RMM - Abstract Algebra Marathon 701 - 800

R M M

ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

Proposed by

Daniel Sitaru – Romania, Bui Hong Suc-Vietnam

Elsen Kerimov-Azerbaijan, Khaled Abd Imouti-Syria

Neculai Stanciu-Romania, Nguyen Hung Cuong-Vietnam

Marin Chirciu-Romania, Nguyen Hung Viet-Vietnam

Kostas Geronikolas-Greece, Dang Ngoc Minh-Vietnam

Netai Chandra Bhar-India, Nguyen Van Canh-Vietnam

Sakthi Vel-India, Mais Hasanov-Azerbaijan

Amin Hajiyev-Azerbaijan, Stewart Ming-Vietnam

Solutions by

Daniel Sitaru – Romania, Hikmat Mammadov-Azerbaijan

Ravi Prakash-India, Mirsadix Muzefferov-Azerbaijan

Abbaszade Yusif-Azerbaijan, Pham Duc Nam-Vietnam

Eric-Dimitrie Cismaru-Romania, Miguel Velasquez Culque-Peru

Khaled Abd Imouti-Syria, Soumava Chakraborty-India

Mohamed Amine Ben Ajiba-Morocco, Tapas Das-India

Togrul Ehmedov-Azerbaijan, Amin Hajiyev-Azerbaijan

Shirvan Tahirov-Azerbaijan, Jenish Rijal-Nepal

701.

$$X \in M_2(\mathbb{R}), X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, X^5 = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$$

Find: $\Omega = a - b + c - d$

Proposed by Daniel Sitaru – Romania

Solution by Hikmat Mammadov-Azerbaijan

$$X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}; X^5 = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix} \overset{say}{\Rightarrow} A = X^5$$

$$\Rightarrow X_A = (3-x)\cdot(5-x) - 8 = x^2 - 8x + 7 = (x-1)\cdot(x-7)$$

$$\Rightarrow (x-1)\cdot(x-7) \Rightarrow D = \begin{pmatrix} 1 & 0 \\ 0 & 7 \end{pmatrix}$$

$$\Rightarrow U_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \to U_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \to P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\Rightarrow A = P \cdot D \cdot P^{-1}$$

$$\Rightarrow X = P \cdot D^{\frac{1}{5}} \cdot P^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \sqrt[5]{7} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4 - \sqrt[5]{7} & -2 - \sqrt[5]{7} \\ 2 + \sqrt[5]{7} & -1 + \sqrt[5]{7} \end{pmatrix}$$

$$\Rightarrow \Omega = 3$$

702. **Solve for natural numbers:**

$$\sum_{i=1}^{x}\sum_{k=1}^{100}|i-k| = 333300$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

For $1 \le i \le x$,

$$\sum_{k=1}^{100}|i-k| = \sum_{k=1}^{i}|i-k| + \sum_{k=i+1}^{100}|k-i| = \sum_{k=1}^{i-1}k + \sum_{k=1}^{100-i}k =$$

$$= \frac{1}{2}(i-1)i + \frac{1}{2}(100-i)(101-i) = \frac{1}{2}[i^2 - i + (100)(101) - 201i + i^2] =$$

$$= i^2 - 101i + 5050 \Rightarrow \sum_{i=1}^{x}\sum_{k=1}^{100}|i-k|$$

$$= \frac{1}{6}x(x+1)(2x+1) - \frac{101x(x+1)}{2} + 5050x$$

$$\frac{1}{6}x(x+1)(2x+1) - \frac{1}{2}(101)x(x+1) + 5050x = 333300$$

$$\Rightarrow 2x^3 + 3x^2 + x - 303x^2 - 303x + 30300x = 1999800$$

$$\Rightarrow 2x^3 - 300x^2 + 29998x - 1999800 = 0$$

$$\Rightarrow x^2(x-100) - 50x(x-100) + 9999(x-100) = 0$$

$$\Rightarrow (x-100)(x^2 - 50x + 9999) = 0 \Rightarrow (x-100)[(x-25)^2 + 9374] = 0 \Rightarrow x = 100$$

Solution 2 by Hikmat Mammadov-Azerbaijan

$$\sum_{k=1}^{100}|i-k| = \sum_{k=1}^{i}|i-k| + \sum_{k=i+1}^{100}|i-k| = \sum_{k=1}^{i}|i-k| + \sum_{k=i+1}^{100}|k-i|$$

$$= \frac{(i-1)\cdot i}{2} + \frac{(100-i)\cdot(101-i)}{2} = \frac{2i^2 + 10100 - 202i}{2} = i^2 - 101i + 5050$$

$$\sum_{i=1}^{x}\sum_{k=1}^{100} = \sum_{i=1}^{x}(i^2 - 101i + 5050) = \frac{x\cdot(x+1)\cdot(2x+1)}{6} - \frac{101x\cdot(x+1)}{2} + 5050x =$$

$$= \frac{x\cdot(x^2 - 150x + 14999)}{3} = 333300 \Rightarrow x = 100$$

$$\sum_{i=1}^{x}\sum_{k=1}^{100}|i-k| = 333300 \Rightarrow x = 100$$

703. **For $a,b,c,x > 0$ and $\log_a(bx) = m, \log_c(ax) = n, \log_b(cx) = p$**

Find $S = \log_{\frac{a^4b^5}{c^6x^3}}\left(\frac{x^8a^7b^5}{c^{20}}\right)$ in terms of m,n,p

Proposed by Bui Hong Suc-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\log_a(bx) = m\ ;\ \log_c(ax) = n\ ;\ \log_c(cx) = p$$

Let's express a,b,c by x

$$\log_c(ax) = n\ \Rightarrow a = \frac{c^n}{x}\ ;\ \textit{Let's write the expression instead of } \log_a(bx) = m\ :$$

$$bx = \left(\frac{c^n}{x}\right)^m \Rightarrow bx^{m+1} = c^{mn}\ \ (1)$$

$$\log_a(bx) = m \Rightarrow b = \frac{a^m}{x} \quad (2)$$

From $\log_c(cx) = p \Rightarrow b = (cx)^{\frac{1}{p}} \quad (3)$

Let's write expression (3) instead of (1):

$$bx^{m+1} = c^{mn} \Rightarrow (cx)^{\frac{1}{p}} \cdot x^{m+1} = c^{m+n} \Rightarrow c = x^{\frac{pm+p+1}{pmn-1}} \ (*)$$

Let's write expression $(*)$ instead of (3):

$$b^p = x^{\frac{pm+p+1}{pmn-1}} \cdot x = x^{\frac{p(mn+m+1)}{pmn-1}} \Rightarrow b = x^{\frac{mn+m+1}{pmn-1}} \ (**)$$

Let's write expression $(**)$ instead of (1):

$$x^{\frac{mn+m+1}{pmn-1}} \cdot x = a^n \Rightarrow x^{\frac{m(pn+n+1)}{pmn-1}} = a^m$$

$$a = x^{\frac{pn+n+1}{pmn-1}} \ (***)$$

Let's use the expressions $(*), (**)$ and $(***)$ in $\left(\frac{a^4b^5}{c^6x^3}\right)$

$$A = \frac{a^4b^5}{c^6x^3} = \frac{x^{\frac{4(pn+n+1)}{pmn-1}} \cdot x^{\frac{5(mn+m+1)}{pmn-1}}}{x^{\frac{6(pm+p+1)}{pmn-1}} \cdot x^3} = x^{\frac{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3}{mpn-1}}$$

$$B = \frac{x^8a^7b^5}{c^{20}} = \frac{x^8 \cdot x^{\frac{7(pn+n+1)}{pmn-1}} \cdot x^{\frac{5(mn+m+1)}{pmn-1}}}{x^{\frac{20(pm+p+1)}{pmn-1}}} = x^{\frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{mpn-1}}$$

$$S = \log_A B = \frac{\frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{mpn-1}}{\frac{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3(mpn-1)}{mpn-1}}$$
$$= \frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3(mpn-1)}$$

$$S = \frac{8(mpn-1)+7(pn+n+1)+5(mn+m+1)-20(pm+p+1)}{4(pn+n+1)+5(mn+m+1)-6(pm+p+1)-3(mpn-1)}$$

704.

Let the sequence $\{a_n\}$ be given

$a_1 = 6, a_2 = 14, \ a_{n+2} = 5a_{n+1} - 6a_n + 2 \ (n \geq 1)$,

$a_{2024} \equiv x \ (mod\ 100)$. Find x.

