{s(s-c)(a-b)^2}{(a+b)^2}\right)^2}{\frac{4ca}{(c+a)^2}.s(s-b).\frac{4ab}{(a+b)^2}.s(s-c)}\overset{?}{\leq}\frac{s(s-a)(s-b)(s-c)a^2}{(s-b)^2(s-c)^2.\frac{4bc}{(b+c)^2}.s(s-a)}$$

$$\underset{\Leftrightarrow}{\overset{\text{via earlier substitution and following simplification}}{}}$$

$$16x^5(y-z)^2(y+z)^3+x^4(y-z)^2(y+z)^2\left(152yz+68(y^2+z^2)\right)+$$
$$x^3(y-z)^2\left(\begin{array}{c}376yz(y^2+yz+z^2)(y+z)+\\104(y^2+yz+z^2)(y^2+z^2)(y+z)+168y^2z^2(y+z)\end{array}\right)+$$
$$x^2(y-z)^2(y+z)^2\left(480y^2z^2+68(y^2+z^2)^2+360yz(y^2+z^2)\right)+$$
$$x(y-z)^2(y+z)\left(\begin{array}{c}16(y^4+y^3z+y^2z^2+yz^3+z^4)(y^2+z^2)+\\136yz(y^2+yz+z^2)(y^2+z^2)+\\344y^2z^2(y+z)(y^2+yz+z^2)+y^3z^3\end{array}\right)+$$
$$(y-z)^2(y+z)^2\left(16yz(y^2+z^2)^2+88y^3z^3+68y^2z^2(y^2+z^2)\right)\overset{?}{\geq}0\to\text{true}$$

$$\therefore \frac{w_b}{w_c}+\frac{w_c}{w_b}\leq\frac{r_b+r_c}{w_a} \text{ and so, } \sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\leq\frac{w_b}{w_c}+\frac{w_c}{w_b}\leq\frac{r_b+r_c}{w_a}\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } b=c \text{ (QED)}$$

3947. In any Δ ABC the following relationship holds :

$$5\sum_{cyc}\frac{r^2}{h_a^2}\leq 1+6\sum_{cyc}\frac{r^3}{h_a^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{3r}{h_a}=x, \frac{3r}{h_b}=y, \frac{3r}{h_c}=z \text{ and then}: \sum_{cyc} x = 3 \to \text{(i)}$$

$$\text{Now, } 5\sum_{cyc}\frac{r^2}{h_a^2} \le 1+6\sum_{cyc}\frac{r^3}{h_a^3} \Leftrightarrow 5\sum_{cyc}\frac{x^2}{9} \overset{?}{\le} 1+6\sum_{cyc}\frac{x^3}{27} \Leftrightarrow 2\sum_{cyc}x^3+9 \overset{?}{\ge} 5\sum_{cyc}x^2$$

$$\Leftrightarrow 2\sum_{cyc}x^3+\frac{9}{27}\left(\sum_{cyc}x\right)^3 \overset{?}{\ge} \frac{5}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\left(\because \sum_{cyc}x=3 \text{ via (i)}\right)$$

$$\Leftrightarrow 6\sum_{cyc}x^3+\left(\sum_{cyc}x\right)^3 \overset{?}{\ge} 5\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$

$$\Leftrightarrow \sum_{cyc}x^3+3xyz \overset{?}{\ge} \sum_{cyc}x^2y+\sum_{cyc}xy^2 \to \text{true via Schur}$$

$$\therefore 5\sum_{cyc}\frac{r^2}{h_a^2} \le 1+6\sum_{cyc}\frac{r^3}{h_a^3}\ \forall\, \Delta\, ABC, ''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3948. In any Δ ABC the following relationship holds :

$$5\sum_{cyc}\frac{r^2}{r_a^2} \le 1+6\sum_{cyc}\frac{r^3}{r_a^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{3r}{r_a}=x, \frac{3r}{r_b}=y, \frac{3r}{r_c}=z \text{ and then}: \sum_{cyc} x = 3 \to \text{(i)}$$

$$\text{Now, } 5\sum_{cyc}\frac{r^2}{r_a^2} \overset{?}{\le} 1+6\sum_{cyc}\frac{r^3}{r_a^3} \Leftrightarrow 5\sum_{cyc}\frac{x^2}{9} \overset{?}{\le} 1+6\sum_{cyc}\frac{x^3}{27} \Leftrightarrow 2\sum_{cyc}x^3+9 \overset{?}{\ge} 5\sum_{cyc}x^2$$

$$\Leftrightarrow 2\sum_{cyc}x^3+\frac{9}{27}\left(\sum_{cyc}x\right)^3 \overset{?}{\ge} \frac{5}{3}\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\left(\because \sum_{cyc}x=3 \text{ via (i)}\right)$$

$$\Leftrightarrow 6\sum_{cyc} x^3 + \left(\sum_{cyc} x\right)^3 \overset{?}{\geq} 5\left(\sum_{cyc} x^2\right)\left(\sum_{cyc} x\right)$$

$$\Leftrightarrow \sum_{cyc} x^3 + 3xyz \overset{?}{\geq} \sum_{cyc} x^2 y + \sum_{cyc} xy^2 \to \text{true via Schur}$$

$$\therefore 5\sum_{cyc} \frac{r^2}{r_a^2} \leq 1 + 6\sum_{cyc} \frac{r^3}{r_a^3}\ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3949. In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{1}{h_a\sqrt{\frac{1}{h_b} + \frac{1}{h_c}}} \geq \sqrt{\frac{3}{2r}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{3r}{h_a} = x, \frac{3r}{h_b} = y, \frac{3r}{h_c} = z \text{ and then : } \sum_{cyc} x = 3 \to (i)$$

$$\text{Now, } \sum_{cyc} \frac{1}{h_a.\sqrt{\frac{1}{h_b} + \frac{1}{h_c}}} = \frac{1}{\sqrt{r}}.\sum_{cyc} \frac{\frac{r}{h_a}}{\sqrt{\frac{r}{h_b} + \frac{r}{h_c}}} = \frac{1}{\sqrt{r}}.\sum_{cyc} \frac{\frac{x}{3}}{\sqrt{\frac{y+z}{3}}} = \frac{1}{\sqrt{3r}}.\sum_{cyc} \frac{x^2}{x.\sqrt{y+z}}$$

$$\overset{\text{Bergstrom}}{\geq} \frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc} x\right)^2}{\sum_{cyc}\left(x.\sqrt{y+z}\right)} = \frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc} x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{xy+zx}\right)} \overset{\text{CBS}}{\geq}$$

$$\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc} x\right)^2}{\sqrt{2\left(\sum_{cyc} x\right)\left(\sum_{cyc} xy\right)}} \geq \frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc} x\right)^2}{\sqrt{2\left(\sum_{cyc} x\right).\frac{1}{3}\left(\sum_{cyc} x\right)^2}} = \frac{\sqrt{\sum_{cyc} x}}{\sqrt{2r}} \overset{\text{via (i)}}{=} \frac{\sqrt{3}}{\sqrt{2r}}$$

$$\text{and so, } \sum_{cyc} \frac{1}{h_a.\sqrt{\frac{1}{h_b} + \frac{1}{h_c}}} \geq \sqrt{\frac{3}{2r}}\ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3950. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{r_a\cdot\sqrt{\frac{1}{r_b}+\frac{1}{r_c}}}\ge\sqrt{\frac{3}{2r}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } \frac{3r}{r_a}=x,\frac{3r}{r_b}=y,\frac{3r}{r_c}=z \text{ and then}: \sum_{cyc}x=3\to(\text{i})$$

$$\text{Now, }\sum_{cyc}\frac{1}{r_a\cdot\sqrt{\frac{1}{r_b}+\frac{1}{r_c}}}=\frac{1}{\sqrt{r}}\cdot\sum_{cyc}\frac{\frac{r}{r_a}}{\sqrt{\frac{r}{r_b}+\frac{r}{r_c}}}=\frac{1}{\sqrt{r}}\cdot\sum_{cyc}\frac{\frac{x}{3}}{\sqrt{\frac{y+z}{3}}}=\frac{1}{\sqrt{3r}}\cdot\sum_{cyc}\frac{x^2}{x.\sqrt{y+z}}$$

$$\overset{\text{Bergstrom}}{\ge}\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(x.\sqrt{y+z}\right)}=\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{xy+zx}\right)}\overset{\text{CBS}}{\ge}$$

$$\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right)\left(\sum_{cyc}xy\right)}}\ge\frac{1}{\sqrt{3r}}\cdot\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{2\left(\sum_{cyc}x\right).\frac{1}{3}\left(\sum_{cyc}x\right)^2}}=\frac{\sqrt{\sum_{cyc}x}}{\sqrt{2r}}\overset{\text{via (i)}}{=}\frac{\sqrt{3}}{\sqrt{2r}}$$

$$\text{and so, }\sum_{cyc}\frac{1}{r_a\cdot\sqrt{\frac{1}{r_b}+\frac{1}{r_c}}}\ge\sqrt{\frac{3}{2r}}\ \forall\,\Delta\,ABC,''=''\ \text{iff }\Delta\,ABC\text{ is equilateral}$$

3951. In any Δ ABC the following relationship holds :

$$\frac{1}{1+\tan^4\frac{A}{2}}+\frac{1}{1+\tan^4\frac{B}{2}}+\frac{1}{1+\tan^4\frac{C}{2}}\le\frac{27}{10}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{1+\tan^4\frac{A}{2}}=\sum_{cyc}\frac{1+\tan^4\frac{A}{2}-\tan^4\frac{A}{2}}{1+\tan^4\frac{A}{2}}=3-\sum_{cyc}\frac{r_a^4}{s^4+r_a^4}\overset{\text{Bergstrom}}{\le}$$

$$3-\frac{\left(\sum_{cyc}r_a^2\right)^2}{3s^4+\sum_{cyc}r_a^4}=3-\frac{((4R+r)^2-2s^2)^2}{3s^4+((4R+r)^2-2s^2)^2-2(s^4-2r(4R+r)s^2)}$$

$$\Leftrightarrow 25s^4-(448R^2+272Rr+40r^2)s^2+7(4R+r)^4 \underbrace{\overset{?}{\geq}}_{(*)} 0$$

$$\text{Now, since } 25s^2(s^2-16Rr+5r^2) \overset{\text{Gerretsen}}{\geq} 0 \therefore \text{in order to prove } (*),$$

$$\text{it suffices to prove : LHS of } (*) \overset{?}{\geq} 25s^2(s^2-16Rr+5r^2)$$

$$\Leftrightarrow 7(4R+r)^4 \underbrace{\overset{?}{\geq}}_{(**)} (448R^2-128Rr+165r^2)s^2 \text{ and again,}$$

$$(448R^2-128Rr+165r^2)s^2 \overset{\text{Gerretsen}}{\leq}$$

$$(448R^2-128Rr+165r^2)(4R^2+4Rr+3r^2) \overset{?}{\leq} 7(4R+r)^4$$

$$\Leftrightarrow 128t^3-205t^2-41t-122 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow (t-2)(128t^2+51t+61) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \frac{1}{1+\tan^4\frac{A}{2}}+\frac{1}{1+\tan^4\frac{B}{2}}+\frac{1}{1+\tan^4\frac{C}{2}}$$

$$\leq \frac{27}{10}\ \forall \Delta ABC, ''='' \text{ iff } \Delta ABC \text{ is equilateral}$$

3952. In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{a+b}{(\cos A+\cos B)^3} \geq 6R\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{b+c}{(\cos B+\cos C)^3} = \sum_{cyc} \frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{8\sin^3\frac{A}{2}.\cos^3\frac{B-C}{2}} = \frac{R}{2}.\sum_{cyc} \frac{\cos\frac{A}{2}}{\sin^3\frac{A}{2}.\cos^2\frac{B-C}{2}} \geq$$

$$\frac{R}{2}.\sum_{cyc} \frac{\cos\frac{A}{2}}{\sin^3\frac{A}{2}} \left(\because \cos^2\frac{B-C}{2} \leq 1\right) = \frac{Rs}{2r^2}.\sum_{cyc}\left(\frac{1}{r_a}.\frac{r^2}{\sin^2\frac{A}{2}}\right)$$

$$= \frac{Rs}{2r^2.rs^2}.\sum_{cyc}(s(s-a).AI^2) = \frac{R}{2r^3}.\sum_{cyc}((s-a).(bc-4Rr))$$

$$= \frac{R}{2r^3}.(s(s^2+4Rr+r^2)-12Rrs-4Rrs) = \frac{Rs(s^2-12Rr+r^2)}{2r^3} \overset{\substack{\text{Gerretsen} \\ \text{and} \\ \text{Mitrinovic}}}{\geq}$$

$$\frac{R.3\sqrt{3}r.(4Rr-4r^2)}{2r^3} = \frac{R.3\sqrt{3}.(2R-2r)}{r} \overset{\text{Euler}}{\geq} \frac{R.3\sqrt{3}.(2r)}{r} = 6R.\sqrt{3} \text{ and so,}$$

$$\sum_{cyc}\frac{a+b}{(\cos A+\cos B)^3}\ge 6R.\sqrt{3}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3953.

In any Δ ABC with $a=max\{a,b,c\}$, the following relationship holds :

$$\frac{a}{b+c}\ge 1-\frac{r}{R}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a}{b+c}\overset{?}{\ge}1-\frac{r}{R}\Leftrightarrow\frac{4(s-a)(s-b)(s-c)}{abc}\overset{?}{\ge}\frac{2(s-a)}{b+c}$$

$$\Leftrightarrow 2yz(2x+y+z)\overset{?}{\ge}(y+z)(z+x)(x+y)$$

$$\left(\begin{matrix}x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y\ and\\ s=x+y+z\end{matrix}\right)$$

$$\Leftrightarrow y^2(z-x)+z^2(y-x)+xy(z-x)+zx(y-x)\overset{?}{\ge}0\to\text{true}$$

$$\because a\ge b,c\Rightarrow y,z\ge x\therefore\frac{a}{b+c}\ge 1-\frac{r}{R}\ \forall\ \Delta\ ABC \text{ with } a=max\{a,b,c\},$$

$$''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3954. **In any Δ ABC the following relationship holds :**

$$g_a\le\sqrt{\frac{R+2r}{4r}}\cdot h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$g_a^2\overset{?}{\le}\left(\frac{R}{4r}+\frac{1}{2}\right).h_a^2\overset{\text{via Bogdan Fustei}}{\Leftrightarrow}s(s-a)-\frac{s-a}{a}.(b-c)^2\overset{?}{\le}$$

$$\left(\frac{abc}{16(s-a)(s-b)(s-c)}+\frac{1}{2}\right)\left(s(s-a)-\frac{s(s-a)}{a^2}.(b-c)^2\right)$$

$$\Leftrightarrow\frac{sa-(b-c)^2}{a}\overset{?}{\le}s.\frac{abc+8(s-a)(s-b)(s-c)}{16(s-a)(s-b)(s-c)}.\frac{4(s-b)(s-c)}{a^2}$$

$$\Leftrightarrow\big((y+z)(z+x)(x+y)+8xyz\big)(x+y+z)\overset{?}{\ge}4(y+z)x\left(\begin{matrix}(y+z)(x+y+z)-\\(y-z)^2\end{matrix}\right)$$

$$\left(x = s - a, y = s - b, z = s - c \Rightarrow a = y + z, b = z + x, c = x + y \text{ and } s = x + y + z\right)$$

$$\Leftrightarrow x^3(y+z) - 2x^2(y+z)^2 + 8x^2.yz + x\left((y+z)^3 - 3yz(y+z)\right) - 4x.yz(y+z) + yz(y+z)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow x^3(mx) - 2x^2(mx)^2 + 8x^2.nx^2 + x\left((mx)^3 - 3nx^2(mx)\right) - 4x.nx^2(mx) + nx^2(mx)^2 \overset{?}{\geq} 0 \left(m = \frac{y+z}{x}, n = \frac{yz}{x^2}\right)$$

$$\Leftrightarrow n(m^2 - 7m + 8) + m(m-1)^2 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and it's trivially true if :}$$

$$m^2 - 7m + 8 \geq 0 \text{ and when : } m^2 - 7m + 8 < 0, \text{then : since } n \overset{AM-GM}{\leq} \frac{m^2}{4}$$

$$\therefore \text{in order to prove } (*), \text{it suffices to prove : } \frac{m^2}{4}.(m^2 - 7m + 8) + m(m-1)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow (m+1)(m-2)^2 \overset{?}{\geq} 0 \rightarrow \text{true} \because m > 0 \text{ as } x, y, z > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore g_a \leq \sqrt{\frac{R+2r}{4r}}.h_a, '' ='' \text{ iff } y = z \text{ and } y + z = 2x \Rightarrow$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3955. In any Δ AB, the following relationship holds

$$m_a \leq \frac{\sqrt{9s^2 - 106Rr + 113r^2}}{12r}.w_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$m_a^2 \overset{?}{\leq} \frac{9s^2 - 106Rr + 113r^2}{144r^2}.w_a^2 \Leftrightarrow s(s-a) + \frac{(b-c)^2}{4} \overset{?}{\leq}$$

$$\left(\frac{9s^3}{144(s-a)(s-b)(s-c)} - \frac{106abc}{144.4(s-a)(s-b)(s-c)} + \frac{113}{144}\right).\frac{4bc.s(s-a)}{(b+c)^2}$$

$$\Leftrightarrow \frac{4x(x+y+z) + (y-z)^2}{4} \overset{?}{\leq}$$

$$\frac{18(x+y+z)^3 - 53(y+z)(z+x)(x+y) + 226xyz}{288xyz}.\frac{4(z+x)(x+y)x(x+y+z)}{(2x+y+z)^2}$$

$$\left(x = s - a, y = s - b, z = s - c \Rightarrow a = y + z, b = z + x, c = x + y \text{ and } s = x + y + z\right)$$

$$\Leftrightarrow 19x^6 + 37x^5(y+z) + 21x^4(y+z)^2 - 44x^4yz + 21x^3\left((y+z)^3 - 3yz(y+z)\right)$$

$$-95x^3yz(y+z)+37x^2(y^2+z^2)^2+42x^2y^2z^2-162x^2yz(y^2+z^2)+$$
$$18(y+z)(y^5+z^5)-88xyz\left((y+z)^3-3yz(y+z)\right)+107xy^2z^2(y+z)+$$
$$19y^2z^2(y+z)^2\overset{?}{\geq}0\Leftrightarrow 18+37m+21m^2-44n+21(m^3-3mn)-$$
$$95mn+37(m^2-2n)^2+42n^2-162n(m^2-2n)+18m((m^2-2n)^2+n^2-nm^2)$$
$$-88n(m^3-3mn)+107n^2m+19n^2m^2\overset{?}{\geq}0\ \left(m=\frac{y+z}{x},n=\frac{yz}{x^2}\right)$$
$$\Leftrightarrow(19m^2+461m+514)n^2-(178m^3+310m^2+158m+514+44)n+$$
$$18m^5+37m^4+21m^3+21m^2+37m+18\ \boxed{\underset{(*)}{\overset{?}{\geq}}}\ 0$$

Now, LHS of $(*)$ is a quadratic polynomial in "n" with discriminant =

$$(178m^3+310m^2+158m+514+44)^2-$$
$$4(19m^2+461m+514)(18m^5+37m^4+21m^3+21m^2+37m+18)$$
$$=-4(m-2)^2(342m^5+2448m^4+7542m^3+11387m^2+8152m+2192)\leq 0$$
$$\because m>0\text{ as }x,y,z>0\Rightarrow(*)\text{ is true}\therefore m_a\leq\frac{\sqrt{9s^2-106Rr+113r^2}}{12r}.w_a\ \forall\ \Delta ABC,$$
$$"="\text{ iff } b+c=2a\ (QED)$$

3956. In any Δ ABC the following relationship holds :

$$p_a\leq\frac{\sqrt{9s^2-48Rr-66r^2}}{9r}.h_a$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$p_a^2\overset{?}{\leq}\frac{9s^2-48Rr-66r^2}{81r^2}.h_a^2\ \underset{\Leftrightarrow}{\overset{\text{via Bogdan Fustei and Mohamed Amine Ben Ajiba}}{}}$$
$$s(s-a)+\frac{s(3s+a)}{(2s+a)^2}.(b-c)^2\overset{?}{\leq}$$
$$\left(\frac{\frac{9s^3}{81(s-a)(s-b)(s-c)}-\frac{48abc}{81.4(s-a)(s-b)(s-c)}}{}-\frac{66}{81}\right)\left(\frac{4s(s-a)(s-b)(s-c)}{a^2}\right)$$
$$\Leftrightarrow\frac{x\left(y+z+2(x+y+z)\right)^2+\left(y+z+3(x+y+z)\right)(y-z)^2}{\left(y+z+2(x+y+z)\right)^2}$$
$$\overset{?}{\leq}\frac{12(x+y+z)^3-16(y+z)(z+x)(x+y)-88xyz}{27(y+z)^2}$$

$$\left(x = s-a, y = s-b, z = s-c \Rightarrow a = y+z, b = z+x, c = x+y \text{ and } s = x+y+z\right)$$

$$\Leftrightarrow 12x^5 + 56x^4(y+z) + 80x^3(y+z)^2 - 88x^3.nx^2 + 36x^2\left((mx)^3 - 3nx^2(mx)\right) - 172x^2.nx^2.mx - 165x.nx^2.(mx)^2 + 72.nx^2.\left((mx)^3 - 3nx^2(mx)\right) + 216n^2x^4.mx \overset{?}{\geq} 0 \left(m = \frac{y+z}{x}, n = \frac{yz}{x^2}\right)$$

$$\Leftrightarrow n(72m^3 - 165m^2 - 280m - 88) + 36m^3 + 80m^2 + 56m + 12 \overset{?}{\underset{(*)}{\geq}} 0 \text{ and it's}$$

trivially true if : $72m^3 - 165m^2 - 280m - 88 \geq 0$ and when :

$$72m^3 - 165m^2 - 280m - 88 < 0, \text{then : since } n \overset{AM-GM}{\leq} \frac{m^2}{4}$$

∴ in order to prove $(*)$, it suffices to prove :

$$\frac{m^2}{4}.(72m^3 - 165m^2 - 280m - 88) + 36m^3 + 80m^2 + 56m + 12 \overset{?}{\geq} 0$$

$$\Leftrightarrow (8m+3)(3m+2)^2(m-2)^2 \overset{?}{\geq} 0 \to \text{true} \because m > 0 \text{ as } x, y, z > 0 \Rightarrow (*) \text{ is true}$$

$$\therefore p_a \leq \frac{\sqrt{9s^2 - 48Rr - 66r^2}}{9r}.h_a \text{ "=" iff } y = z \text{ and } y + z = 2x \Rightarrow$$

"=" iff Δ ABC is equilateral (QED)

3957. In any Δ ABC the following relationship holds :

$$8 \leq \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \left(\frac{R}{r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{\prod_{cyc} \frac{m_b + m_c}{m_a}} \overset{AM-GM}{\leq} \frac{2}{3}\left(\sum_{cyc} m_a\right).\frac{1}{\sqrt[3]{m_a m_b m_c}} \overset{Bager}{\leq} \frac{2}{3}.\frac{4R+r}{\sqrt[3]{h_a h_b h_c}} \overset{Euler}{\leq}$$

$$\frac{2}{3}.\frac{\frac{9R}{2}}{\sqrt[3]{\frac{2r^2s^2}{R}}} \overset{Gerretsen + Euler}{\leq} \frac{2}{3}.\frac{\frac{9R}{2}}{\sqrt[3]{\frac{r^2.27Rr}{R}}} = \frac{R}{r} \therefore \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \left(\frac{R}{r}\right)^3 \text{ and}$$

$$\text{via Cesaro, } \prod_{cyc} \frac{m_b + m_c}{m_a} \geq 8 \text{ and so, } 8 \leq \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \left(\frac{R}{r}\right)^3 \ \forall \Delta ABC,$$

"=" iff Δ ABC is equilateral (QED)

3958. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a^3+b^3}{h_c^2}\le\sum_{cyc}\frac{a^3+b^3}{r_c^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{WLOG we may assume}: a\ge b\ge c \text{ and then}:$$

$$b^3+c^3\le c^3+a^3\le a^3+b^3, \frac{1}{h_a^2}\ge\frac{1}{h_b^2}\ge\frac{1}{h_b^2} \text{ and } \frac{1}{r_a^2}\le\frac{1}{r_b^2}\le\frac{1}{r_b^2} \text{ and}$$

$$\text{hence via Chebyshev, }\sum_{cyc}\frac{b^3+c^3}{h_a^2}-\sum_{cyc}\frac{b^3+c^3}{r_a^2}\le$$

$$\frac{1}{3}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(\sum_{cyc}\frac{1}{h_a^2}\right)-\frac{1}{3}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(\sum_{cyc}\frac{1}{r_a^2}\right)$$

$$=\frac{1}{3}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(\frac{1}{4r^2s^2}\left(\sum_{cyc}a^2-4\sum_{cyc}(s-a)^2\right)\right)$$

$$=\frac{1}{6r^2s^2}\cdot\left(\sum_{cyc}(b^3+c^3)\right)\left(s^2-4Rr-r^2-2\left(3s^2-4s^2+2(s^2-4Rr-r^2)\right)\right)$$

$$=\frac{-1}{6r^2s^2}\cdot\left(\sum_{cyc}(b^3+c^3)\right)(s^2-12Rr-3r^2)\le 0 \because s^2-12Rr-3r^2$$

$$=s^2-16Rr+5r^2+4r(R-2r)\overset{\substack{\text{Gerretsen}\\ \text{and}\\ \text{Euler}}}{\ge}0 \therefore \sum_{cyc}\frac{a^3+b^3}{h_c^2}\le\sum_{cyc}\frac{a^3+b^3}{r_c^2}\ \forall\,\Delta\,ABC,$$

$$''=''\text{ iff }\Delta\,ABC\text{ is equilateral (QED)}$$

3959. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a+b}{(\cos A+\cos B)^2}\ge 6R\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{b+c}{(\cos B+\cos C)^2}=\sum_{cyc}\frac{4R\cos\frac{A}{2}\cos\frac{B-C}{2}}{4\sin^2\frac{A}{2}\cdot\cos^2\frac{B-C}{2}}=R.\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}\cdot\cos\frac{B-C}{2}}\geq$$

$$R.\sum_{cyc}\frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}}\left(\because\cos\frac{B-C}{2}\leq 1\right)\overset{AM-GM}{\geq}3R.\sqrt[3]{\prod_{cyc}\frac{\cos\frac{A}{2}}{\sin^2\frac{A}{2}}}=3R.\sqrt[3]{\frac{s}{4R}\cdot\frac{16R^2}{r^2}}$$

$$=3R.\sqrt[3]{\frac{4Rs}{r^2}}\overset{\substack{Euler\\and\\Mitrinovic}}{\geq}3R.\sqrt[3]{\frac{8r.3\sqrt{3}.r}{r^2}}=6R.\sqrt{3}\text{ and so,}$$

$$\sum_{cyc}\frac{a+b}{(\cos A+\cos B)^2}\geq 6R.\sqrt{3}\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