Proposed by Elsen Kerimov-Azerbaijan

Solution by Abbaszade Yusif-Azerbaijan

$$Let: \ a_n = x^n, \quad x^{n+2} = 5x^{n+1} - 6x^n$$

$$x^n(x^2 - 5x + 6) = 0, \quad x_1 = 2 \quad x_2 = 3$$
$$a_n = c_1 2^n + c_2 3^n + c_3$$

$$Let\ a_n = y \ => \ y - 5y + 6y = 2 \ => \ y = 1 = c_1 \ => \ a_n = c_1 2^n + c_2 3^n + 1$$

$$6 = 2c_1 + 3c_2 + 1 \ => \ 4c_1 + 6c_2 = 10 \ => \ c_1 = 1$$
$$14 = 4c_1 + 9c_2 + 1 \ => \ 4c_1 + 9c_2 = 13 \ => \ c_2 = 1$$

$$a_n = 2^n + 3^n + 1 \ => \ a_{2024} = 2^{2024} + 3^{2024} + 1$$

$$2^{2024} + 3^{2024} + 1 \equiv x\ (mod\ 100) =>$$
$$2^{24} \times (2^{40})^{50} + 3^{24} \times (3^{40})^{50} + 1 \equiv x\ (mod\ 100)$$
$$2^{24} + 3^{24} + 1 \equiv x\ (mod\ 100)$$
$$x = \left\{\frac{5 \times 4 + 1}{10}\right\} \times 10 + 6 + \left\{\frac{8 - 5 \times 2}{10}\right\} \times 10 + 1 + 1 = 98$$

705. Solve for real numbers:

$$4^{\cos^2\left(\frac{x}{2}\right)} + \cos x = 4$$

Proposed by Khaled Abd Imouti-Syria

Solution by Pham Duc Nam-Vietnam

$$4^{\cos^2\left(\frac{x}{2}\right)} + \cos(x) = 4 \Leftrightarrow 2^{1+\cos(x)} + \cos(x) = 4, t = \cos(x), -1 \le t \le 1$$

$\Leftrightarrow 2^{1+t} + t = 4, f(t) = 2^{1+t} + t, f'(t) = 2^{t+1}\ln(2) + 1 > 0, \forall t \in [-1,1] \Rightarrow f(t)$ is strictly increasing on $[-1,1] \Rightarrow 2^{1+t} + t = 4$ has unique root, and $g(t) = 2^{1+t} + t - 4, g(0) = -2, g(1) = 1 \Rightarrow$ The root lies on $(0,1)$.

We have: $2^{1+t} + t = 4 \Leftrightarrow 2^{1+t} = 4 - t \Leftrightarrow \ln(2)\, 2^{1+t} = \ln(2)\,(4-t) \Leftrightarrow$

$$\Leftrightarrow 2^{4-t}\ln(2)\,(4-t) = 32\ln(2) \Leftrightarrow$$

$$\Leftrightarrow \ln(2)\,(4-t)e^{(\ln(2)(4-t))} = 32\ln(2) \overset{xe^x = z \Rightarrow x = W(z)}{\rightarrow} W\left(\ln(2)(4-t)e^{(\ln(2)(4-t))}\right) =$$
$$= W(32\ln(2))$$

$$\Leftrightarrow \ln(2)\,(4-t) = W(32\ln(2)) \Leftrightarrow t = 4 - \frac{W(32\ln(2))}{\ln(2)} \approx 0.7156207332755864$$

$\Rightarrow x = \pm\arccos\left(4 - \frac{W(32\ln(2))}{\ln(2)}\right) + k2\pi, k \in \mathbb{Z}. W(z)$ is the Lambert W function.

706. **Solve for positive real numbers the following system:**

$$\begin{cases} \frac{x^3}{(1+y)(1+z)} = \frac{6y-x-z-2}{8} \\ \frac{y^3}{(1+z)(1+x)} = \frac{6z-y-x-2}{8} \\ \frac{z^3}{(1+x)(1+y)} = \frac{6x-z-y-2}{8} \end{cases}$$

Proposed by Neculai Stanciu – Romania

Solution 1 by Pham Duc Nam-Vietnam

Sum 3 equations we get: $\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} = \frac{4(x+y+z)-6}{8} = \frac{2(x+y+z)-3}{4}$ (1)

Now, we have: $\frac{x^3}{(1+y)(1+z)} + \frac{1+y}{8} + \frac{1+z}{8} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{x^3}{(1+y)(1+z)} \frac{1+y}{8} \frac{1+z}{8}} = \frac{3x}{4}$

Similarly, $\frac{y^3}{(1+z)(1+x)} + \frac{1+z}{8} + \frac{1+x}{8} \geq \frac{3y}{4}, \frac{z^3}{(1+x)(1+y)} + \frac{1+x}{8} + \frac{1+y}{8} \geq \frac{3z}{4}$

$$\Rightarrow \frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} + \frac{1+x}{4} + \frac{1+y}{4} + \frac{1+z}{4} \geq$$

$$\geq \frac{3(x+y+z)}{4}$$

$$\Leftrightarrow \frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} \geq$$

$$\geq \frac{3(x+y+z)}{4} - \frac{(x+y+z)}{4} - \frac{3}{4} = \frac{2(x+y+z)-3}{4} \quad (2)$$

From (1), (2): $\frac{x^3}{(1+y)(1+z)} + \frac{y^3}{(1+z)(1+x)} + \frac{z^3}{(1+x)(1+y)} = \frac{2(x+y+z)-3}{4} \Rightarrow x = y = z$

* Replace $x = y = z$ to the first equation (or any equation) we have: $\frac{x^3}{(1+x)^2} = \frac{2x-1}{4}$

$$\Leftrightarrow 4x^3 - (2x-1)(1+x)^2 = 0 \Leftrightarrow 2x^3 - 3x^2 + 1 = 0 \Leftrightarrow (2x+1)(x-1)^2 = 0 \overset{x \in \mathbb{R}^+}{\rightarrow}$$

$$x = 1 = y = z = 1$$

$\Rightarrow (x, y, z) = (1, 1, 1)$ is the unique solution of the given system of equations.

Solution 2 by Eric-Dimitrie Cismaru-Romania

Adding the three equations, we obtain $\sum_{cyc}\frac{x^3}{(1+y)(1+z)}=\frac{x+y+z}{2}-\frac{3}{4}$. We prove that

$$LHS \geq RHS$$

The triplets $\{x^3,y^3,z^3\}$ and $\left\{\frac{1}{(1+y)(1+z)},\frac{1}{(1+z)(1+x)},\frac{1}{(1+x)(1+y)}\right\}$ are ordered the same. By Chebyshev's Inequality,

$$LHS \geq \frac{x^3+y^3+z^3}{3}\cdot\sum_{cyc}\frac{1}{(1+y)(1+z)} \overset{Holder}{\geq} \left(\frac{x+y+z}{3}\right)^3\cdot\frac{x+y+z+3}{\prod(1+x)}=\frac{3\alpha^3(\alpha+1)}{\prod(1+x)},$$

where $\alpha=\frac{x+y+z}{3}$. Denote $a=x+1$, $b=y+1$ and $c=z+1$. Then, by AM-GM,

$$(a+b+c)^3 \geq \left(3\sqrt[3]{abc}\right)^3=27abc, \text{ or, in other words,}$$

$$\prod_{cyc}(1+x) \leq \frac{(a+b+c)^3}{27}=\frac{(x+y+z+3)^3}{27}=\left(\frac{x+y+z}{3}+1\right)^3=(\alpha+1)^3. \text{ Therefore, we get}$$

$$LHS \geq \frac{3\alpha^3(\alpha+1)}{(\alpha+1)^3}=\frac{3\alpha^3}{(\alpha+1)^2}\geq RHS=\frac{3\alpha}{2}-\frac{3}{4} \Leftrightarrow \frac{(\alpha-1)^2(2\alpha+1)}{(\alpha+1)^2}\geq 0 \Leftrightarrow LHS \geq RHS.$$

Since $RHS=LHS\geq RHS$, equality holds in all the previous inequalities, which is, $\alpha=1$ and $x=y=z$, so $S=\{1,1,1\}$ is the only solution for the given system of equations.

707. If $2a^2+b=2b^2+c=2c^2+a=\frac{1}{2}$ then find:

$$\Omega=a+b+c$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\begin{cases}2a^2+b=\frac{1}{2}\\ 2b^2+c=\frac{1}{2}\\ 2c^2+a=\frac{1}{2}\end{cases} \Rightarrow \begin{cases}4a^2+2b=1\\ 4b^2+2c=1\\ 4c^2+2a=1\end{cases} \Rightarrow \begin{cases}4a^2=1-2b\\ 4b^2=1-2c\\ 4c^2=1-2a\end{cases} \Rightarrow$$

$$a,b,c \in \left(-\infty;\frac{1}{2}\right]$$

$$(*)\begin{cases}4a^2+2b=1\\ 4b^2+2c=1\\ 4c^2+2a=1\end{cases} \Rightarrow \textit{Summarize the system of equatons side by side:}$$

$$(4a^2+2a)+(4b^2+2b)+(4c^2+2c)=3$$

$$\left(2a+\frac{1}{2}\right)^2+\left(2b+\frac{1}{2}\right)^2+\left(2c+\frac{1}{2}\right)^2=\frac{15}{4}\ \ (1)$$

$(*)$*The system of equations is symmetric with respect to the variables* a, b, c .

Therefore $a = b = c$.

$$\text{Or from } (*)\begin{cases}4(a+b)(a-b)=2(c-b)\\4(a+c)(a-c)=2(a-b)\Rightarrow\\4(b+c)(b-c)=2(a-c)\end{cases}$$

$$64(a+b)(a-b)(a-c)(a+c)(b-c)(b+c)=8(c-b)(a-b)(a-c)$$

$$8(a-b)(b-c)(a-c)(8(a+c)(b+c)(a+c)+1)=0\Rightarrow a=b=c,$$

$$\text{Then by } (1)\ :\ \ 3\left(2a+\frac{1}{2}\right)^2=\frac{15}{4}$$

$$a_1=\frac{-1-\sqrt{5}}{4}\ ,\ a_2=\frac{-1+\sqrt{5}}{4}$$

$$\text{So}\quad a_1=b_1=c_1=\frac{-1-\sqrt{5}}{4}\ ;a_2=b_2=c_2=\frac{-1+\sqrt{5}}{4}$$

$$\text{Then}\quad a_1+b_1+c_1=\frac{-3-3\sqrt{5}}{4}\ ,\quad a_2+b_2+c_2=\frac{-3+3\sqrt{5}}{4}$$

708. Solve for real numbers:

$$\sqrt{x-y}+3\sqrt{y-z}+5\sqrt{z+x}=x+\frac{35}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Miguel Velasquez Culque-Cajamarca-Peru

$$\text{Change of variable}: \sqrt{x-y}=a\ ;\ \sqrt{y-z}=b\ ;\ \sqrt{z+x}=c$$

$$x-y=a^2\ ;\ y-z=b^2\ ;\ z+x=c^2\Longrightarrow a^2+b^2+c^2=2x$$

Then we have:

$$a+3b+5c=x+\frac{35}{2}\Longrightarrow 2a+6b+10c-35=2x$$

The equality is:

$$a^2+b^2+c^2=2a+6b+10c-35$$

$$a^2-2a+1+b^2-6b+9+c^2-10c+25=0$$

$$(a-1)^2+(b-3)^2+(c-5)^2=0\Leftrightarrow$$

$$(a-1)^2=0\ \wedge\ (b-3)^2=0\ \wedge\ (c-5)^2=0\Leftrightarrow$$

Then we have:

$$a=1\ \wedge b=3\ \wedge c=5\Longrightarrow$$

$$\sqrt{x-y}=1\ ;\ \sqrt{y-z}=3\ ;\ \sqrt{z+x}=5\Longrightarrow$$

$$x-y=1\,;\ y-z=9\,;\ z+x=25\Rightarrow$$

$$\textit{Solving the system}:$$

$$\textit{Answer}: x=\frac{35}{2}\,;y=\frac{33}{2}\,;z=\frac{15}{2}$$

709. **Solve for real numbers:**

$$x^2-2x+29=2\sqrt{x^2-x+1}+4\sqrt{x+3}+6\sqrt{11-2x}$$

Proposed by Nguyen Hung Viet-Vietnam

Solution by Miguel Velasquez Culque-Peru

$$\textit{Change of variables}:$$

$$\sqrt{x^2-x+1}=a\longrightarrow x^2-x+1=a^2$$

$$\sqrt{x+3}=b\longrightarrow x+3=b^2$$

$$\sqrt{11-2x}=c\longrightarrow 11-2x=c^2$$

$\textit{Sum of values}:\ x^2-2x+15=a^2+b^2+c^2$

$$x^2-2x+15+14=a^2+b^2+c^2+14$$

$$x^2-2x+15+29=a^2+b^2+c^2+14$$

$\textit{Replace the values}: a^2+b^2+c^2+14=2a+4b+6c$

$$a^2-2a+1+b^2-4b+4+c^2-6c+9=0$$

$$(a-1)^2+(b-2)^2+(c-3)^2=0\longrightarrow$$

$$a-1=0\ \vee\ b-2=0\ \vee c-3=3$$

$$a=1\ \vee\ b=2\ \vee c=3$$

$$\textit{But we know}: \sqrt{x^2-x+1}=1\ \vee\ \sqrt{x+3}=2\ \vee\ \sqrt{11-2x}=3$$

$$x^2-x+1=1\ \vee\ x+3=4\ \vee\ 11-2x=9$$

$$x(x-1)=0\ \vee\ x=1\ \vee\ 2=2x$$

$x=0\ \vee\ x=1\ \vee\ x=1\ \ \vee\ x=1.\ \textit{Answer}: x=1$

710. **Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that**

$$(9x)+f(16x)=2f(12x)\qquad,\quad \forall\in\mathbb{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Khaled Abd Imouti-Syria

$$f(9x)+f(16x)=2f(12x)\qquad \forall\in\mathbb{R}$$

$$f\left(\frac{5}{12}\right)+f\left(\frac{16}{12}\right)=2f(x)$$

$$\boxed{2f(x)=f\left(\frac{3}{4}x\right)+f\left(\frac{4}{3}x\right)}$$

$$f(x)=\frac{1}{2}\left[f\left(\frac{3}{4}x\right)+f\left(\frac{4}{3}x\right)\right]$$
$$f\left(\frac{4}{3}x\right)=\frac{1}{2}\left[f(x)+f\left(\frac{16}{9}x\right)\right]$$
$$\boxed{f(x)=2f\left(\frac{4}{3}x\right)-f\left(\frac{16}{9}x\right)}$$

In similar way:

$$\boxed{f(x)=2f\left(\frac{3}{4}x\right)-f\left(\frac{9}{16}x\right)}$$
$$2f(x)=2\left[f\left(\frac{4}{3}x\right)+f\left(\frac{3}{4}x\right)\right]-\left[f\left(\frac{16}{9}x\right)+f\left(\frac{9}{16}x\right)\right]$$
$$2f(x)=2\big(2f(x)\big)-\left(f\left(\frac{16}{9}x\right)+f\left(\frac{9}{16}x\right)\right)$$
$$\boxed{2f(x)=f\left(\frac{16}{9}x\right)+f\left(\frac{9}{16}x\right)}$$

So from (*) and (**) :

$$2f(x)=f\left(\left(\tfrac{4}{3}\right)^n x\right)+f\left(\left(\tfrac{3}{4}\right)^n x\right)\quad,\ \forall n\geq 1$$
$$2f\left(\left(\frac{3}{4}\right)^n x\right)=f(x)+f\left(\left(\frac{3}{4}\right)^{2n} x\right)\qquad (***)$$

because f is continuous by taking limits of two sides:

$$from\ (***): 2f(0)=f(x)+f(0)$$
$$\text{so}: f(x)=f(0)=c$$

so : f is constant

711. ***Find x,y,z and $t\in Z^+$:***

$$\frac{y+z+t}{x}+\frac{z+t}{x+y}+\frac{t}{x+y+z}=x+y+z+t-4$$

Proposed by Bui Hong Suc-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\left(\frac{y+z+t}{x}+1\right)+\left(\frac{z+t}{x+y}+1\right)+\left(\frac{t}{x+y+z}+1\right)=x+y+z+t-1$$
$$\frac{y+x+z+t}{x}+\frac{y+x+z+t}{x+y}+\frac{y+x+z+t}{x+y+z}=x+y+z+t-1$$
$$x+y+z+t>0$$
$$\frac{1}{x}+\frac{1}{x+y}+\frac{1}{x+y+z}+\frac{1}{x+y+z+t}=1$$
$$1)\ case\ \ x=1\ \ cannot\ \ be\ \ solved:$$

$$\frac{1}{x}>\frac{1}{x+y}>\frac{1}{x+y+z}>\frac{1}{x+y+z+t}\rightarrow\frac{4}{x}>1\rightarrow\boxed{x<4}$$

2) case iff $x=2$. Then:

$$\frac{1}{2+y}+\frac{1}{2+y+z}+\frac{1}{2+y+z+t}=\frac{1}{2}$$

$$\frac{3}{2+y}>\frac{1}{2}\rightarrow y+2<6\rightarrow\boxed{y<4}$$

iff $x=2;\ y=1$. Then:

$$\frac{1}{3+z}+\frac{1}{3+z+t}=\frac{1}{6}\rightarrow\frac{2}{3+z}>\frac{1}{6}\rightarrow\boxed{z<9}$$

The set of solutions satisfying the condition:

$x=2;\ y=1,\quad \boxed{z<9}\rightarrow\{(2;1;4;35),(2;1;5;16),(2;1;6;9),(2;1;7;5)\}$

3) case. For the condition $x=2$. Let's look at the case $y=2$

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{4+z}+\frac{1}{4+z+t}=1\rightarrow\frac{1}{4+z}+\frac{1}{4+z+t}=\frac{1}{4}\rightarrow\frac{2}{4+z}>\frac{1}{4}\rightarrow\boxed{z<4}$$

So, $x=2,\quad y=2$ and $z<4$

In this case, a set of solutions: $\{(2;2;1;15),(2;2;2;6)\}$

4) case. $x=2,\quad y=3$

$$\frac{1}{2}+\frac{1}{5}+\frac{1}{7+z}+\frac{1}{7+z+t}=1,\qquad\frac{1}{7+z}+\frac{1}{7+z+t}=\frac{10}{3}\rightarrow\frac{2}{7+z}>\frac{3}{10}\rightarrow z<-\frac{1}{3}$$

So, in this case there is no solution.