3960. In any Δ ABC the following relationship holds :

$$\frac{g_a}{r}+\frac{n_a}{r_a}=\frac{4(h_a-r)}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$g_a-n_a+\sqrt{4r^2+(b-c)^2}\overset{\text{Bogdan Fustei}}{=}g_a-n_a+\frac{an_a}{s}=g_a-\frac{r.n_a(s-a)}{rs}$$

$$=g_a-\frac{r.n_a}{r_a}=\frac{g_ar_a-r.n_a}{r_a}\therefore\left(\frac{g_a}{r}+\frac{n_a}{r_a}\right)\left(g_a-n_a+\sqrt{4r^2+(b-c)^2}\right)=$$

$$\left(\frac{g_ar_a+r.n_a}{r.r_a}\right)\left(\frac{g_ar_a-r.n_a}{r_a}\right)=\frac{g_a^2r_a^2-r^2n_a^2}{r.r_a^2}$$

$$\overset{\text{Bogdan Fustei}}{=}\frac{((s-a)^2+2r.h_a)r_a^2-r^2(s^2-2h_ar_a)}{r.r_a^2}$$

$$=\frac{(s-a)^2.\frac{r^2s^2}{(s-a)^2}+2r.h_ar_a^2-r^2s^2+2r^2.h_ar_a}{r.r_a^2}=\frac{2h_ar_a+2r.h_a}{r_a}=\frac{2h_a\left(\frac{rs}{s-a}+\frac{rs}{s}\right)}{r_a}$$

$$=\frac{2h_a(s-a)\frac{rs(2s-a)}{s(s-a)}}{rs}=2h_a\left(\frac{2s-a}{s}\right)=4h_a-\frac{4rs}{s}=4(h_a-r)$$

$$\therefore\frac{g_a}{r}+\frac{n_a}{r_a}=\frac{4(h_a-r)}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}\ \forall\ \Delta\ ABC\text{ (QED)}$$

3961. In any acute Δ ABC the following relationship holds :

$$p_a p_b p_c \ge \frac{(R+4r)s^2}{6}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$p_a = \frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}$$

$$\left(\begin{matrix}\text{Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba} - 68;\\ \text{relation } (\bullet\bullet\bullet)\text{; published at www.ssmrmh.ro}\end{matrix}\right)$$

$$\Rightarrow p_a \overset{\text{Walker}}{\ge} \frac{2s}{2s+a}\cdot\sqrt{2R^2+8Rr-16Rr\sin^2\frac{A}{2}} \text{ and analogs}$$

$$\Rightarrow p_a p_b p_c \ge \frac{8s^3}{2s(9s^2+6Rr+r^2)}\cdot\sqrt{\prod_{cyc}\left(2R^2+8Rr-16Rr\sin^2\frac{A}{2}\right)} \to (1)$$

$$\text{Now, } \prod_{cyc}\left(2R^2+8Rr-16Rr\sin^2\frac{A}{2}\right) =$$

$$(2R^2+8Rr)^3 - 16Rr(2R^2+8Rr)^2.\frac{2R-r}{2R} +$$

$$256R^2r^2(2R^2+8Rr)\left(\frac{(2R-r)^2}{4R^2} - \frac{2}{16R^2}\sum_{cyc} AI^2\right) -$$

$$(16)(256)R^3r^3.\frac{r^2}{16R^2}\left(\because \sum_{cyc}\sin^4\frac{A}{2} = \left(\overset{\frac{2R-r}{2R}}{\overbrace{\sum_{cyc}\sin^2\frac{A}{2}}}\right)^2 - \frac{2r^2}{16R^2}\text{cosec}^2\frac{A}{2}\right)$$

$$= (2R^2+8Rr)^3 - 8r(2R-r)(2R^2+8Rr)^2 +$$

$$32r^2(2R^2+8Rr)\left(2(2R-r)^2-(s^2-8Rr+r^2)\right) - 256Rr^5 \overset{\text{Gerretsen}}{\ge}$$

$$(2R^2+8Rr)^3 - 8r(2R-r)(2R^2+8Rr)^2 +$$

$$32r^2(2R^2+8Rr)\left(2(2R-r)^2-(4R^2-4Rr+4r^2)\right) - 256Rr^5$$

$$\overset{\text{via (1) and Gerretsen}}{\Rightarrow} p_a p_b p_c \ge$$

$$\frac{4s^2}{9(4R^2+4Rr+3r^2)+6Rr+r^2}\cdot\sqrt{\begin{matrix}(2R^2+8Rr)^3-8r(2R-r)(2R^2+8Rr)^2+\\ 32r^2(2R^2+8Rr)(2(2R-r)^2-(4R^2-4Rr+4r^2))-\\ 256Rr^5\end{matrix}}$$

$$\overset{?}{\ge} \frac{(R+4r)s^2}{6} \overset{\text{squaring and simplifying}}{\Leftrightarrow}$$

$$828t^6+1260t^5+10863t^4+53484t^3-112180t^2-121568t-3136\overset{?}{\ge}0$$

$$\left(t=\frac{R}{r}\right)\Leftrightarrow(t-2)\left(828t^5+2916t^4+16695t^3+86874t^2+61568t+1568\right)\overset{?}{\ge}0$$

$$\to \text{true}\because t\overset{\text{Euler}}{\ge}2\therefore p_ap_bp_c\ge\frac{(R+4r)s^2}{6}\ \forall\ \textit{acute}\ ABC,$$

$$''=''\ \text{iff}\ \Delta\ ABC\ \text{is equilateral (QED)}$$

3962. In *any* Δ ABC the following relationship holds :

$$|b-c|\le\min\left\{\sqrt{s^2-8Rr-11r^2},\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$|b-c|\overset{?}{\le}\sqrt{s^2-8Rr-11r^2}$$

$$\Leftrightarrow(y-z)^2\overset{?}{\le}(x+y+z)^2-\frac{8(y+z)(z+x)(x+y)}{4(x+y+z)}-\frac{11xyz}{x+y+z}$$

$$\left(\begin{array}{c}x=s-a,y=s-b,z=s-c\Rightarrow a=y+z,b=z+x,c=x+y\ \text{and}\\ s=x+y+z\end{array}\right)$$

$$\Leftrightarrow x^3+(y+z)x^2-7xyz+2yz(y+z)\overset{?}{\ge}0$$

$$\Leftrightarrow x^3+(mx)x^2-7x.nx^2+2nx^2(mx)\overset{?}{\ge}0\ \left(m=\frac{y+z}{x},n=\frac{yz}{x^2}\right)$$

$$\Leftrightarrow 1+m+n(2m-7)\boxed{\overset{?}{\underset{(*)}{\ge}}}0\ \textit{and it's trivially true if}:2m-7\ge0\ \ \textit{and when}:$$

$$2m-7<0,\textit{then}:\text{since } n\overset{AM-GM}{\le}\frac{m^2}{4}\therefore\text{in order to prove }(*),$$

$$\text{it suffices to prove}:1+m+\frac{m^2}{4}.(2m-7)\overset{?}{\ge}0\Leftrightarrow(2m+1)(m-2)^2\overset{?}{\ge}0$$

$$\to\text{true}\because m>0\ \textit{as}\ x,y,z>0\Rightarrow(*)\text{ is true}\therefore|b-c|\le\sqrt{s^2-8Rr-11r^2}\to①$$

$$\textit{and again},|b-c|\overset{?}{\le}\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}$$

$$\Leftrightarrow16(y-z)^2\overset{?}{\le}32(x+y+z)^2-\frac{437(y+z)(z+x)(x+y)}{4(x+y+z)}+\frac{10xyz}{x+y+z}$$

$$\Leftrightarrow128x^3-53(y+z)x^2-117x((y+z)^2-2yz)+62xyz+$$

$$64\left((y+z)^3-3yz(y+z)\right)+11yz(y+z)\overset{?}{\ge}0$$

$$\Leftrightarrow 128x^3-53(mx)x^2-117x((mx)^2-2nx^2)+62x.nx^2+$$
$$64\left((mx)^3-3nx^2(mx)\right)+11nx^2(mx)\overset{?}{\geq}0$$
$$\Leftrightarrow 64m^3-117m^2-53m+128+n(296-181m)\boxed{\underset{(**)}{\overset{?}{\geq}}}0$$

We shall now prove that : $64m^3-117m^2-53m+128>0\ \forall\ m>0\rightarrow$ (i)

Case 1 $m\leq\frac{761}{535}$ and then : $64m^3-117m^2-53m+128$
$$=\frac{1}{125}\left((320m+311)(5m-7)^2+761-535m\right)>0$$
$\left(\text{inequality is strict because "m" cannot be simultaneously equal to } \frac{7}{5} \text{ and } \frac{761}{535}\right)$

Case 2 $m>\frac{761}{535}$ and then : $64m^3-117m^2-53m+128$
$$=\frac{1}{729}\left((576m+611)(9m-13)^2+6993m-9947\right)>0$$
$\left(\because m>\frac{761}{535}>\frac{9947}{6993}\text{ as }(761)(6993)-(535)(9947)=28>0\right)$ and so,
combining both cases, (i) is true and hence, $(**)$ it's trivially true if :
$296-181m\geq 0$ and when : $296-181m<0$, then : since $n\overset{AM-GM}{\leq}\frac{m^2}{4}$
$\therefore$ in order to prove $(**)$, it suffices to prove :
$$64m^3-117m^2-53m+128+\frac{m^2}{4}.(296-181m)\overset{?}{\geq}0$$
$$\Leftrightarrow (75m+128)(m-2)^2\overset{?}{\geq}0\rightarrow\text{true}\because m>0\text{ as }x,y,z>0\Rightarrow(**)\text{ is true}$$
$$\therefore |b-c|\leq\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}\rightarrow ② \therefore ① \text{ and } ② \Rightarrow$$
$$|b-c|\leq\sqrt{s^2-8Rr-11r^2},\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}$$
$$\Rightarrow |b-c|\leq\min\left\{\sqrt{s^2-8Rr-11r^2},\frac{1}{4}.\sqrt{32s^2-437Rr+10r^2}\right\}$$
$\forall\ \Delta ABC$, " = " iff $y=z$ and $y+z=2x\Rightarrow$ " = " iff ΔABC is equilateral (QED)

3963. *In any ΔABC, $a=\max\{a,b,c\}$ prove that:*

$$\frac{r_b}{r_c}+\frac{r_c}{r_b}\leq\frac{s^2}{9r^2}-1$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\text{Let } s-a=x, s-b=y, s-c=z \text{ then } x+y+z=s \text{ and}$$

$$F=Area=\sqrt{(x+y+z)xyz}$$

$$a=y+z, b=x+z, c=x+y$$

$$\frac{s^2}{9r^2}=\frac{s^4}{9F^2}=\frac{(x+y+z)^3}{9xyz}\ \&\ \frac{r_b}{r_c}+\frac{r_c}{r_b}=\frac{s-c}{s-b}+\frac{s-b}{s-c}=\frac{y}{z}+\frac{z}{y}=\frac{y^2+z^2}{yz}$$

We need to show:

$$\frac{y^2+z^2}{yz}\le\frac{(x+y+z)^3}{9xyz}-1 \text{ or}, (x+y+z)^3\ge 9x(y^2+yz+z^2)\ (1)$$

$$\text{as } a=\max\{a,b,c\} \text{ then } y+z\ge x+y\ \&\ y+z\ge z+x \text{ or}, x\le y, z$$

$$\text{let } y=x+a\ \&\ z=x+b \text{ where } a,b\ge 0$$

From (1) we need to show:

$$(3x+a+b)^3\ge 9x((x+a)^2+(x+a)(x+b)+(x+b)^2)$$

$$L.H.S=(3x)^3+3(3x)^2(a+b)+3.(3x)(a+b)^2+(a+b)^3$$

$$R.H.S=9x(3x^2+3x(a+b)+(a^2+ab+b^2))$$

$$L.H.S-R.H.S=9xab+(a+b)^3\ge 0 \text{ as } a,b\ge 0$$

$$(3x+a+b)^3\ge 9x((x+a)^2+(x+a)(x+b)+(x+b)^2)$$

$$\text{Equality occurs when } a=b=0 \text{ or } x=y=z \text{ or } a=b=c.$$

3964. In ΔABC the following relationship holds:

$$\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\le\sqrt{\frac{2R}{r}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Tapas Das-India

$$\text{let } x=s-a, y=s-b, z=s-c \text{ then } a=y+z, b=x+z, c=x+y\ \&\ s=x+y+z$$

$$\frac{h_a}{r_a}+\frac{r_a}{h_a}=\frac{2(s-a)}{a}+\frac{a}{2(s-a)}=\frac{2x}{y+z}+\frac{y+z}{2x}$$

$$\frac{R}{r}=\frac{\frac{abc}{4F}}{\frac{F}{s}}=\frac{sabc}{4F^2}=\frac{(x+y)(y+z)(z+x)}{4xyz} \text{ then } \frac{2R}{r}=\frac{(x+y)(y+z)(z+x)}{2xyz}$$

We need to show:

$$\sqrt{\frac{h_a}{r_a}}+\sqrt{\frac{r_a}{h_a}}\le\sqrt{\frac{2R}{r}}\, or,\frac{h_a}{r_a}+\frac{r_a}{h_a}+2\overset{squaring}{\le}\frac{2R}{r}$$

$$\frac{2x}{y+z}+\frac{y+z}{2x}+2\le\frac{(x+y)(y+z)(z+x)}{2xyz}$$

$$2xyz\left(\frac{2x}{y+z}+\frac{y+z}{2x}+2\right)\le(x+y)(y+z)(z+x)$$

$$R.H.S=(x+y)(y+z)(z+x)=x^2y+xy^2+y^2z+yz^2+z^2x+zx^2+2xyz$$

$$L.H.S=2xyz\left(\frac{2x}{y+z}+\frac{y+z}{2x}+2\right)=\frac{4x^2yz}{y+z}+y^2z+yz^2+4xyz$$

$$L.H.S-R.H.S=x^2y+xy^2+z^2x+zx^2-\frac{4x^2yz}{y+z}-2xyz$$

$$=x^2(y+z)+x(y^2+z^2)-2xyz-\frac{4x^2yz}{y+z}=x^2(y+z)+x(y+z)^2-4xyz-\frac{4x^2yz}{y+z}$$

$$=x(y+z)(x+y+z)-4xyz\left(\frac{x+y+z}{y+z}\right)=x(x+y+z)\left((y+z)-\frac{4yz}{y+z}\right)$$

$$=\frac{x(x+y+z)((y+z)^2-4yz)}{y+z}=\frac{x(x+y+z)(y-z)^2}{y+z}\ge 0$$

Equality holds for b=c.