5) case. $x=3$

$$\frac{1}{3}+\frac{1}{3+y}+\frac{1}{3+y+z}+\frac{1}{3+y+z+t}=1\rightarrow\frac{1}{3+y}+\frac{1}{3+y+z}+\frac{1}{3+y+z+t}=\frac{2}{3}$$

$$\rightarrow\frac{3}{3+y}>\frac{2}{3}\rightarrow 6+2y<9\rightarrow\boxed{y<1.5}$$

$$\text{iff } y=1.\text{ Then }\frac{1}{3}+\frac{1}{4}+\frac{1}{4+z}+\frac{1}{4+z+t}=1\rightarrow\frac{1}{4+z}+\frac{1}{4+z+t}=\frac{5}{12}$$

$$\rightarrow\frac{2}{4+z}>\frac{5}{12}\rightarrow 20+5z<24\rightarrow 5z<4\rightarrow\boxed{z<\frac{4}{5}}$$

The resulting answer:

$\{(2;1;4;35),(2;1;5;16),(2;1;6;9),(2;1;7;5),(2;2;1;15),(2;2;2;6)\}$

712. Solve for real numbers:

$$(1+x)\sqrt{1-x}+(1-x)\sqrt{1+x}=2\sqrt{1+x^2}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$(1+x)\sqrt{1-x}+(1-x)\sqrt{1+x}=2\sqrt{1+x^2}$$

$$\sqrt{1-x^2}\left(\sqrt{1+x}+\sqrt{1-x}\right)=2\sqrt{1+x^2}\ ,-1\le x\le 1$$

$$(1-x^2)\left(2+2\sqrt{1-x^2}\right)=4(1+x^2)$$

$$Let\ \ x=\sin(t)$$

$$1.\ \ \left(1-sin^2(t)\right)(1+\cos(t))=2\left(2-cos^2(t)\right)$$

$$case\ \ \cos(t)\ge 0$$

$$cos^3(t)+3cos^2(t)-4=0,\qquad (\cos(t)-1)(\cos(t)+2)^2=0$$

$$\cos(t)=1\ ;\ \cos(t)\ne 2$$

$$\cos(t)=1\to\ \ x=0$$

$$2.\ \ \left(1-sin^2(t)\right)(1-\cos(t))=2\left(2-cos^2(t)\right)$$

$$case\ \ \cos(t)<0$$

$$cos^3(t)-3cos^2(t)+4=0,\qquad (\cos(t)+1)(\cos(t)-2)^2=0$$

$$\cos(t)=-1\ ;\ \cos(t)\ne 2$$

$$\cos(t)=-1\to\ \ x=0$$

$$So\ \ answer\ \ \{0\}$$

713. Find *all* values of $x,y,z\in\mathbb{R}$ such that :

$$xyz(xy+yz+zx)=x^2+y^2+z^2=3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$a^2+b^2+c^2-ab-bc-ca=\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)\ge 0$$

$$\Rightarrow a^2+b^2+c^2\ge ab+bc+ca\ \forall\ a,b,c\in\mathbb{R}\to(1)$$

$$\text{Putting } a=xy, b=yz, c=zx \text{ in } (1), \sum_{cyc}x^2y^2\ge xyz\sum_{cyc}x$$

$$\forall\ x,y,z\in\mathbb{R}\to\ (2)\ ('' = '' \text{ iff } x=y=z)\ and$$

$$\text{putting } a=x^2, b=y^2, c=z^2 \text{ in } (1), \sum_{cyc}x^4\ge\sum_{cyc}x^2y^2\overset{\text{via }(2)}{\ge}xyz\sum_{cyc}x$$

$$\forall\ x,y,z\in\mathbb{R}\to(3)\ ('' = '' \text{ iff } x=y=z)$$

$$\therefore (2)+(3)\Rightarrow\sum_{cyc}x^4+\sum_{cyc}x^2y^2\ge 2xyz\sum_{cyc}x\Rightarrow\sum_{cyc}x^4+2\sum_{cyc}x^2y^2\ge$$

$$\sum_{cyc}x^2y^2+2xyz\sum_{cyc}x\Rightarrow\left(\sum_{cyc}x^2\right)^2\boxed{\ge}\left(\sum_{cyc}xy\right)^2$$

$$\forall\ x,y,z\in\mathbb{R}\ \to(4)\ ('' = '' \text{ iff } x=y=z)$$

$$\text{Now}, \forall\ a,b,c\ge 0, \sum_{cyc}a\ge 3.\sqrt[3]{abc}\ \therefore\ assigning\ a=x^2, b=y^2, c=z^2,$$

$$\sum_{cyc} x^2 \ge 3.\sqrt[3]{x^2y^2z^2} \Rightarrow \left(\sum_{cyc} x^2\right)^3 \boxed{\ge} 27x^2y^2z^2$$

$$\forall\, x,y,z \in \mathbb{R} \rightarrow (5)\ ('' = ''\text{ iff } x^2 = y^2 = z^2)$$

$$\therefore (2)\bullet(3) \Rightarrow \left(\sum_{cyc} x^2\right)^5 \ge 27x^2y^2z^2\left(\sum_{cyc} xy\right)^2 \quad \forall\, x,y,z \in \mathbb{R}$$

$$('' = ''\text{ iff } x = y = z) \overset{x^2+y^2+z^2=3}{\Rightarrow} 27\left(\sum_{cyc} x^2\right)^2 \ge 27x^2y^2z^2\left(\sum_{cyc} xy\right)^2$$

$$\Rightarrow \boxed{\left(\sum_{cyc} x^2\right)^2 \ge x^2y^2z^2\left(\sum_{cyc} xy\right)^2 \quad \forall\, x,y,z \in \mathbb{R}\ ('' = ''\text{ iff } x = y = z)}\text{ but,}$$

$$\left(\sum_{cyc} x^2\right)^2 = x^2y^2z^2\left(\sum_{cyc} xy\right)^2 \quad (\because xyz(xy+yz+zx) = x^2+y^2+z^2)$$

$$\therefore xyz\left(\sum_{cyc} xy\right) = \sum_{cyc} x^2 = 3 \Rightarrow x = y = z \therefore x^3.3x^2 = 3$$

$$\left(\text{using } xyz\left(\sum_{cyc} xy\right) = 3\right) \Rightarrow x = 1$$

$\therefore$ $(x = y = z = 1)$ **is the only desired set of values** (ans)

714. Find *all values* of $x,y,z \in \mathbb{R}$ such that :

$$x^3y^3z^3(x^2y^2+y^2z^2+z^2x^2) = x^2+y^2+z^2 = 3$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\forall\, a,b,c \ge 0, \sum_{cyc} a \ge 3.\sqrt[3]{abc} \therefore \text{assigning } a = x^2, b = y^2, c = z^2,$$

$$3 = \sum_{cyc} x^2 \ge 3.\sqrt[3]{x^2y^2z^2} \Rightarrow x^2y^2z^2 \le 1 \Rightarrow xyz \le 1 \rightarrow (1)$$

$$\left(\because x^3y^3z^3\left(\sum_{cyc} x^2y^2\right) = 3 \Rightarrow x^3y^3z^3 > 0 \Rightarrow xyz > 0\right)$$

$$\text{Now, } \sum_{cyc} x^2 = x^3y^3z^3\left(\sum_{cyc} x^2y^2\right) \overset{via\ (2)}{\leq} \sum_{cyc} x^2y^2 \overset{x^2+y^2+z^2=3}{\Rightarrow}$$

$$\left(\sum_{cyc} x^2\right)^2 \leq 3\sum_{cyc} x^2y^2 \Rightarrow \sum_{cyc} x^4 - \sum_{cyc} x^2y^2 \leq 0 \Rightarrow \boxed{\frac{1}{2}\sum_{cyc}(x^2-y^2)^2 \leq 0};$$

$$\text{but } \frac{1}{2}\sum_{cyc}(x^2-y^2)^2 \geq 0 \therefore \sum_{cyc}(x^2-y^2)^2 = 0 \Rightarrow x^2 = y^2 = z^2$$

$$\Rightarrow 3x^2 = 3\left(\text{using } \sum_{cyc} x^2 = 3\right) \therefore \boxed{x = \pm 1, y = \pm 1, z = \pm 1} \to (2)$$

$$\text{Using } x^3y^3z^3\left(\sum_{cyc} x^2y^2\right) = 3, \text{we get} : 3xyz = 3\ (\because x^2 = y^2 = z^2 = 1)$$

$$\Rightarrow \boxed{xyz = 1} \to (3) \therefore (2) \text{ and } (3) \Rightarrow$$

$$\begin{pmatrix} x=1 \\ y=1 \\ z=1 \end{pmatrix}, \begin{pmatrix} x=1 \\ y=-1 \\ z=-1 \end{pmatrix}, \begin{pmatrix} x=-1 \\ y=1 \\ z=-1 \end{pmatrix} \text{ and } \begin{pmatrix} x=-1 \\ y=-1 \\ z=1 \end{pmatrix}$$

$$\text{are the only set of values} \mid x^3y^3z^3(x^2y^2+y^2z^2+z^2x^2) = x^2+y^2+z^2 = 3\ (ans)$$