3965. In any Δ ABC the following relationship holds :

$$\max\{a,b,c\}-\min\{a,b,c\}\ge\frac{\sqrt{s^2-27r^2}}{2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(\max\{a,b,c\}-\min\{a,b,c\})^2=\left(\sum_{cyc}\frac{|b-c|}{2}\right)^2$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 45; published at www.ssmrmh.ro)

$$=\frac{1}{4}\sum_{cyc}(b-c)^2+\frac{2}{4}\sum_{cyc}(|b-c||c-a|)\overset{Triangle\ Inequality}{\ge}$$

$$\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\frac{1}{2}\cdot\left|\sum_{cyc}((b-c)(c-a))\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\frac{1}{2}\cdot\left|\sum_{cyc}bc-\sum_{cyc}ab-\sum_{cyc}c^2+\sum_{cyc}ca\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)+\frac{1}{2}\left(\sum_{cyc}a^2-\sum_{cyc}ab\right)\left(\because\sum_{cyc}a^2\geq\sum_{cyc}ab\right)$$

$$=2(s^2-4Rr-r^2)-(s^2+4Rr+r^2)=s^2-12Rr-3r^2\overset{?}{\geq}\frac{s^2-27r^2}{4}$$

$$\Leftrightarrow s^2\overset{?}{\geq}16Rr-5r^2\to\text{true via Gerretsen}$$

$$\therefore max\{a,b,c\}-\min\{a,b,c\}\geq\frac{\sqrt{s^2-27r^2}}{2}\ \forall\Delta ABC,$$

$$''=''\ \text{iff }\Delta ABC\text{ is equilateral (QED)}$$

3966. In any Δ ABC the following relationship holds :

$$max\{r_a,r_b,r_c\}-\min\{r_a,r_b,r_c\}\geq 2\sqrt{R^2-Rr-2r^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(max\{r_a,r_b,r_c\}-\min\{r_a,r_b,r_c\})^2=\left(\sum_{cyc}\frac{|r_b-r_c|}{2}\right)^2$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 45; published at www.ssmrmh.ro)

$$=\frac{1}{4}\sum_{cyc}(r_b-r_c)^2+\frac{2}{4}\sum_{cyc}(|r_b-r_c||r_c-r_a|)\overset{\text{Triangle Inequality}}{\geq}$$

$$\frac{1}{2}\left(\sum_{cyc}r_a^2-\sum_{cyc}r_ar_b\right)+\frac{1}{2}\cdot\left|\sum_{cyc}((r_b-r_c)(r_c-r_a))\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}r_a^2-\sum_{cyc}r_ar_b\right)+\frac{1}{2}\cdot\left|\sum_{cyc}r_br_c-\sum_{cyc}r_ar_b-\sum_{cyc}r_c^2+\sum_{cyc}r_cr_a\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}r_a^2-\sum_{cyc}r_ar_b\right)+\frac{1}{2}\left(\sum_{cyc}r_a^2-\sum_{cyc}r_ar_b\right)\left(\because\sum_{cyc}r_a^2\geq\sum_{cyc}r_ar_b\right)$$

$$= (4R+r)^2 - 3s^2 \overset{\text{Gerretsen}}{\geq} (4R+r)^2 - 3(4R^2+4Rr+3r^2) = 4(R^2 - Rr - 2r^2)$$

$$\therefore max\{r_a, r_b, r_c\} - \min\{r_a, r_b, r_c\} \geq 2\sqrt{R^2 - Rr - 2r^2}\ \forall \Delta\ ABC,$$

$$'' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3967. In any Δ ABC the following relationship holds :

$$8 \leq \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \frac{8}{729}\left(\frac{4R}{r} + 1\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc} \frac{m_b + m_c}{m_a} = \frac{(\sum_{cyc} m_a)(\sum_{cyc} m_a m_b) - m_a m_b m_c}{m_a m_b m_c}$$

$$= \left(\sum_{cyc} m_a\right)\left(\sum_{cyc} \frac{1}{m_a}\right) - 1 \leq \left(\sum_{cyc} m_a\right)\left(\sum_{cyc} \frac{1}{h_a}\right) - 1 \overset{\text{Bager}}{\leq} \frac{4R+r}{r} - 1$$

$$\therefore \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \frac{4R}{r} \to (1) \text{ and again, } \frac{8}{729}\left(\frac{4R}{r}+1\right)^3 \overset{\text{Euler}}{\geq} \frac{8}{81}\left(\frac{4R}{r}+1\right)^2 \overset{?}{\geq} \frac{4R}{r}$$

$$\Leftrightarrow 2(4R+r)^2 \overset{?}{\geq} 81Rr \Leftrightarrow 32R^2 - 65Rr + 2r^2 \overset{?}{\geq} 0 \Leftrightarrow (32R - r)(R - 2r) \overset{?}{\geq} 0$$

$$\to \text{true via Euler } \therefore \frac{8}{729}\left(\frac{4R}{r}+1\right)^3 \geq \frac{4R}{r} \overset{\text{via }(1)}{\geq} \prod_{cyc} \frac{m_b + m_c}{m_a} \text{ and since}$$

$$\prod_{cyc} \frac{m_b + m_c}{m_a} \overset{\text{Cesaro}}{\geq} 8 \text{ and so, } 8 \leq \prod_{cyc} \frac{m_b + m_c}{m_a} \leq \frac{8}{729}\left(\frac{4R}{r}+1\right)^3\ \forall \Delta\ ABC,$$

$$'' ='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3968. In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{bc}{m_a^2} \geq \frac{8r}{R}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{bc}{m_a^2}\overset{Panaitopol}{\geq}\sum_{cyc}\frac{bc}{\left(\frac{R^2s^2}{a^2}\right)}=\frac{abc}{R^2s^2}\cdot\sum_{cyc}a=\frac{(4Rrs)(2s)}{R^2s^2}=\frac{8r}{R}\ \forall\Delta ABC,$$

$$''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

3969.

In any Δ ABC the following relationship holds :

$$-1-\frac{2r^2}{R^2}\leq\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\leq 3-\frac{18r^2}{R^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}=\sum_{cyc}\frac{\cos^2 A-\sin^2 A}{\cos^2 A+\sin^2 A}=\sum_{cyc}\cos 2A$$

$$=-1-4\cos A\cos B\cos C=-1-\frac{s^2-4R^2-4Rr-r^2}{R^2}$$

$$\therefore\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}=\frac{3R^2+4Rr+r^2-s^2}{R^2}\ \text{and}\ \because s^2\overset{Gerretsen}{\geq}16Rr-5r^2\ \text{and}$$

$$s^2\overset{Gerretsen}{\leq}4R^2+4Rr+3r^2\ \therefore\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\leq\frac{3R^2+4Rr+r^2-16Rr+5r^2}{R^2}$$

$$=3-\frac{12Rr-6r^2}{R^2}\overset{Euler}{\leq}3-\frac{24r^2-6r^2}{R^2}=3-\frac{18r^2}{R^2}\ \text{and}$$

$$\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\geq\frac{3R^2+4Rr+r^2-(4R^2+4Rr+3r^2)}{R^2}=-1-\frac{2r^2}{R^2}\ \text{and so,}$$

$$-1-\frac{2r^2}{R^2}\leq\sum_{cyc}\frac{1-\tan^2 A}{1+\tan^2 A}\leq 3-\frac{18r^2}{R^2}\ \forall\Delta ABC,$$

$$''=''\ \text{iff}\ \Delta ABC\ \text{is equilateral (QED)}$$

3970. In any Δ ABC the following relationship holds :

$$max\{h_a,h_b,h_c\}-\min\{h_a,h_b,h_c\}\geq\frac{s^2-12Rr-3r^2}{2R}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(max\{h_a,h_b,h_c\}-\min\{h_a,h_b,h_c\})^2=\left(\sum_{cyc}\frac{|h_b-h_c|}{2}\right)^2$$

$\left(\text{Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba} - 45;\ \text{published at www.ssmrmh.ro}\right)$

$$=\frac{1}{4}\sum_{cyc}(h_b-h_c)^2+\frac{2}{4}\sum_{cyc}(|h_b-h_c||h_c-h_a|)\overset{\text{Triangle Inequality}}{\geq}$$

$$\frac{1}{2}\left(\sum_{cyc}h_a^2-\sum_{cyc}h_ah_b\right)+\frac{1}{2}\cdot\left|\sum_{cyc}((h_b-h_c)(h_c-h_a))\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}h_a^2-\sum_{cyc}h_ah_b\right)+\frac{1}{2}\cdot\left|\sum_{cyc}h_bh_c-\sum_{cyc}h_ah_b-\sum_{cyc}h_c^2+\sum_{cyc}h_ch_a\right|$$

$$=\frac{1}{2}\left(\sum_{cyc}h_a^2-\sum_{cyc}h_ah_b\right)+\frac{1}{2}\left(\sum_{cyc}h_a^2-\sum_{cyc}h_ah_b\right)\left(\because\sum_{cyc}h_a^2\geq\sum_{cyc}h_ah_b\right)$$

$$=\frac{(s^2+4Rr+r^2)^2-16Rrs^2-8Rrs^2}{4R^2}\overset{?}{\geq}\frac{(s^2-12Rr-3r^2)^2}{4R^2}$$

$$\Leftrightarrow (R+r)s^2\underbrace{\overset{?}{\geq}}_{(*)}r(4R+r)^2$$

$$\text{Now, }(R+r)s^2\overset{\text{Gerretsen}}{\geq}(R+r)(16Rr-5r^2)\overset{?}{\geq}r(4R+r)^2\Leftrightarrow 3r^2(R-2r)\overset{?}{\geq}0$$

$$\to \text{true} \because R\overset{\text{Euler}}{\geq}2r\Rightarrow(*)\text{ is true}\therefore max\{h_a,h_b,h_c\}-\min\{h_a,h_b,h_c\}\geq$$

$$\frac{s^2-12Rr-3r^2}{2R}\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

3971. In any Δ ABC holds :

$$|r_b-r_c|\leq 4\sqrt{R^2-(\sqrt{2}+1)Rr+2(\sqrt{2}-1)r^2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via }(r_b-r_c)^2=\frac{s(s-a)}{(s-b)(s-c)},\ R=\frac{abc}{4.\sqrt{s(s-a)(s-b)(s-c)}}\text{ and}$$

$$r=\frac{\sqrt{s(s-a)(s-b)(s-c)}}{s},\ (r_b-r_c)^2\overset{?}{\leq}R^2-(\sqrt{2}+1)Rr+2(\sqrt{2}-1)r^2$$

$$\Leftrightarrow \frac{16\big(sabc-8s(s-a)(s-b)(s-c)\big)\big(sabc+4s(s-a)(s-b)(s-c)\big)}{4s.4s.s(s-a)(s-b)(s-c)}-$$

$$\frac{s^2(s-a)^2(b-c)^2}{s(s-a)(s-b)(s-c)}\overset{?}{\geq}16\sqrt{2}.\frac{F}{s}.\frac{sabc-8s(s-a)(s-b)(s-c)}{4s.F}$$

$$\Leftrightarrow 16x^8+32x^7(y+z)-256x^6.yz-32x^5\left((y+z)^3-3yz(y+z)\right)+$$
$$+128x^5yz(y+z)-12x^4(((y+z)^2-2yz)^2-2y^2z^2)+248x^4.yz((y+z)^2-2yz)$$
$$-504x^4.y^2z^2+8x^3(y+z)(((y+z)^2-2yz)^2-yz(y+z)^2+y^2z^2)-$$
$$160x^3.yz\left((y+z)^3-3yz(y+z)\right)+248x^3.y^2z^2(y+z)+$$
$$4x^2((y+z)^2-2yz)(((y+z)^2-2yz)^2-3y^2z^2)+$$
$$12x^2.yz(((y+z)^2-2yz)^2-2y^2z^2)-68x^2.y^2z^2(y+z)^2-16x^2.y^3z^3+$$
$$4x.yz(y+z)(((y+z)^2-2yz)^2-yz(y+z)^2+y^2z^2)+$$
$$12x.y^2z^2\left((y+z)^3-3yz(y+z)\right)+16x.y^3z^3(y+z)+$$
$$y^2z^2(((y+z)^2-2yz)^2-2y^2z^2)+4y^3z^3((y+z)^2-2yz)+6y^4z^4\overset{?}{\geq}0$$

$$\left(\begin{matrix}x=s-a,y=s-b,z=s-c\Rightarrow\\ a=y+z,b=z+x,c=x+y\ \textit{and}\ s=x+y+z\end{matrix}\right)$$

$$\Leftrightarrow\Big(\overbrace{m^4-8m^3-80m^2+768m-1024}^{\sigma_1}\Big)n^2+$$

$$\Big(\overbrace{4m^5-12m^4-200m^3+296m^2+224m-256}^{\sigma_2}\Big)n+$$

$$\overbrace{4m^6+8m^5-12m^4-32m^3+32m+16}^{\sigma_3}\boxed{\overset{?}{\geq}_{(*)}}0\ \left(m=\frac{y+z}{x},n=\frac{yz}{x^2}\right)$$

$$\textit{and}\because \text{LHS of }(*)=(m-8)^2(m^2+8(m-8)+48)n^2+$$
$$(m-8)\left((m-8)(4m^3+52m^2+376m+2984)+23904\right)n+$$
$$4\left((m-8)\big(m^5+10m^4+77m^3+608m^2+4864m+38920\big)+311364\right)>0$$

whenever $m\geq 8\ \therefore$ we now focus on the case when : $m<8$

Case 1 $m^4-8m^3-80m^2+768m-1024>0$ (*and* $m<8$) *and* then,

$\because\ \delta=\sigma_2^2-4\sigma_1\sigma_3=128(m+1)^2(m-2)^4(m-8)^2\geq 0\ \therefore$ in order to prove $(*)$,

it suffices to prove : $2\sigma_1.n\overset{?}{\leq}-\sigma_2-\sqrt{\delta}$ *and* $\because n\overset{AM-GM}{\leq}\frac{m^2}{4}\ \therefore$ it suffices to prove :

$$2\sqrt{\delta}\overset{?}{\leq}-2\sigma_2-m^2\sigma_1=(m-2)^2\overbrace{(8-m)}^{>0}(m^3+12m^2+4m+16)$$

$$\Leftrightarrow(m-2)^4(m-8)^2(m^3+12m^2+4m+16)\overset{?}{\geq}512(m+1)^2(m-2)^4(m-8)^2$$
$$\Leftrightarrow(m-2)^4(m-8)^2\big(m^6+24m^5+152m^4+128m^3-112m^2-896m-256\big)$$
$$\overset{?}{\geq}0\Leftrightarrow(m-2)^4(m-8)^2.\frac{\left(\begin{matrix}m^4-8m^3-80m^2+\\768m-1024\end{matrix}\right)\left(\begin{matrix}m^4+16m^3+\\40m^2+64m+16\end{matrix}\right)}{(m-8)^2}\overset{?}{\geq}0$$

$\rightarrow$ true $\because m^4-8m^3-80m^2+768m-1024>0$ and $m<8$

Case 2 $\sigma_1=m^4-8m^3-80m^2+768m-1024<0$ (and $m<8$) and then :

$(*)\Leftrightarrow(-\sigma_1)n^2-\sigma_2 n-\sigma_3 \underset{(**)}{\overset{?}{\geq}} 0$ and then $\because$ discriminiant $=\sigma_2^2-4\sigma_1\sigma_3=\delta\geq 0$

$\therefore$ to prove $(**)$, suffices to prove : $-2\sigma_1.n \underset{①}{\overset{?}{\leq}} \sigma_2+\sqrt{\delta}$ AND $-2\sigma_1.n \underset{②}{\overset{?}{\geq}} \sigma_2-\sqrt{\delta}$

$\because -\sigma_1>0$ and $n\leq\frac{m^2}{4}$ $\therefore$ to prove ①, it suffices to prove : $-2\sigma_2-m^2\sigma_1\overset{?}{\leq}2\sqrt{\delta}$

$$\Leftrightarrow(m-2)^2\overbrace{(8-m)}^{>0}(m^3+12m^2+4m+16)\overset{?}{\leq}2\sqrt{\delta}\Leftrightarrow$$

$$(m-2)^4(m-8)^2(m^3+12m^2+4m+16)^2\overset{?}{\leq}512(m+1)^2(m-2)^4(m-8)^2$$

$$\Leftrightarrow(m-2)^4(m-8)^2.\frac{\begin{pmatrix}m^4-8m^3-80m^2+\\768m-1024\end{pmatrix}\begin{pmatrix}m^4+16m^3+\\40m^2+64m+16\end{pmatrix}}{(m-8)^2}\overset{?}{\leq}0\rightarrow\text{true}$$

$\because m^4-8m^3-80m^2+768m-1024<0$ and $m<8\Rightarrow$ ① is true

and again, since : $-2\sigma_1.n>0$ $\therefore$ *in order to prove* ②, it suffices to prove :

$\sqrt{\delta}\overset{?}{\geq}\sigma_2$ and it's trivially true if $\sigma_2<0$ and when : $\sigma_2\geq 0$, it suffices to prove :

$$128(m+1)^2(m-2)^4(m-8)^2\overset{?}{\geq}\begin{pmatrix}4m^5-12m^4-200m^3+296m^2+\\224m-256\end{pmatrix}^2$$

$$\Leftrightarrow-16(m+1)^2(m-8)^2(m^6+8m^5-20m^4-32m^3+68m^2+32m-64)\overset{?}{\geq}0$$

$$\Leftrightarrow-16(m+1)^2(m-8)^2.\frac{(m^4-8m^3-80m^2+768m-1024)(m^2-2)^2}{(m-8)^2}\overset{?}{\geq}0$$

$\rightarrow$ true $\because m^4-8m^3-80m^2+768m-1024<0$ and $m<8\Rightarrow$ ② is true

$\therefore$ combining cases ① and ②, $(*)$ is true $\forall\, m,n>0\,|\,m^2\geq 4n$ and so,

$$|r_b-r_c|\leq 4.\sqrt{R^2-(\sqrt{2}+1)Rr+2(\sqrt{2}-1)r^2}\ \forall\,\Delta\,ABC,$$

$''=''$ iff $y=z$ and $y+z=2x\Rightarrow''=''$ iff Δ ABC is equilateral (QED)

3972. In any Δ ABC the following relationship holds :

$$\left(\sum_{cyc}\frac{m_a}{w_b+w_c}\right)\left(\sum_{cyc}\frac{w_b+w_c}{\sqrt{r_br_c}}\right)\geq 9$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\frac{m_a}{w_b+w_c}\right)\left(\sum_{cyc}\frac{w_b+w_c}{\sqrt{r_br_c}}\right)\overset{\text{Reverse CBS}}{\geq}\left(\sum_{cyc}\frac{\sqrt{m_a}}{\sqrt[4]{r_br_c}}\right)^2\overset{\text{AM-GM}}{\geq}$$

$$9.\left(\frac{\sqrt{m_am_bm_c}}{\sqrt[4]{s(s-a).s(s-b).s(s-c)}}\right)^{\frac{2}{3}}\overset{\text{Lascu + AM-GM}}{\geq}9.\left(\frac{\sqrt[4]{s(s-a).s(s-b).s(s-c)}}{\sqrt[4]{s(s-a).s(s-b).s(s-c)}}\right)^{\frac{2}{3}}$$

$$=9\text{ and so},\left(\sum_{cyc}\frac{m_a}{w_b+w_c}\right)\left(\sum_{cyc}\frac{w_b+w_c}{\sqrt{r_br_c}}\right)\geq 9\ \forall\,\Delta\,ABC,$$

$$''=''\ \text{iff }\Delta\,ABC\text{ is equilateral (QED)}$$

3973. In ΔABC the following relationship holds:

$$\frac{12}{a+b+c}\leq\sum\frac{a}{r_a^2}\leq\frac{4}{a+b+c}\left(\frac{2R}{r}-1\right)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{a}{r_a^2}=\frac{1}{F^2}\sum a(s-a)^2=\frac{1}{F^2}\sum a(s^2-2sa+a^2)=$$

$$=\frac{1}{r^2s^2}\left(2s^3-4s(s^2-r^2-4Rr)+2s(s^2-3r^2-6Rr)\right)=$$

$$=\frac{2s(2Rr-r^2)}{r^2s^2}=\frac{4}{2s}\left(\frac{2Rr-r^2}{r^2}\right)=\frac{4}{a+b+c}\left(\frac{2R}{r}-1\right)$$

$$\sum\frac{a}{r_a^2}=\frac{4}{a+b+c}\left(\frac{2R}{r}-1\right)\overset{EULER}{\geq}\frac{4}{a+b+c}(2\times 2-1)=\frac{12}{a+b+c}$$

Equality holds for an equilateral triangle.

3974. In ΔABC the following relationship holds:

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3}\geq 8+\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

$$\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\overset{CBS}{\le}\sqrt{\left(\sum m_a^2\right)\left(\sum\frac{1}{m_a^2}\right)}\overset{m_a\ge\sqrt{s(s-a)}}{\le}\sqrt{\left(\frac{3}{4}\sum a^2\right)\left(\sum\frac{1}{s(s-a)}\right)}\le$$

$$\overset{Leibniz}{\le}\sqrt{\left(\frac{3}{4}\times 9R^2\right)\left(\frac{4R+r}{s^2r}\right)}\overset{Doucet}{\le}\sqrt{\left(\frac{3}{4}\times 9R^2\right)\left(\frac{4R+r}{3r(4R+r)r}\right)}=\frac{3R}{2r}$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\overset{AM-GM}{\ge}3$$

$$We\ need\ to\ show:$$

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}+\frac{R^3}{r^3}.\ge 8+\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}$$

$$or, 3+\frac{R^3}{r^3}\ge 8+\frac{3R}{2r}\ or, 2x^3-3x-10\overset{\substack{\frac{R}{r}=x\ge 2\\ by\ Euler}}{\ge}0\ or, (x-2)(2x^2+4x+5)\ge 0\ true$$

Equality holds for an equilateral triangle.

3975. In ΔABC the following relationship holds:

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=2$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{C}{2}\cos\frac{A}{2}}+\frac{\sin\frac{C}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=$$

$$=\sum_{cy}\frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}}=\sum_{cyc}\sqrt{\frac{(s-b)(s-c)}{bc}}\cdot\sqrt{\frac{ac\cdot ab}{s(s-b)\cdot s(s-c)}}=$$

$$=\sum_{cyc}\sqrt{\frac{a^2}{s^2}}=\frac{1}{s}\sum_{cyc}a=\frac{a+b+c}{s}=\frac{2s}{s}=2$$

3976. **In ΔABC the following relationship holds:**

$$\sin^2 A\sin 2B+\sin^2 B\sin 2A=2\sin A\sin B\sin C$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin^2 A\sin 2B+\sin^2 B\sin 2A=$$

$$=\sin^2 A\cdot 2\sin B\cos B+\sin^2 B\cdot 2\sin A\cos A=$$

$$=2\sin A\sin B\,(\sin A\cos B+\sin B\cos A)=$$

$$=2\sin A\sin B\cdot\sin(A+B)=2\sin A\sin B\cdot\sin(\pi-C)=2\sin A\sin B\sin C$$

3977. **In ΔABC the following relationship holds:**

$$\cot A+\cot B+\cot C=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{\sin 2A+\sin 2B+\sin 2C}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin 2A+\sin 2B+\sin 2C=2\sin\frac{2A+2B}{2}\cos\frac{2A-2B}{2}+\sin 2C=$$

$$=2\sin(A+B)\cos(A-B)+2\sin C\cos C=$$

$$=2\sin(\pi-C)\cos(A-B)+2\sin C\cos C=2\sin C\,(\cos(A-B)+\cos C)=$$

$$=2\sin C\cdot 2\cos\frac{A-B+C}{2}\cos\frac{A-B-C}{2}=$$

$$=4\sin C\cos\frac{\pi-2B}{2}\cos\frac{2A-\pi}{2}=4\sin C\cos\left(\frac{\pi}{2}-B\right)\cos\left(\frac{\pi}{2}-A\right)=$$

$$=4\sin C\sin B\sin A$$

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C\quad (1)$$

$$\cot A+\cot B+\cot C=\sum_{cyc}\cot A=\sum_{cyc}\frac{\cos A}{\sin A}=\sum_{cyc}\frac{b^2+c^2-a^2}{2bc\sin A}=$$

$$=\frac{1}{4F}\sum_{cyc}(b^2+c^2-a^2)=\frac{1}{4F}\sum_{cyc}a^2=\frac{1}{4F}\sum_{cyc}4R^2\sin^2 A=$$

$$=\frac{4R^2(\sin^2 A+\sin^2 B+\sin^2 C)}{4\cdot 2R^2\sin A\sin B\sin C}=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{4\sin A\sin B\sin C}\overset{(1)}{=}$$

$$=\frac{2(\sin^2 A+\sin^2 B+\sin^2 C)}{\sin 2A+\sin 2B+\sin 2C}$$

3978.