715. Find all values of $x, y, z \in \mathbb{Z}$ such that :

$$x^4 + 9y^2 + 25z^2 = x^2 + 6xy + 2022$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Mohamed Amine Ben Ajiba-Tanger-Morocco

The equation is equivalent to :

$$9y^2 - 6xy + x^4 - x^2 - 2022 + 25z^2 = 0.$$

This equation has integral solutions if and only if its discriminant

$$36[2023 - (x^2-1)^2 - 25z^2]$$

is a perfect square. It follows that

$$2023 - (x^2-1)^2 - 25z^2 = t^2 \text{ or } (x^2-1)^2 + 25z^2 + t^2 = 2023.$$

For any integer n, we have $n^2 \equiv 0,1,4\ (mod 8)$, then

$$(x^2-1)^2+25z^2+t^2 \equiv 0,1,2,3,4,5,6\ (mod 8),$$

but we have $2023 \equiv 7\ (mod 8)$. So the equation has no solution in integers.

716. If $A,B \in M_2(\mathbb{R})$ then:

$$det((AB-BA)^{1000}+(AB+BA)^{1000}) \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Let be:

$$C=(AB-BA)^{500}, \quad D=(AB+BA)^{500}, \quad z=det(C+iD)$$

$$C,D \in M_2(\mathbb{R}) \Rightarrow det(C-iD)=\overline{det(C+iD)}=\bar{z}$$

$$det((AB-BA)^{1000}+(AB+BA)^{1000})=det(C^2+D^2)=$$

$$=det(C^2-i^2D^2)=det\big((C+iD)(C-iD)\big)=$$

$$=det(C+iD)\cdot det(C-iD)=z\cdot\bar{z}=|z|^2 \geq 0$$

717. Solve for $x \in \left(0,\frac{\pi}{2}\right)$:

$$1+\frac{1}{2}\cos x+\frac{1}{4}\cos 2x+\frac{1}{8}\cos 3x+\frac{1}{16}\cos 4x+\cdots=\frac{23+3\sqrt{5}}{22}$$

Proposed by Netai Chandra Bhar-India

Solution by Tapas Das-India

$$We\ know\ that\ e^{i\theta}=\cos\theta+i\sin\theta\ and\ \cos\theta=\frac{\left(e^{i\theta}+e^{-i\theta}\right)}{2}$$

$$Componendo\ dividendo\ if\ \frac{a}{b}=\frac{c}{d}\ then\ \frac{a+b}{a-b}=\frac{c+d}{c-d}$$

$$\cos\frac{\pi}{5}=\frac{\sqrt{5}+1}{4}$$

$$1+\frac{1}{2}\cos x+\frac{1}{4}\cos 2x+\frac{1}{8}\cos 3x+\frac{1}{16}\cos 4x+\cdots=$$

$$=\left(\frac{1}{2}+\frac{1}{2}\right)+\frac{1}{2}\frac{e^{ix}+e^{-ix}}{2}+\frac{1}{4}\frac{e^{i2x}+e^{-i2x}}{2}+\frac{1}{8}\frac{e^{i3x}+e^{-i3x}}{2}+\frac{1}{16}\frac{e^{i4x}+e^{-i4x}}{2}\ \ldots=$$

$$=\frac{1}{2}\left(1+\frac{1}{2}e^{ix}+\frac{1}{4}e^{2ix}+\frac{1}{8}e^{3ix}+\frac{1}{16}e^{4ix}+\cdots\right)+$$

$$+\frac{1}{2}\left(1+\frac{1}{2}e^{-ix}+\frac{1}{4}e^{-2ix}+\frac{1}{8}e^{-3ix}+\frac{1}{16}e^{-4ix}+\cdots\right)$$

$$=\frac{1}{2}\frac{1}{1-\frac{1}{2}e^{ix}}+\frac{1}{2}\frac{1}{1-\frac{1}{2}e^{-ix}}(\,infinite\ G.P\ series)$$

$$=\frac{1}{2-e^{ix}}+\frac{1}{2-e^{-ix}}=\frac{4-(e^{ix}+e^{-ix})}{(2-e^{ix})(2-e^{-ix})}=$$

$$=\frac{4-2\cos x}{(2-\cos x+i\sin x)(2-\cos x+i\sin x)}=$$

$$=\frac{4-2\cos x}{(2-\cos x)^2+\sin^2 x}=\frac{4-2\cos x}{5-4\cos x}$$

$$Now\ by\ the\ problem:\ \frac{4-2\cos x}{5-4\cos x}=\frac{23+3\sqrt{5}}{22}$$

$$\frac{8-4\cos x}{5-4\cos x}=\frac{23+3\sqrt{5}}{11}$$

$$\frac{13-8\cos x}{3}=\frac{(34+3\sqrt{5})}{12+3\sqrt{5}}(using\ Componendo\ dividendo)$$

$$or\ 13-8\cos x=\frac{(34+3\sqrt{5})}{4+\sqrt{5}}\ or\ 8\cos x=13-\frac{(34+3\sqrt{5})}{4+\sqrt{5}}$$

$$or\ 8\cos x=\frac{18+10\sqrt{5}}{4+\sqrt{5}}\ or\ 4\cos x=\frac{9+4\sqrt{5}}{4+\sqrt{5}}=\frac{(4+\sqrt{5})(\sqrt{5}+1)}{4+\sqrt{5}}$$

$$or\ \cos x=\frac{(\sqrt{5}+1)}{4}=\cos\frac{\pi}{5}\ or\ x=\frac{\pi}{5}$$

718. Prove that:

$$\sin\frac{\pi}{14}+6\sin^2\frac{\pi}{14}-8\sin^4\frac{\pi}{14}=\frac{1}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$We\ know\ that:$$

$$\sin 3\theta=3\sin\theta-4\sin^3\theta\quad(1)$$

$$2\sin A\cos B=\sin(A+B)+\sin(A-B)\ (2)$$

$$Let\ \frac{\pi}{14}=\theta\ then\ 14\theta=\pi\ and\ 7\theta=\frac{\pi}{2}$$

$$\sin\frac{\pi}{14}+6\sin^2\frac{\pi}{14}-8\sin^4\frac{\pi}{14}=\sin\theta+6\sin^2\theta-8\sin^4\theta=$$

$$=\sin\theta+2\sin\theta(3\sin\theta-4\sin^3\theta)=\sin\theta+2\sin\theta\sin 3\theta=$$

$$=\sin\theta+(\cos 2\theta-\cos 4\theta)$$

$$=\sin\theta+\cos(7\theta-5\theta)-\cos(7\theta-3\theta)=\sin\theta+\cos\left(\frac{\pi}{2}-5\theta\right)-\cos\left(\frac{\pi}{2}-3\theta\right)$$

$$=\sin\theta+\sin 5\theta-\sin 3\theta=\frac{1}{2\cos\theta}(2\cos\theta\sin\theta+2\cos\theta\sin 5\theta-2\cos\theta\sin 3\theta)=$$

$$=\frac{1}{2\cos\theta}(\sin 2\theta+\sin 6\theta+\sin 4\theta-\sin 4\theta-\sin 2\theta)=$$

$$=\frac{\sin 6\theta}{2\cos\theta}=\frac{\sin(7\theta-\theta)}{2\cos\theta}=\frac{\sin\left(\frac{\pi}{2}-\theta\right)}{2\cos\theta}=\frac{\cos(\theta)}{2\cos\theta}=\frac{1}{2}$$

719.

Find the last digit of $A=2024^{2025^{2026}}+2026^{2027^{2028}}+2028^{2029^{2030}}$

Proposed by Nguyen Van Canh-Vietnam

Solution by Tapas Das-India

Now we need to find the last digit of $2024^{2025^{2026}}$
last digit of 2024 *is* 4
now, 4^1 *ends in* 4, 4^2 *ends in* 6, 4^3 *ends in* 4
the pattern is $4,6$ *which repeats every* 2 *powers* (1)

We know that any psitive integer power of an odd number odd
2025 *is an odd then* 2025^{2026} *is an odd number, if expnent is odd, the last digit of* $2024^{2025^{2026}}$ *is* 4 *(using* (1) *)* (2)

now we need to find the last digit of $2026^{2027^{2028}}$
clearly, $6^1=6, 6^2=36, 6^3=216, \dots\dots$
so clearly last digit of $2026^{2027^{2028}}$ *is* 6 (3)

now we need to find the last digit of $2028^{2029^{2030}}$
for this we can focus on the last digt of 2028,
which is 8 *and*
$8^1=8$, 8^2 *ends in* 4, 8^3 *ends in* 2, 8^4 *ends in* 6
the pattern repeats every 4 *power*
now, $2029\equiv 1(mod\ 4), (2029)^{2030}\equiv 1^{2030}(mod\ 4), 2029^{2030}\equiv 1(mod\ 4)$

So, last digit of $2028^{2029^{2030}}$ *is* $8^1=8$ (4)

Now using result $(2),(3),(4)$ *we get* $4+6+8=18$
last digit of A *is* 8

720.

$$\text{If}\begin{cases} a+b+c=0 \\ ab+ac+bc=-3 \\ bac=2 \end{cases} \text{ then find } \frac{\sum a^2\sum a^6}{\sum a^3\sum a^5}$$

Proposed by Marin Chirciu-Romania

Solution by Togrul Ehmedov-Azerbaijan

We know that:

$$\sum a^2=\left(\sum a\right)^2-2\sum ab\Rightarrow\sum a^2=6$$

a,b and c are the roots of the equation:

$$x^3-3x-2=0$$

$$\begin{cases} \sum a^3-3\sum a-6=0 \\ \sum a^4-3\sum a^2-2\sum a=0 \\ \sum a^5-3\sum a^3-2\sum a^2=0 \\ \sum a^6-3\sum a^4-2\sum a^3=0 \end{cases}\Rightarrow\begin{cases} \sum a^3=6 \\ \sum a^4=18 \\ \sum a^5=30 \\ \sum a^6=66 \end{cases}\Rightarrow\frac{\sum a^2\sum a^6}{\sum a^3\sum a^5}=2,2$$

721. Solve for real numbers:

$$a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c},\qquad ab+bc+ca=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ a+b+c=p,ab+bc+ca=q,abc=r$$

$$then\ \ a+b+c=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ or,a+b+c=\frac{ab+bc+ca}{abc}\ or,p=\frac{q}{r}\ \ or,pr=q\ (1)$$

$$ab+bc+ca=\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\ \ or,ab+bc+ca=\frac{a+b+c}{abc}\ or,$$

$$q=\frac{p}{r}\ or,qr=p\ (2)$$

$$From\ (2),qr=p\ or,(pr).r\overset{(1)}{=}p\ \ or,p(r^2-1)=0\ which\ implies\ r=1,-1$$

$$p=0\ not\ possible\ because$$

$$p = a + b + c = 0 \text{ gives from } (1)\ q = ab + bc + ca = 0$$
$$\text{as a result } (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca) \text{ or}, a^2 + b^2 + c^2 = 0$$
$$\Rightarrow a = b = c = 0, \text{for this value reciprocal are not defined}, \text{so } p \neq 0$$

$$\text{Case} - 1:\ r = 1 \text{ then from}(1)\text{we get } p = q$$
$$\text{known result } (a-1)(b-1)(c-1) = abc - (ab + bc + ca) + (a+b+c) - 1$$
$$= r - q + p - 1 = 1 - p + p - 1 = 0$$
$$\text{or}, (a-1)(b-1)(c-1)$$
$$= 0 \text{ which implies that at least one of the number } a, b, c$$
$$\text{is equal to } 1.\ \text{If } a = 1 \text{ then } r = abc = 1 \Rightarrow bc = 1 \text{ or}, c = \frac{1}{b}$$
$$\text{solution for Case} - 1, \text{all the permutation } \left(1, t, \frac{1}{t}\right), t \in R - \{0\}$$
$$\text{Case} - 2: r = -1 \text{ then from}(1)\text{we get } q = -p$$
$$\text{Known result } (a+1)(b+1)(c+1) = abc + (ab + bc + ca) + (a+b+c) + 1$$
$$= r + q + p + 1 = -1 - p + p + 1 = 0 \text{ or}, (a+1)(b+1)(c+1) = 0$$
$$\text{which implies that at least one of the number } a, b, c \text{ is equal to } -1$$
$$\text{If } a = -1 \text{ then } r = abc \Rightarrow -1 = -bc \text{ or}, c = \frac{1}{b}$$
$$\text{Ssolution for Case} - 2, \text{all the permutation } \left(-1, t, \frac{1}{t}\right), t \in R - \{0\}$$

722.

If $a, b, c > 0, b^2 + c^2 = a^2$ and the equation : $-ax^2 + bx + c = 0$ has 2 real roots $x_1 < x_2$, then prove that : $-\sqrt{2} < x_1 < x_2 < \sqrt{2}$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Easy to see that}: x_1 = \frac{b - \sqrt{b^2 + 4ac}}{2a} \text{ and } x_2 = \frac{b + \sqrt{b^2 + 4ac}}{2a}$$
$$\therefore x_1 > \frac{-\sqrt{b^2 + 4ac}}{2a} \overset{?}{>} -\sqrt{2} \Leftrightarrow 2 \overset{?}{>} \frac{b^2 + 4ac}{4a^2} \overset{b^2+c^2=a^2}{\Leftrightarrow} 8a^2 \overset{?}{>} a^2 - c^2 + 4ac$$
$$\Leftrightarrow (2a - c)^2 + 3a^2 \overset{?}{>} 0 \to \text{true} \therefore x_1 > -\sqrt{2} \text{ and again}, x_2 = \frac{b + \sqrt{b^2 + 4ac}}{2a} \overset{?}{<} \sqrt{2}$$
$$\overset{b^2+c^2=a^2}{\Leftrightarrow} \sqrt{\frac{a^2 - c^2}{a^2}} + \sqrt{\frac{a^2 - c^2 + 4ac}{a^2}} \overset{?}{<} 2\sqrt{2}$$
$$\Leftrightarrow \left(\sqrt{1 - t^2} + \sqrt{1 - t^2 + 4t}\right)^2 \overset{?}{<} \left(2\sqrt{2}\right)^2 \left(t = \frac{c}{a}\right)$$
$$\Leftrightarrow 2 - 2t^2 + 4t + 2.\sqrt{(1 - t^2)(1 - t^2 + 4t)} \overset{?}{<} 8$$

$$\Leftrightarrow t^2-2t+3 \overset{?}{>} \sqrt{(1-t^2)(1-t^2+4t)} \Leftrightarrow (t^2-2t+3)^2 \overset{?}{>} (1-t^2)(1-t^2+4t)$$

$$(\because t^2-2t+3=(t-1)^2+2>0) \Leftrightarrow 3t^2-4t+2 \overset{?}{>} 0 \Leftrightarrow 2(t-1)^2+t^2 \overset{?}{>} 0$$

$$\therefore x_2<\sqrt{2} \therefore -\sqrt{2}<x_1<x_2<\sqrt{2}\ \forall\, a,b,c>0, b^2+c^2=a^2 \text{ such that}$$

the equation : $-ax^2+bx+c=0$ has 2 real roots $x_1<x_2$ (QED)

723.

If $a,b>0, n\geq 0$ and the equation : $x^3-ax^2+bx-a=0$ has 3 roots

$$x_1,x_2,x_3>1, \text{then prove that : } \frac{b^n-3^n}{a^n}\geq 3^{\frac{n}{2}}-\left(\frac{1}{\sqrt{3}}\right)^n$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

Via Viete, $x_1+x_2+x_3=a, x_1x_2+x_2x_3+x_3x_1=b$ and $x_1x_2x_3=a$ and $\because (x_1+x_2+x_3)^2\geq 3(x_1x_2+x_2x_3+x_3x_1)$ and

$$(x_1x_2+x_2x_3+x_3x_1)^2\geq 3x_1x_2x_3(x_1+x_2+x_3) \therefore a^2\geq 3b \text{ and } b^2 \overset{①}{\geq} 3a^2$$

$$\Rightarrow b^2\geq 3a^2\geq 9b \Rightarrow b \overset{②}{\geq} 9 \therefore \frac{b^n-3^n}{a^n} \overset{\text{via ①}}{\geq} \frac{b^n-3^n}{\left(\frac{b}{\sqrt{3}}\right)^n} (\because n\geq 0) = 3^{\frac{n}{2}}-\frac{3^n.3^{\frac{n}{2}}}{b^n} \overset{\text{via ②}}{\geq}$$

$$3^{\frac{n}{2}}-\frac{3^n.3^{\frac{n}{2}}}{9^n} (\because n\geq 0) = 3^{\frac{n}{2}}-\frac{1}{3^{\frac{n}{2}}} \therefore \frac{b^n-3^n}{a^n}\geq 3^{\frac{n}{2}}-\left(\frac{1}{\sqrt{3}}\right)^n\ \forall\, a,b>0, n\geq 0$$

such that the equation : $x^3-ax^2+bx-a=0$ has 3 roots $x_1,x_2,x_3>1$,

$$''='' \text{ iff } \left(a=3\sqrt{3}, b=9\right) \text{ (QED)}$$

724. ***If the equation $ax^3+bx^2+cx+d=0, (a\neq 0)$ has 3 roots $x_1,x_2,x_3>0$ then prove that:***

$$x_1^7+x_2^7+x_3^7\geq -\frac{b^3c^2}{81a^5}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$ax^3+bx^2+cx+d=0 \ (1)$$