In any Δ ABC the following relationship holds :

$$\left(\frac{s}{r}\right)^2\le\left(\frac{l_a l_b l_c}{g_a g_b g_c}\right)^{\frac{1}{2}}\prod_{cyc}\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}-(4r^2+(b-c)^2)^{\frac{1}{2}}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$m_a n_a\overset{?}{\ge}p_a^2+\frac{(b-c)^2}{18}\overset{\substack{\text{Fustei}\\ \text{and}\\ \text{Ajiba}}}{\Leftrightarrow}$$

$$\left(s(s-a)+\frac{(b-c)^2}{4}\right)\left(s(s-a)+\frac{s(b-c)^2}{a}\right)\overset{?}{\ge}\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)^2$$

$$+\frac{(b-c)^4}{324}+\frac{(b-c)^2}{9}.\left(s(s-a)+\frac{s(3s+a)(b-c)^2}{(2s+a)^2}\right)$$

$$\Leftrightarrow s(s-a)(b-c)^2\left(\frac{s}{a}+\frac{1}{4}\right)+\frac{s(b-c)^4}{4a}\overset{?}{\ge}\frac{s^2(3s+a)^2(b-c)^4}{(2s+a)^4}+$$

$$2s(s-a).\frac{s(3s+a)(b-c)^2}{(2s+a)^2}+\frac{(b-c)^4}{324}+s(s-a).\frac{(b-c)^2}{9}+\frac{s(3s+a)(b-c)^4}{9(2s+a)^2}$$

$$\Leftrightarrow s(s-a)\left(\frac{s}{a}+\frac{1}{4}-\frac{2s(3s+a)(b-c)^2}{(2s+a)^2}-\frac{1}{9}\right)+$$

$$\left(\frac{s}{4a}-\frac{s^2(3s+a)^2}{(2s+a)^4}-\frac{1}{324}-\frac{s(3s+a)}{9(2s+a)^2}\right)(b-c)^2\overset{?}{\ge}0\ (\because (b-c)^2\ge 0)$$

$$\Leftrightarrow\frac{s(s-a)(144s^3-52s^2a-16sa^2+5a^3)}{36a(2s+a)^2}+$$

$$\frac{1296s^5-772s^4a-608s^3a^2+48s^2a^3+37sa^4-a^5}{324a(2s+a)^4}.(b-c)^2\overset{?}{\ge}0$$

$$\Leftrightarrow \frac{s(s-a)\left((s-a)(144s^2+92sa+76a^2)+81a^3\right)}{36a(2s+a)^2}+$$

$$\frac{(s-a)\left(\begin{matrix}(s-a)(1296s^3+1820s^2a+1736sa^2+1700a^3)+\\1701a^4\end{matrix}\right)}{324a(2s+a)^4}.(b-c)^2 \overset{?}{\geq} 0$$

$$\rightarrow \text{true (strict inequality)} \therefore m_a n_a \geq p_a^2 + \frac{(b-c)^2}{18} \geq p_a^2 \Rightarrow m_a n_a \geq p_a^2 \rightarrow ①$$

$$\text{and } n_a g_a \overset{\text{Bogdan Fustei}}{\geq} m_a l_a \rightarrow ② \therefore ①\bullet② \Rightarrow (m_a n_a)(n_a g_a) \geq p_a^2.m_a l_a$$

$$\Rightarrow n_a \geq p_a.\sqrt{\frac{l_a}{g_a}} \Rightarrow \left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}.\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}-(4r^2+(b-c)^2)^{\frac{1}{2}}} = \frac{1}{1-\frac{\sqrt{4r^2+(b-c)^2}}{p_a.\sqrt{\frac{l_a}{g_a}}}}$$

$$\overset{\text{Bogdan Fustei}}{=} \frac{1}{1-\frac{\left(\frac{an_a}{s}\right)}{p_a.\sqrt{\frac{l_a}{g_a}}}} \geq \frac{1}{1-\frac{\left(\frac{an_a}{s}\right)}{n_a}} = \frac{s}{s-a} \therefore \left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}.\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}-(4r^2+(b-c)^2)^{\frac{1}{2}}} \geq$$

$$\frac{s}{s-a}\prod_{cyc}\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}-(4r^2+(b-c)^2)^{\frac{1}{2}}} \geq \prod_{cyc}\frac{s}{s-a} = \frac{s^3}{r^2 s} = \frac{s^2}{r^2} \text{ and so,}$$

$$\left(\frac{s}{r}\right)^2 \leq \left(\frac{l_a l_b l_c}{g_a g_b g_c}\right)^{\frac{1}{2}} \prod_{cyc}\frac{p_a}{p_a\left(\frac{l_a}{g_a}\right)^{\frac{1}{2}}-(4r^2+(b-c)^2)^{\frac{1}{2}}} \;\forall\, \Delta\, ABC,$$

$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3980. In any Δ ABC the following relationship holds :

$$1 \geq \frac{2m_a - g_a - \sqrt{4r^2+(b-c)^2}}{n_a} + \frac{2m_b - g_b - \sqrt{4r^2+(c-a)^2}}{n_b} +$$

$$+\frac{2m_c - g_c - \sqrt{4r^2+(a-b)^2}}{n_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{2m_a-g_a-\sqrt{4r^2+(b-c)^2}}{n_a}\overset{\text{Bogdan Fustei}}{\le}$$

$$\sum_{cyc}\frac{n_a+g_a-g_a-\sqrt{4r^2+(b-c)^2}}{n_a}\overset{\text{Bogdan Fustei}}{=}\sum_{cyc}\frac{n_a-\frac{an_a}{s}}{n_a}=\sum_{cyc}\left(1-\frac{a}{s}\right)$$

$$=3-\frac{2s}{s}=1 \text{ and so, } 1\ge\frac{2m_a-g_a-\sqrt{4r^2+(b-c)^2}}{n_a}+$$

$$\frac{2m_b-g_b-\sqrt{4r^2+(c-a)^2}}{n_b}+\frac{2m_c-g_c-\sqrt{4r^2+(a-b)^2}}{n_c}\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3981.

In any triangle ABC with the area F the following inequality holds :

$$\frac{a^4+b^4}{m_c^2}+\frac{b^4+c^4}{m_a^2}+\frac{c^4+a^4}{m_b^2}\ge\frac{32\sqrt{3}}{3}F$$

Proposed by D.M.Bătinețu-Giurgiu, Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$LHS=\sum_{cyc}\frac{b^4+c^4}{m_a^2}\ge\sum_{cyc}\frac{(b^2+c^2)^2}{2m_a^2}\overset{\text{Bergstrom}}{\ge}\frac{\left(\sum_{cyc}(b^2+c^2)\right)^2}{2\sum_{cyc}m_a^2}=$$

$$=\frac{2\left(\sum_{cyc}a^2\right)^2}{\frac{3}{4}\sum_{cyc}a^2}=\frac{8}{3}\cdot\sum_{cyc}a^2\overset{\text{Ionescu-Weitzenbock}}{\ge}\frac{8}{3}.4\sqrt{3}.F=\frac{32\sqrt{3}}{3}F \text{ and so,}$$

$$\frac{a^4+b^4}{m_c^2}+\frac{b^4+c^4}{m_a^2}+\frac{c^4+a^4}{m_b^2}\ge\frac{32\sqrt{3}}{3}F\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3982. In any Δ ABC, the following relationship holds :

$$\frac{\sqrt{s^2+37r^2}}{4r}\ge\frac{m_a}{h_a}+\frac{h_a}{m_a}\ge max\left\{\frac{w_a}{h_a}+\frac{h_a}{w_a},\frac{m_a}{w_a}+\frac{w_a}{m_a}\right\}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{\sqrt{s^2+37r^2}}{4r} \overset{?}{\geq} \frac{m_a}{h_a}+\frac{h_a}{m_a} \Leftrightarrow \frac{s^2+37r^2}{16r^2}-4 \overset{?}{\geq} \frac{m_a^2}{h_a^2}+\frac{h_a^2}{m_a^2}-2$$

$$\Leftrightarrow \frac{s^2-27r^2}{16r^2} \overset{?}{\geq} \frac{\left(\left(s(s-a)+\frac{(b-c)^2}{4}\right)-\left(s(s-a)-s(s-a).\frac{(b-c)^2}{a^2}\right)\right)^2}{h_a^2m_a^2}$$

$$\Leftrightarrow \frac{s^2-27r^2}{16r^2} \overset{?}{\geq} \frac{(b-c)^4((b+c)^2-a^2+a^2)^2}{16a^4.\frac{4r^2s^2}{a^2}.m_a^2} \Leftrightarrow s^2-27r^2 \overset{?}{\underset{(*)}{\geq}} \frac{(b-c)^4(b+c)^4}{4a^2m_a^2s^2}$$

and now, $4a^2m_a^2-(b^2-c^2)^2 = a^2(2b^2+2c^2-a^2)-b^4-c^4+2b^2c^2 = 2\sum_{cyc}a^2b^2-\sum_{cyc}a^4 = 16r^2s^2 > 0 \Rightarrow 4a^2m_a^2 > (b-c)^2(b+c)^2$ *and* so,

in order to prove $(*)$, it suffices to prove : $s^2(s^2-27r^2) \overset{?}{\geq} (b^2-c^2)^2$

$$\Leftrightarrow (x+y+z)^4-27xyz(x+y+z) \overset{?}{\geq} ((z+x)^2-(x+y)^2)^2$$

$$\left(\begin{matrix} x=s-a, y=s-b, z=s-c \Rightarrow \\ a=y+z, b=z+x, c=x+y \text{ and } s=x+y+z \end{matrix}\right)$$

$$\Leftrightarrow x^4+4x^3(y+z)+2x^2(y+z)^2-11x^2yz-11xyz(y+z)+4yz(y+z)^2 \overset{?}{\geq} 0$$

$$\Leftrightarrow x^4+4x^3.mx+2x^2(mx)^2-11x^2.nx^2-11x.nx^2(mx)+4nx^2(mx)^2 \overset{?}{\geq} 0$$

$$\left(m=\frac{y+z}{x}, n=\frac{yz}{x^2}\right) \Leftrightarrow 2m^2+4m+1+n(4m^2-11m-11) \overset{?}{\underset{(**)}{\geq}} 0$$

and it's trivi*ally* true if : $4m^2-11m-11 \geq 0$ *and* when : $4m^2-11m-11 < 0$,

then, *as* $n \overset{AM-GM}{\leq} \frac{m^2}{4}$ ∴ in order to prove $(**)$, it suffices to prove :

$$2m^2+4m+1+\frac{m^2}{4}(4m^2-11m-11) \overset{?}{\geq} 0 \Leftrightarrow \frac{(m+1)(4m+1)(m-2)^2}{4} \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because m > 0 \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \boxed{\frac{m_a}{h_a}+\frac{h_a}{m_a} \geq \frac{\sqrt{s^2+37r^2}}{4r}}$$

$$\text{Now, } \frac{m_a}{h_a}+\frac{h_a}{m_a} \overset{?}{\geq} \frac{w_a}{h_a}+\frac{h_a}{w_a} \Leftrightarrow \frac{m_a^2+h_a^2}{m_ah_a} \overset{?}{\geq} \frac{w_a^2+h_a^2}{w_ah_a} \Leftrightarrow \frac{m_a^2+h_a^2}{w_a^2+h_a^2}-1 \overset{?}{\geq} \frac{m_a}{w_a}-1$$

$$\Leftrightarrow \frac{(m_a-w_a)(m_a+w_a)}{w_a^2+h_a^2} \overset{?}{\geq} \frac{m_a-w_a}{w_a} \Leftrightarrow w_a(m_a+w_a) \overset{?}{\geq} w_a^2+h_a^2$$

$$(\because m_a-w_a \geq 0) \Leftrightarrow m_aw_a \overset{?}{\geq} h_a^2 \rightarrow \text{true} \because m_a, w_a \geq h_a \therefore \frac{m_a}{h_a}+\frac{h_a}{m_a} \overset{①}{\geq} \frac{w_a}{h_a}+\frac{h_a}{w_a}$$

$$\text{Again, } \frac{m_a}{h_a}+\frac{h_a}{m_a} \overset{?}{\geq} \frac{m_a}{w_a}+\frac{w_a}{m_a} \Leftrightarrow \frac{m_a^2+h_a^2}{m_ah_a} \overset{?}{\geq} \frac{m_a^2+w_a^2}{w_am_a} \Leftrightarrow \frac{w_a}{h_a}-1 \overset{?}{\geq} \frac{m_a^2+w_a^2}{m_a^2+h_a^2}-1$$

$$\Leftrightarrow \frac{w_a-h_a}{h_a} \overset{?}{\geq} \frac{(w_a-h_a)(w_a+h_a)}{m_a^2+h_a^2} \Leftrightarrow m_a^2+h_a^2 \overset{?}{\geq} h_a(w_a+h_a) \ (\because w_a-h_a \geq 0)$$

$$\Leftrightarrow m_a^2 \overset{?}{\geq} h_a w_a \to \text{true} \because m_a \geq h_a, w_a \therefore \frac{m_a}{h_a} + \frac{h_a}{m_a} \overset{②}{\geq} \frac{m_a}{w_a} + \frac{w_a}{m_a} \text{ and so,}$$

$$① \text{ and } ② \Rightarrow \boxed{\frac{m_a}{h_a} + \frac{h_a}{m_a} \geq \max\left\{\frac{w_a}{h_a} + \frac{h_a}{w_a}, \frac{m_a}{w_a} + \frac{w_a}{m_a}\right\}}$$

$$\therefore \frac{\sqrt{s^2 + 37r^2}}{4r} \geq \frac{m_a}{h_a} + \frac{h_a}{m_a} \geq \max\left\{\frac{w_a}{h_a} + \frac{h_a}{w_a}, \frac{m_a}{w_a} + \frac{w_a}{m_a}\right\} \forall \Delta ABC,$$

$'' ='' $ for uppper bound iff $y = z \wedge y + z = 2x \Rightarrow '' ='' $ iff ΔABC is equilateral and $'' ='' $ for lower bound iff $b = c$ (QED)

3983. In ΔABC the following relationship holds:

$$\frac{9r^2}{2R^2} \leq \sum_{cyc} \frac{1 - \cos(A)}{1 + \tan^2\left(\frac{A}{2}\right)} \leq 1 + \frac{r^2}{2R^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{1 - \cos(A)}{1 + \tan^2\left(\frac{A}{2}\right)} = \frac{2\sin^2\left(\frac{A}{2}\right)}{\frac{1}{\cos^2\left(\frac{A}{2}\right)}} = 2\left(\sin^2\left(\frac{A}{2}\right).\cos^2\left(\frac{A}{2}\right)\right) = \frac{1}{2}(\sin(A))^2 \quad *$$

$$\sum_{cyc} \frac{s^2 - r^2 - 4Rr}{2R^2} \overset{Gerretsen}{\leq}$$

$$\leq \frac{4R^2 + 4Rr + 3r^2 - r^2 - 4Rr}{2R^2} = \frac{4R^2 + 2r^2}{2R^2} = 2 + \frac{r^2}{R^2} \quad (1)$$

$$\sum_{cyc} \sin^2(A) = \frac{s^2 - r^2 - 4Rr}{2R^2} \geq \frac{16Rr - 5r^2 - r^2 - 4Rr}{2R^2} =$$

$$= \frac{12Rr - 2r^2}{2R^2} \overset{Euler}{\geq} \frac{18r^2}{2R^2} = \frac{9r^2}{R^2} \quad (2)$$

From $(*), (1)$ *and* (2) *we get that* :

$$\frac{9r^2}{2R^2} \leq \sum_{cyc} \frac{1 - \cos(A)}{1 + \tan^2\left(\frac{A}{2}\right)} \leq 1 + \frac{r^2}{2R^2}$$

Equality holds for $a = b = c$.