Relationship between roots and coefficient we get

$$x_1+x_2+x_3=-\frac{b}{a}$$

$$x_1x_2+x_2x_3+x_3x_1=\frac{c}{a}$$

$$x_1x_2x_3 = -\frac{d}{a}$$

$$x_1^7 + x_2^7 + x_3^7 \overset{Chebyshev}{\geq} \frac{1}{3}(x_1^3 + x_2^3 + x_3^3)(x_1^4 + x_2^4 + x_3^4) \overset{CBS}{\geq}$$

$$\geq \frac{1}{3} \cdot \frac{(x_1 + x_2 + x_3)^3}{9} \cdot \frac{(x_1^2 + x_2^2 + x_3^2)^2}{3} \geq$$

$$\geq \frac{1}{81}(x_1 + x_2 + x_3)^3(x_1x_2 + x_2x_3 + x_3x_1)^2 = \frac{1}{81}\left(-\frac{b}{a}\right)^3\left(\frac{c}{a}\right)^2 = -\frac{b^3c^2}{81a^5}$$

725. ***If the equation*** $x^5 + ax^4 + bx^3 + cx^2 + dx + 1 = 0$ ***has 5 real roots are different in pairs then prove that*** $2(a^2 + d^2) > 5(b + c)$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^5 + ax^4 + bx^3 + cx^2 + dx + 1 = 0 \quad (1)$$

Let r_1, r_2, r_3, r_4, r_5 ***be the roots of the equation*** (1)

Relationships between roots and coefficient we get:

$$p_1 = \sum_{i=1}^{5} r_i = -a$$

$$p_2 = \sum_{1 \leq i < j \leq 5} r_ir_j = b$$

$$p_3 = \sum_{1 \leq i < j < k \leq 5} r_ir_jr_k = -c$$

$$p_4 = \sum_{1 \leq i < j < k < l \leq 5} r_ir_jr_kr_l = d$$

$$p_5 = \prod_{i=1}^{5} r_i = -1$$

$$\sum_{i=1}^{5} \frac{1}{r_i} = \left(\sum_{1 \leq i < j < k < l \leq 5} r_ir_jr_kr_l\right) \times \frac{1}{\prod_{i=1}^{5} r_i} = \frac{p_4}{p_5} = -d \quad (1)$$

$$\left(\sum_{i=1}^{5} \frac{1}{r_i^2}\right) = \left(\sum_{i=1}^{5} \frac{1}{r_i}\right)^2 - 2\sum_{1 \leq i < j \leq 5} \frac{1}{r_ir_j} =$$

$$=\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2-2\left(\sum_{1\le i<j<k\le 5}r_ir_jr_k\right)\times\frac{1}{\prod_{i=1}^{5}r_i}\overset{(1)}{=}d^2-2\left(\frac{p_3}{p_5}\right)=d^2-2c\ \ (2)$$

$$\left(\sum_{i=1}^{5}r_i^2\right)\left(\sum_{i=1}^{5}1^2\right)\overset{Cauchy-Schwarz}{>}\left(\sum_{i=1}^{5}r_i\right)^2$$

$$or 5\left(\sum_{i=1}^{5}r_i^2\right)>(p_1)^2\ or\ 5\left(\left(\sum_{i=1}^{5}r_i\right)^2-2\sum_{1\le i<j\le 5}r_ir_j\right)>(p_1)^2$$

$$or\ 5(p_1^2-2p_2)>(p_1)^2\ \ or\ 5(a^2-2b)>a^2\ \ or\ 4a^2>10b\ or\ 2a^2>5b\ (3)$$

$$\left(\sum_{i=1}^{5}\frac{1}{r_i^2}\right)\left(\sum_{i=1}^{5}1^2\right)\overset{Cauchy-Schwarz}{>}\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2\ \ or, 5\left(\sum_{i=1}^{5}\frac{1}{r_i^2}\right)>\left(\sum_{i=1}^{5}\frac{1}{r_i}\right)^2$$

$$or\ 5(d^2-2c)\overset{(1)\&(2)}{>}d^2\ or, 4d^2>10c\ \ or, 2d^2>5c\ \ (4)$$

$$Adding\ (3)\ \&\ (4) we\ get:$$

$$2a^2+2d^2>5b+5c\ \ or, 2(a^2+d^2)>5(b+c)$$

726. Solve for naturals:

$$4(x+y+z)+xyz=2(xy+yz+zx)+9$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$We\ know\ that:$$

$$(x-2)(y-2)(z-2)=xyz-2(xy+yz+zx)+4(x+y+z)-8$$

$$\Rightarrow 4(x+y+z)+xyz-2(xy+yz+zx)=(x-2)(y-2)(z-2)+8\ \ (1)$$

$$from\ the\ given\ condition, 4(x+y+z)+xyz=2(xy+yz+zx)+9$$

$$\Rightarrow 4(x+y+z)+xyz-2(xy+yz+zx)=9$$

$$\Rightarrow (x-2)(y-2)(z-2)+8\overset{(1)}{=}9\Rightarrow (x-2)(y-2)(z-2)=1$$

$$Thus\ we\ need\ integer\ tripletes\ whose\ product\ is\ 1$$

$Case-1$

$$1.1.1=1, for\ this\ x-2=y-2=z-2=1\ or, x=y=z=3$$
$$solution: (x,y,z)=(3,3,3)$$

$Case-2$

$$(-1).(-1).(1)=1\ (and\ permutation)$$
$$for\ this\ x-2=-1, y-2=-1, z-2=1\ which\ gives\ x=1, y=1, z=3$$
$$solutions: (x,y,z)=(1,1,3),(1,3,1),(3,1,1)$$

727. Solve for natural numbers:

$$x^3-y^3=2(x^2+y^2)+3xy+17$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^3-y^3=2(x^2+y^2)+3xy+17\ \ (1)$$
$$Clearly, x,y\in N\ for\ this\ R.H.S\ part>0, consequently\ L.H.S>0, x>y$$

$$let\ d=x-y\overset{(2)}{>}0\ then\ x=d+y\ \ whth\ this\ substitution\ from(1)we\ get$$
$$(d+y)^3-y^3=2((d+y)^2+y^2)+3(d+y)y+17$$

$$d(d^2+2dy+y^2+dy+y^2+y^2)=2(d^2+2dy+2y^2)+3y(d+y)+17$$

$$or, y^2(3d-7)+y(3d^2-7d)+(d^3-2d^2-17)=0\ \ (2)$$

$$we\ note\ that\ x,y\in N$$

$Case-1$:

$d=1\ then\ equation\ (2)becomes, -4y^2-4y-18=0\ or, 2y^2+2y+9=0, real\ root\ does\ not\ exist\ as\ discriminant=2^2-4.2.9=-68<0$

$Case-2$:

$d=2\ then\ equation\ (2)becomes, -y^2-2y-17=0\ or$
$y^2+2y+17=0, real\ root\ does\ not\ exist\ as\ discriminant=2^2-4.1.17=-64<0$

$Case-3$:

$$d=3\ then\ equation\ (2)becomes, 2y^2+6y-8=0\ or, (y+4)(y-1)=0$$
$$or, y=1\ (as\ y\in N, so\ y\neq-4), x=y+d=1+3=4$$
$$now\ from(2)\ for\ d\geq 4\ coeffcient\ of\ y^2\geq 3\times 4-7=5>0$$
$$coeffcient\ of\ y\geq 3\times 4^2-7\times 4=20>0$$
$$constant\ term\geq 4^3-32-17=64-49=15>0$$

$$clearly\ for\ d \geq 4\ equation\ (2)\ never\ be\ zero.$$

$$Required\ solution\ in\ N\ is\ x = 4, y = 1$$

728. Solve for natural numbers:

$$\sqrt[3]{x^3 + 4x^2 + 3} + \sqrt{x^2 + x + 3} = x + 4$$

Proposed by Sakthi Vel-India

Solution by Tapas Das-India

$$Let\ f(x) = \sqrt[3]{x^3 + 4x^2 + 3} + \sqrt{x^2 + x + 3} - (x + 4)$$

$$We\ are\ going\ to\ find\ x \in N\ for\ which\ f(x) = 0$$

$$for\ x = 1: f(1) = \sqrt[3]{8} + \sqrt{5} - (1 + 4) = \sqrt{5} - 3 \neq 0$$
$$for\ x = 2: f(2) = \sqrt[3]{27} + \sqrt{9} - (2 + 4) = 3 + 3 - 6 = 0$$

$$so\ x = 2\ is\ a\ solution\ of\ f(x) = 0,$$
$$Now\ we\ will\ show\ their\ does\ not\ exist\ any\ x \in N\ which\ satisfy\ f(x) = 0$$
$$for\ x \geq 3 \in N$$
$$We\ have\ (x+1)^3 = x^3 + 3x^2 + 3x + 1$$
$$x^3 + 4x^2 + 3 - (x+1)^3 = x^2 - 3x + 2 = (x-1)(x-2) > 0\ as\ x \geq 3$$
$$x^3 + 4x^2 + 3 > (x+1)^3 \Rightarrow \sqrt[3]{x^3 + 4x^2 + 3} > x + 1$$
$$Again\ for\ x \geq 3, x^2 + x + 3 \geq 3^2 + 3 + 3 = 15 > 9 \Rightarrow \sqrt{x^2 + x + 3} > \sqrt{9} = 3$$

$$Adding\ above\ two\ result\ we\ get:$$
$$\sqrt[3]{x^3 + 4x^2 + 3} + \sqrt{x^2 + x + 3} > x + 1 + 3 = x + 4$$
$$Clearly\ for\ x \geq 3 (\in N), f(x) \neq 0$$

$$Required\ solution\ x = 2.$$

729.

$$If\ a, b > 0\ and\ the\ equation$$
$$x^3 - ax^2 + bx - a = 0\ has\ 3\ roots\ x_1, x_2, x_3 \geq 1$$
$$then\ show\ that:$$
$$a \geq \left(\frac{1}{4} + \frac{\sqrt{2}}{8}\right)(b + 3)$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^3-ax^2+bx-a=0 \text{ has } 3 \text{ roots } x_1,x_2,x_3\geq 1$$

$$\text{then } x_1+x_2+x_3=a, x_1x_2+x_2x_3+x_3x_1=b, \quad x_1x_2x_3=a$$

$$\text{According to the condition } x_1+x_2+x_3=x_1x_2x_3 \ (1)$$

$$\frac{a}{b+3}=\frac{x_1+x_2+x_3}{x_1x_2+x_2x_3+x_3x_1+3} \text{ now we are going to find Min value of } \frac{a}{b+3}$$

$$\text{Since the function symmetric to get Min value we take:}$$
$$x_2=x_3=t\geq 1 \text{ and } x_1=x$$
$$\text{From (1) we get } x+2t=xt^2 \text{ or } x=\frac{2t}{t^2-1} \ (2)$$
$$a=x_1+x_2+x_3=2t+x=2t+\frac{2t}{t^2-1}=\frac{2t^3}{t^2-1}$$
$$b=x_1x_2+x_2x_3+x_3x_1=t^2+2xt=t^2+2t.\frac{2t}{t^2-1}=t^2+\frac{4t^2}{t^2-1}=\frac{t^2(t^2+3)}{t^2-1}$$
$$\frac{a}{b+3}=\frac{\frac{2t^3}{t^2-1}}{\frac{t^2(t^2+3)}{t^2-1}+3}=\frac{2t^3}{t^4+6t^2-3}$$

$$\text{Let } f(t)=\frac{2t^3}{t^4+6t^2-3}, f'(t)=\frac{-2t^2(t^2-3)^2}{(t^4+6t^2-3)^2}<0 \text{ as } t\geq 1 \text{ so } f(t) \text{ is decreasing}$$
$$\text{We declared } x=\frac{2t}{t^2-1} \text{ so } t\neq 1, t>1 \text{ now } x\geq 1 \text{ (given)}$$
$$\text{For this } \frac{2t}{t^2-1}\geq 1 \text{ or } t^2-2t-1\leq 0 \text{ or } \left(t-\left(\sqrt{2}+1\right)\right)\left(t-\left(1-\sqrt{2}\right)\right)\leq 0$$
$$\text{or } \left(1-\sqrt{2}\right)\leq t\leq\left(\sqrt{2}+1\right) \text{ as } t>1 \text{ then } 1<t<\left(\sqrt{2}+1\right)$$

$$\text{Since, } f(t) \text{ is decreasing so } f(t) \min \text{ at } t=\sqrt{2}+1$$
$$f\left(\sqrt{2}+1\right)=\frac{2\left(\sqrt{2}+1\right)^3}{\left(\sqrt{2}+1\right)^4+6\left(\sqrt{2}+1\right)^2-3}=\frac{2\left(2\sqrt{2}+7+3\sqrt{2}\right)}{\left(3+2\sqrt{2}\right)^2+6\left(3+2\sqrt{2}\right)-3}$$
$$=\frac{2\left(5\sqrt{2}+7\right)}{32+24\sqrt{2}}=\frac{1}{4}\frac{\left(5\sqrt{2}+7\right)}{\left(3\sqrt{2}+4\right)}=\frac{1}{4}\frac{\left(3\sqrt{2}-4\right)\left(5\sqrt{2}+7\right)}{\left(3\sqrt{2}-4\right)\left(3\sqrt{2}+4\right)}=\frac{1}{4}\frac{2+\sqrt{2}}{2}=\left(\frac{1}{4}+\frac{\sqrt{2}}{8}\right)$$

$$\text{Therefore:}$$

$$\frac{a}{b+3} \ge \left(\frac{1}{4}+\frac{\sqrt{2}}{8}\right) \text{ or } a \ge \left(\frac{1}{4}+\frac{\sqrt{2}}{8}\right)(b+3)$$
$$\text{when } x_1 = 1, x_2 = x_3 = \sqrt{2}+1$$

730. ***If*** $x^3 - 6x^2 + ax - b = 0$ ***has 3 roots*** $x_1, x_2, x_3 \ge 0$ ***prove that:***

$$8a - 3b \le 72$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^3 - 6x^2 + ax - b = 0 \text{ then } \sum x_1 = 6, \sum x_1x_2 = a, x_1x_2x_3 = b$$
$$8a - 3b = 8\left(\sum x_1x_2\right) - 3x_1x_2x_3$$

$$\text{Let } F(x_1, x_2, x_3) = 8\left(\sum x_1x_2\right) - 3x_1x_2x_3$$
$$F \text{ is symmetric, to find its Max value under}$$
$$\text{the constraint } x_1 + x_2 + x_3 = 6 \text{, we consider}$$
$$x_1 = x_2 = t, x_3 = 6 - 2t \text{ and } t \in [0,3] \text{ as }, x_i \ge 0$$

$$a = \sum x_1x_2 = t^2 + 2(6-2t), b = x_1x_2x_3 = t^2(6-2t)$$
$$F(t) = 8\left(t^2 + 2(6-2t)\right) - 3t^2(6-2t) = 6t^3 - 42t^2 + 96t$$
$$F'(t) = 18t^2 - 84t + 96 \text{ to get critical point } F'(t) = 0 \text{ gives}$$
$$18t^2 - 84t + 96 = 0 \text{ or } (3t-8)(t-2) = 0 \text{ or } t = \frac{8}{3}, 2$$

$$\text{We find value of } F(t) \text{ a critical point and boundary } [0,3]$$

$$F(0) = 0, F(2) = 6 \times 2^3 - 42 \times 2^2 + 96 \times 2 = 72, F\left(\frac{8}{3}\right) = 71.11, F(3) = 72$$

$$\text{so, Max value of } F(t) \text{ is } 72 \text{ and } 8a - 3b \le 72 \text{ at } x_1 = x_2 = 2, x_3 = 2$$

731. **Solve for real numbers:**

$$\sqrt[3]{x+6} + \sqrt{x-1} = x^2 - 1$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\textit{Since } \sqrt{x-1} \textit{ is define on } x \geq 1,$$
$$\textit{so we are going to solve above equation} \in [1,\infty)$$

$$\textit{Let } f(x) = \sqrt[3]{x+6} + \sqrt{x-1} - (x^2-1), x \geq 1$$
$$f(1) = \sqrt[3]{7} + 0 - 0 = \sqrt[3]{7} > 0, f(2) = \sqrt[3]{8} + 1 - 3 = 0$$
$$\textit{so } x = 2 \textit{ is a solution}$$

$$\textit{Now we check nature of the function in } x \in [1,2)$$
$$f'(x) = \frac{1}{3}(x+6)^{-\frac{2}{3}} + \frac{1}{2\sqrt{x-1}} - 2x$$
$$f''(x) = -\frac{2}{9}(x+6)^{-\frac{5}{3}} - \frac{1}{4}(x-1)^{-\frac{3}{2}} - 2 = -\left(\frac{2}{9}(x+6)^{-\frac{5}{3}} + \frac{1}{4}(x-1)^{-\frac{3}{2}} + 2\right) < 0$$
$$\textit{so } f \textit{ is concave in } [1,2], \textit{for a concave function we have standard inequality}$$
$$\textit{for } x \in [1,2) \Rightarrow x = (2-x).1 + (x-1).2$$
$$\textit{then } f\big((2-x).1 + (x-1).2\big) \geq (2-x)f(1) + (x-1)f(2)$$
$$\textit{or } f(x) \geq (2-x)f(1) > 0 \textit{ as } x \in [1,2) \; \& \; f(1) = \sqrt[3]{7} > 0$$
$$\textit{so } f(x) > 0 \textit{ in } [1,2) \textit{ and } x = 2 \; f(2) = 0$$

$$\textit{Now we check nature of the function in } x > 2$$
$$f'(x) = \frac{1}{3}(x+6)^{-\frac{2}{3}} + \frac{1}{2\sqrt{x-1}} - 