3984. In any acute ΔABC the following relationship holds :

$$\sum_{cyc} \frac{\cot^2 A \cot^2 B}{\cot C(\cot A + \cot B)} \geq \frac{1}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan^2\frac{B}{2}\tan^2\frac{C}{2}}{\tan\frac{A}{2}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}=\frac{1}{s^2}\cdot\sum_{cyc}\frac{r_b^2r_c^2}{r_a(r_b+r_c)}\overset{Bergstrom}{\geq}\frac{1}{s^2}\cdot\frac{\left(\sum_{cyc}r_ar_b\right)^2}{2\sum_{cyc}r_ar_b}$$

$$=\frac{s^2}{2s^2}=\frac{1}{2}\therefore\sum_{cyc}\frac{\tan^2\frac{B}{2}\tan^2\frac{C}{2}}{\tan\frac{A}{2}\cdot\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)}\geq\frac{1}{2}\to (1)$$

Let us consider $a\ \Delta\ A'B'C'$ with $angles$ $A'\equiv(\pi-2A), B'\equiv(\pi-2B)$ and $C'\equiv(\pi-2C)$ & then : $\cos A'\cos B'\cos C'=\cos(\pi-2A)\cos(\pi-2B)\cos(\pi-2C)$ $=-\cos 2A\cos 2B\cos 2C=1+4\cos A\cos B\cos C>0$ ($\because$ Δ ABC being $acute$ $\Rightarrow\cos A\cos B\cos C>0)\Rightarrow\Delta\ A'B'C'$ is $acute$ and hence, imp$l$ementing (1) on $\Delta\ A'B'C'$, we get :

$$\sum_{cyc}\frac{\tan^2\frac{\pi-2B}{2}\tan^2\frac{\pi-2C}{2}}{\tan\frac{\pi-2A}{2}\cdot\left(\tan\frac{\pi-2B}{2}+\tan\frac{\pi-2C}{2}\right)}\geq\frac{1}{2}$$

$$\Rightarrow\sum_{cyc}\frac{\cot^2B\cot^2C}{\cot A\,(\cot B+\cot C)}\geq\frac{1}{2}\ \forall\ acute\ \Delta\ ABC,$$

'' ='' iff Δ ABC is equi$lateral$ (QED)

3985. In ΔABC the following relationship holds:

$$3\leq\sqrt{\sum_{cyc}m_a\cdot\sum_{cyc}\frac{1}{m_a}}\leq\frac{3R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}m_a\cdot\sum_{cyc}\frac{1}{m_a}\geq 9\Rightarrow\sqrt{\sum_{cyc}m_a\cdot\sum_{cyc}\frac{1}{m_a}}\geq 3$$

$$\sum_{cyc}m_a\leq\frac{9R}{2}\quad(Gotman)$$

$$m_a\geq h_a\Rightarrow\frac{1}{m_a}\leq\frac{1}{h_a}\Rightarrow\sum_{cyc}\frac{1}{m_a}\leq\sum_{cyc}\frac{1}{h_a}=\frac{1}{r}$$

$$\sqrt{\sum_{cyc}m_a\cdot\sum_{cyc}\frac{1}{m_a}}\leq\sqrt{\frac{9R}{2}\cdot\frac{1}{r}}\leq\frac{3R}{2r}\Leftrightarrow\frac{9R}{2r}\leq\frac{9R^2}{4r^2}\Leftrightarrow 2r\leq R\ (Euler)$$

$$3\leq\sqrt{\sum_{cyc}m_a\cdot\sum_{cyc}\frac{1}{m_a}}\leq\frac{3R}{2r}$$

Equality holds for $a=b=c.$

3986. In ΔABC the following relationship holds:

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}\geq 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}=\sum_{cyc}\frac{r_a}{b}=\sum_{cyc}\frac{F}{b(s-a)}=$$

$$=F\sum_{cyc}\frac{1}{b(s-a)}\overset{AM-GM}{\geq}\frac{3F}{\sqrt[3]{abc(s-a)(s-b)(s-c)}}=$$

$$=\frac{3F}{\sqrt[3]{4Rrs(s-a)(s-b)(s-c)}}=\frac{3F}{\sqrt[3]{4RrF^2}}=$$

$$=\frac{3F}{\sqrt[3]{4Rrr^2s^2}}=\frac{3F}{r\cdot\sqrt[3]{4Rs^2}}=\frac{3rs}{r\cdot\sqrt[3]{4Rs^2}}\overset{MITRINOVIC}{\geq}\frac{3s}{\sqrt[3]{4R\left(\frac{3\sqrt{3}R}{2}\right)^2}}\overset{MITRINOVIC}{\geq}$$

$$\geq\frac{3\cdot 3\sqrt{3}r}{\sqrt[3]{4R\cdot\frac{27R^2}{4}}}=\frac{9\sqrt{3}r}{3R}=3\sqrt{3}\cdot\frac{r}{R}$$

Equality holds for $a=b=c$.

3987. In ΔABC the following relationship holds:

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\geq\frac{3r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\overset{JENSEN}{\geq}3\sin\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}=3\sin\frac{\pi}{6}=3\cdot\frac{1}{2}\geq 3\cdot\frac{r}{R}$$

Equality holds for $A=B=C$.

3988. In ΔABC the following relationship holds:

$$\sin A+\sin B+\sin C\ge 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\sin A+\sin B+\sin C=\sum_{cyc}\sin A=\sum_{cyc}\frac{a}{2R}=$$

$$=\frac{1}{2R}(a+b+c)=\frac{2s}{2R}=\frac{s}{R}\overset{MITRINOVIC}{\ge}\frac{3\sqrt{3}r}{R}$$

Equality holds for $a=b=c$.

3989. In ΔABC the following relationship holds:

$$h_ah_bh_c\ge 27r^3$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$h_ah_bh_c=\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}=\frac{8F^3}{abc}=\frac{8F^3}{4RF}=\frac{2F^2}{R}=\frac{2r^2s^2}{R}$$

We must prove that:

$$\frac{2r^2s^2}{R}\ge 27r^3\Leftrightarrow\frac{2s^2}{R}\ge 27r\Leftrightarrow 27r\Leftrightarrow 2s^2\ge 27Rr$$

$$2s^2\overset{GERRETSEN}{\ge}2(16Rr-5r^2)\ge 27Rr$$

$$32Rr-10r^2\ge 27Rr$$

$$5Rr\ge 10r^2$$

$$R\ge 2r\quad\text{(Euler)}$$

Equality holds for $a=b=c$.

3990. In ΔABC the following relationship holds:

$$\frac{h_a+h_b}{a}+\frac{h_b+h_c}{b}+\frac{h_c+h_a}{c}\ge 6\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

Lemma: In ΔABC holds:

$$h_a h_b h_c \ge 27r^3$$

Proof:

$$h_a h_b h_c = \frac{8F^3}{abc} = \frac{8F^3}{4RF} = \frac{2F^2}{R} = \frac{2r^2s^2}{R}$$

$$\frac{2r^2s^2}{R} \ge 27r^3 \Leftrightarrow \frac{2s^2}{R} \ge 27r \Leftrightarrow 2s^2 \ge 27Rr$$

$$2s^2 \overset{GERRETSEN}{\ge} 2(16Rr - 5r^2) \ge 27Rr$$

$$32Rr - 10r^2 \ge 27Rr \Leftrightarrow R \ge 2r \quad \text{(Euler)}$$

Back to the problem:

$$\sum_{cyc} \frac{h_a + h_b}{a} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{(h_a+h_b)(h_b+h_c)(h_c+h_a)}{abc}} \ge$$

$$\overset{CESARO}{\ge} 3\sqrt[3]{\frac{8h_a h_b h_c}{4RF}} \overset{Lemma}{\ge} 3\sqrt[3]{\frac{8\cdot 27r^3}{4Rrs}} = 3\cdot 2\cdot 3r\cdot\sqrt[3]{\frac{1}{4rs}} = \frac{18r}{\sqrt[3]{2R^2s}} \overset{EULER}{\ge}$$

$$\ge \frac{18r}{\sqrt[3]{4R\cdot\frac{R}{2}\cdot s}} = \frac{18r}{\sqrt[3]{2R^2s}} \overset{MITRINOVIC}{\ge}$$

$$\ge \frac{18r}{\sqrt[3]{2R^2\cdot\frac{3\sqrt{3}}{2}\cdot R}} = \frac{18r}{\sqrt[3]{\left(R\sqrt{3}\right)^3}} = \frac{18r}{\sqrt{3}R} = \frac{18\sqrt{3}r}{3R} = 6\sqrt{3}\cdot\frac{r}{R}$$

Equality holds for $a = b = c$.

3991. In ΔABC the following relationship holds:

$$\frac{csc^5\left(\frac{A}{2}\right)}{a} + \frac{csc^5\left(\frac{B}{2}\right)}{b} + \frac{csc^5\left(\frac{C}{2}\right)}{c} \ge \frac{32\sqrt{3}}{R}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{csc^5\left(\frac{A}{2}\right)}{a}+\frac{csc^5\left(\frac{B}{2}\right)}{b}+\frac{csc^5\left(\frac{C}{2}\right)}{c}\overset{Holder}{\geq}\frac{\left(csc\frac{A}{2}+csc\frac{B}{2}+csc\frac{C}{2}\right)^5}{3^{5-2}(a+b+c)}\overset{AM-GM}{\geq}$$

$$\geq\frac{\left(3\left(\prod_{cyc}csc\frac{A}{2}\right)^{\frac{1}{3}}\right)^5}{27.2s}=\frac{9\left(\prod\frac{1}{sin\frac{A}{2}}\right)^{\frac{5}{3}}}{2s}=\frac{9\left(\frac{4R}{r}\right)^{\frac{5}{3}}}{2s}\overset{Euler}{\geq}$$

$$\geq\frac{9.(8)^{\frac{5}{3}}}{2s}\overset{Mitrinovic}{\geq}\frac{9\cdot 32}{3\sqrt{3}R}=\frac{32\sqrt{3}}{R}$$

$$Equality\ \ holds\ \ for\ \ a=b=c.$$

3992. Prove that:

$$\frac{1}{\sin^2\frac{4\pi}{9}}-\frac{1}{\sin^2\frac{\pi}{9}}=\frac{8\sqrt{3}}{3}\left(-2\sin\frac{4\pi}{9}+\sin\frac{\pi}{9}\right)$$

Proposed by Vasile Mircea Popa-Romania

Solution by Tapas Das-India

$$Let\frac{\pi}{9}=A\ then\ \ 9A=\pi, sin5A=sin(9A-4A)=sin(\pi-4A)=sin4A\ (1)$$

$$L.H.S=\frac{1}{\sin^2\frac{4\pi}{9}}-\frac{1}{\sin^2\frac{\pi}{9}}=\frac{1}{\sin^2 4A}-\frac{1}{\sin^2 A}=\frac{\sin^2 A-\sin^2 4A}{.\sin^2 4A\sin^2 A}=-\frac{sin5A.sin3A}{.\sin^2 4A\sin^2 A}$$

$$\overset{(1)}{=}-\frac{.sin3A}{sin4A\sin^2 A}\overset{\frac{\pi}{9}=A}{=}-\frac{.\frac{\sqrt{3}}{2}}{sin4A\sin^2 A}=\frac{\sqrt{3}}{2sin4A\sin^2 A}$$

$$R.H.S=\frac{8\sqrt{3}}{3}\left(-2\sin\frac{4\pi}{9}+\sin\frac{\pi}{9}\right)=\frac{8\sqrt{3}}{3}(-2\sin 4A+\sin A)$$

$$We\ need\ to\ show:$$

$$\frac{\sqrt{3}}{2sin4A\sin^2 A}=\frac{8\sqrt{3}}{3}(-2\sin 4A+\sin A)\ \ or$$

$$-3=16\sin^3 A\,sin4A-32\sin^2 A\sin^2 4A\ \ (1)$$

$$16\sin^3 A\,sin4A=4(3sinA-sin3A)sin4A\overset{\frac{\pi}{9}=A}{=}4\left(3sinA-\frac{\sqrt{3}}{2}\right)sin4A=$$

$$=12sinAsin4A-2\sqrt{3}\,sin4A$$

$$=6(cos3A-cos5A)-2\sqrt{3}\,sin4A\overset{\frac{\pi}{9}=A}{=}3+6cos4A-2\sqrt{3}\,sin4A$$

$$32\sin^2 A\sin^2 4A=8(2sinAsin4A)^2=8(cos3A-cos5A)^2\overset{\frac{\pi}{9}=A}{=}8\left(\frac{1}{2}-cos5A\right)^2=$$

$$=2-8cos5A+8\cos^2 5A\overset{\frac{\pi}{9}=A}{=}2+8cos4A+8\cos^2 4A\ =6+8cos5A+4cos8A$$

$$16\sin^3 A\,sin4A-32\sin^2 A\sin^2 4A$$
$$=\left(3+6cos4A-2\sqrt{3}\,sin4A\right)-(6+8cos4A+4cos8A)$$
$$=-3-2cos4A-2\sqrt{3}sin4A-4cos8A$$
$$=-3-4\left(\frac{1}{2}cos4A+\frac{\sqrt{3}}{2}sin4A\right)-4cos8A$$
$$\overset{\frac{\pi}{9}=A}{=}-3-4\cos\left(\frac{4\pi}{9}-\frac{\pi}{3}\right)-4\cos\left(\frac{8\pi}{9}\right)=-3-4\cos\left(\frac{\pi}{9}\right)-4\cos\left(\pi-\frac{\pi}{9}\right)$$
$$=-3-4\cos\left(\frac{\pi}{9}\right)+4\cos\left(\frac{\pi}{9}\right)=-3$$
$$so\ relation\ (1)\ is\ true$$

3993. In any Δ ABC the following relationship holds :

$$2\sum_{cyc}\frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\sum_{cyc}\frac{h_a}{n_a+s}=\frac{g_a+g_b+g_c+s}{2r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{2(h_a-r)}{g_a-n_a+\sqrt{4r^2+(b-c)^2}}+\frac{h_a}{n_a+s}\overset{?}{=}\frac{g_a+s-a}{2r}\overset{\text{Bogdan Fustei}}{\Leftrightarrow}$$
$$\frac{2\left(\frac{2rs}{a}-r\right)}{g_a-n_a+\frac{an_a}{s}}+\frac{h_a(s-n_a)}{s^2-n_a^2}\overset{?}{=}\frac{g_a+s-a}{2r}\overset{\text{Bogdan Fustei}}{\Leftrightarrow}$$
$$\frac{2rs\left(\frac{b+c}{a}\right)}{sg_a-(s-a)n_a}+\frac{h_a(s-n_a)}{2h_ar_a}\overset{?}{=}\frac{g_a+s-a}{2r}$$
$$\Leftrightarrow\frac{2rs\left(\frac{b+c}{a}\right)}{sg_a-(s-a)n_a}\overset{?}{=}\frac{g_a+s-a}{2r}-\frac{(s-a)(s-n_a)}{2rs}$$
$$\Leftrightarrow\frac{2rs\left(\frac{b+c}{a}\right)}{sg_a-(s-a)n_a}\overset{?}{=}\frac{sg_a+s(s-a)-s(s-a)+(s-a)n_a}{2rs}$$
$$\Leftrightarrow 4r^2s^2\left(\frac{b+c}{a}\right)\underset{(*)}{\underbrace{\overset{?}{=}}}(sg_a+(s-a)n_a)(sg_a-(s-a)n_a)$$

Now, $(sg_a+(s-a)n_a)(sg_a-(s-a)n_a)=s^2g_a^2-(s-a)^2n_a^2\overset{\text{Bogdan Fustei}}{=}$
$$\left(s^2((s-a)^2+2rh_a)-(s-a)^2(s^2-2h_ar_a)\right)=2rs^2h_a+2h_a(s-a)^2.\frac{rs}{s-a}$$
$$=2rsh_a(s+s-a)=2rs.\frac{2rs}{a}.(b+c)=4r^2s^2\left(\frac{b+c}{a}\right)\Rightarrow(*)\text{ is true}$$

$$\therefore \frac{2(h_a - r)}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \frac{h_a}{n_a + s} = \frac{g_a + s - a}{2r} \text{ and analogs}$$

$$\Rightarrow 2\sum_{cyc} \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \sum_{cyc} \frac{h_a}{n_a + s} = \frac{1}{2r} \cdot \sum_{cyc} g_a + \frac{1}{2r}\sum_{cyc}(s - a)$$

$$= \frac{1}{2r} \cdot \sum_{cyc} g_a + \frac{s}{2r} \text{ and so, } 2\sum_{cyc} \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \sum_{cyc} \frac{h_a}{n_a + s} =$$

$$\frac{g_a + g_b + g_c + s}{2r} \quad \forall \Delta ABC \text{ (QED)}$$

3994. **In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{1}{a}\sqrt{\frac{b^2 + c^2}{b^2 + bc + c^2}} \le \sqrt{2}\frac{R}{4r^2}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

It is known that:

$$\sum_{cyc} \frac{1}{a^2} \le \frac{1}{4r^2}$$

$$\frac{b^2 + c^2}{b^2 + bc + c^2} \overset{A-G}{\le} \frac{b^2 + c^2}{3bc} = \frac{1}{3}\left(\frac{b}{c} + \frac{c}{b}\right)$$

$$Analogous:$$

$$\frac{b^2 + a^2}{a^2 + ba + b^2} \overset{A-G}{\le} \frac{1}{3}\left(\frac{a}{b} + \frac{b}{a}\right); \quad \frac{c^2 + a^2}{a^2 + ca + c^2} \overset{A-G}{\le} \frac{1}{3}\left(\frac{a}{c} + \frac{c}{a}\right) \text{ (1)}$$

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \overset{CBS}{\le} \sqrt{\sum_{cyc} a^2} \cdot \sqrt{\sum_{cyc} \frac{1}{a^2}} \overset{LEIBNIZ}{\le} \sqrt{9R^2} \cdot \sqrt{\frac{1}{4r^2}} = \frac{3R}{2r}$$

$$Analogous:$$

$$\frac{b}{a} + \frac{a}{c} + \frac{c}{b} \le \frac{3R}{2r} \text{ (2)}$$

$$\sum_{cyc} \frac{1}{a}\sqrt{\frac{b^2 + c^2}{b^2 + bc + c^2}} \overset{C-B-S}{\le} \sqrt{\sum_{cyc} \frac{1}{a^2}} \cdot \sqrt{\sum_{cyc} \frac{b^2 + c^2}{b^2 + bc + c^2}} \overset{(1)}{\le}$$

$$\le \sqrt{\frac{1}{4r^2}} \cdot \sqrt{\frac{1}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + \frac{1}{3}\left(\frac{b}{a} + \frac{a}{c} + \frac{c}{b}\right)} \overset{(2)}{\le}$$

$$\le \frac{1}{2r}\sqrt{\frac{1}{3}\cdot\frac{3R}{2r}+\frac{1}{3}\cdot\frac{3R}{2r}}=\frac{1}{2r}\sqrt{\frac{R}{r}}=\frac{1}{2r}\sqrt{\frac{4Rr}{4r^2}}=\frac{1}{4r^2}\sqrt{4Rr}\overset{Euler}{\le}\frac{1}{4r^2}\sqrt{2R^2}=\sqrt{2}\frac{R}{4r^2}$$

Equality holds for $a=b=c$.

3995. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\tan\frac{A}{2}}{\tan^2\frac{B}{2}}\ge 3\sqrt{3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\tan\frac{A}{2}}{\tan^2\frac{B}{2}}=s\sum_{cyc}\frac{r_a}{r_b^2}=s.\sum_{cyc}\frac{\left(\frac{1}{r_b}\right)^2}{\frac{1}{r_a}}\overset{Bergstrom}{\ge}s.\frac{\left(\sum_{cyc}\frac{1}{r_a}\right)^2}{\sum_{cyc}\frac{1}{r_a}}=\frac{s.\frac{1}{r^2}}{\frac{1}{r}}=\frac{s}{r}$$

$$\overset{Mitrinovic}{\ge}3\sqrt{3}\text{ and so, }\sum_{cyc}\frac{\tan\frac{A}{2}}{\tan^2\frac{B}{2}}\ge 3\sqrt{3}\ \forall\ \Delta\ ABC,$$

$'' ='' $ iff Δ ABC is equilateral (QED

3996. In ΔABC the following relationship holds:

$$\sum\frac{1}{h_a^2}\le\sum\frac{1}{r_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{1}{h_a^2}=\frac{1}{4F^2}\sum a^2=\frac{s^2-r^2-4Rr}{2s^2r^2}$$

$$\sum\frac{1}{r_a^2}=\left(\sum\frac{1}{r_a}\right)^2-2\sum\frac{1}{r_ar_b}=\frac{1}{r^2}-\frac{2(4R+r)}{s^2r}=\frac{s^2-2r(4R+r)}{s^2r^2}$$

We need to show:

$$\frac{s^2-2r(4R+r)}{s^2r^2}\ge\frac{s^2-r^2-4Rr}{2s^2r^2}$$

$$2s^2-4r(4R+r)\ge s^2-r^2-4Rr$$

$$s^2\ge\ 4r(4R+r)-r^2-4Rr$$

$$s^2 \geq 16Rr + 4r^2 - r^2 - 4Rr$$

$$s^2 \geq 12Rr + 3r^2$$

$$s^2 \overset{GERRETSEN}{\overbrace{\geq}} 16Rr - 5r^2 \geq 12Rr + 3r^2$$

$$4Rr \geq 8r^2$$

$$R \overset{EULER}{\overbrace{\geq}} 2r$$

Equality holds for an equilateral triangle.

3997. In ΔABC the following relationship holds:

$$\frac{AI \cdot BI \cdot CI}{abc} \leq \frac{1}{3\sqrt{3}}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{AI \cdot BI \cdot CI}{abc} = \frac{\frac{r}{\sin\frac{A}{2}} \cdot \frac{r}{\sin\frac{B}{2}} \cdot \frac{r}{\sin\frac{C}{2}}}{abc} = \frac{r^3}{abc\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} =$$

$$= \frac{r^3}{abc\sqrt{\frac{(s-b)(s-c)(s-a)(s-c)(s-a)(s-b)}{bc \cdot ac \cdot ab}}} =$$

$$= \frac{r^3}{abc \cdot \frac{(s-a)(s-b)(s-c)}{abc}} = \frac{r^3 s}{s(s-a)(s-b)(s-c)} =$$

$$= \frac{r^3 s}{F^2} = \frac{r^3 s}{r^2 s^2} = \frac{r}{s} \overset{MITRINOVIC}{\leq} \frac{r}{3\sqrt{3}r} = \frac{1}{3\sqrt{3}}$$

Equality holds for $a = b = c$.

3998. In ΔABC the following relationship holds:

$$\frac{a+b}{h_a} + \frac{b+c}{h_b} + \frac{c+a}{h_c} \geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\frac{a+b}{h_a}+\frac{b+c}{h_b}+\frac{c+a}{h_c}=\sum_{cyc}\frac{a+b}{h_a}\overset{AM-GM}{\geq}$$

$$\geq 3\sqrt[3]{\frac{(a+b)(b+c)(c+a)}{h_ah_bh_c}}\overset{CESARO}{\geq}3\sqrt[3]{\frac{8abc}{\frac{2F}{a}\cdot\frac{2F}{b}\cdot\frac{2F}{c}}}=$$

$$=3\sqrt[3]{\frac{(abc)^2}{F^3}}=3\sqrt[3]{\frac{16R^2F^2}{F^3}}=3\sqrt[3]{\frac{16R^2}{F}}\overset{MITRINOVIC}{\geq}3\sqrt[3]{\frac{16\cdot\frac{4}{27}s^2}{rs}}=3\cdot\frac{4}{3}\sqrt[3]{\frac{s}{r}}\geq$$

$$\overset{MITRINOVIC}{\geq}4\sqrt[3]{\frac{3\sqrt{3}r}{r}}=4\sqrt[3]{\left(\sqrt{3}\right)^3}=4\sqrt{3}$$

Equality holds for $a=b=c$.

3999. **In ΔABC the following relationship holds:**

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\geq 3\sqrt{3}\cdot\frac{r}{R}$$

Proposed by Nguyen Hung Cuong – Vietnam

Solution by Daniel Sitaru – Romania

$$\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\overset{AM-GM}{\geq}3\sqrt[3]{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}=$$

$$=3\sqrt[3]{\sqrt{\frac{s(s-a)\cdot s(s-b)\cdot s(s-c)}{bc\cdot ac\cdot ab}}}=$$

$$=3\sqrt[6]{\frac{s^3(s-a)(s-b)(s-c)}{(abc)^2}}=3\sqrt[6]{\frac{s^2F^2}{(abc)^2}}=3\sqrt[3]{\frac{sF}{abc}}=3\sqrt[3]{\frac{sF}{4RF}}=3\sqrt[3]{\frac{s}{4R}}$$

Remains to prove:

$$3\sqrt[3]{\frac{s}{4R}}\geq 3\sqrt{3}\cdot\frac{r}{R}\Leftrightarrow\frac{s}{4R}\geq 3\sqrt{3}\cdot\frac{r^3}{R^3}$$

$$sR^3 \geq 12\sqrt{3}Rr^3 \Leftrightarrow sR^2 \geq 12\sqrt{3}r^3 \quad \text{(to prove)}$$

$$sR^2 \overset{EULER}{\geq} s \cdot (2r)^2 \overset{MITRINOVIC}{\geq} 3\sqrt{3}r \cdot 4r^2 = 12\sqrt{3}r^3$$

Equality holds for $A = B = C$.

4000. If in any Δ ABC the following relationship holds :

$$\sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq 1 + \frac{r}{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\left(\sum_{cyc}\sin\frac{A}{2}\right)^2 = \sum_{cyc}\sin^2\frac{A}{2} + 2\sum_{cyc}\left(\sin\frac{B}{2}\sin\frac{C}{2}\right) = \frac{2R-r}{2R} + \frac{2r}{4R}\cdot\sum_{cyc}\csc\frac{A}{2}$$

$$\overset{\text{Jensen}}{\geq} \frac{2R-r}{2R} + \frac{r}{2R}.3\csc\frac{\pi}{6}\left(\because f(x) = \csc\frac{x}{2}\ \forall x \in (0,\pi) \text{ is convex}\right) = \frac{2R-r}{2R} + \frac{6r}{2R}$$

$$= \frac{2R+5r}{2R} \overset{?}{\geq} \frac{(R+r)^2}{R^2} \Leftrightarrow 5Rr \overset{?}{\geq} 4Rr + 2r^2 \Leftrightarrow R \overset{?}{\geq} 2r \rightarrow \text{true via Euler}$$

$$\therefore \left(\sum_{cyc}\sin\frac{A}{2}\right)^2 \geq \left(1+\frac{r}{R}\right)^2 \Rightarrow \sin\frac{A}{2} + \sin\frac{B}{2} + \sin\frac{C}{2} \geq 1 + \frac{r}{R}\ \forall\,\Delta\,\text{ABC},$$

$'' ='' \text{ iff } \Delta\,\text{ABC is equilateral (QED)}$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru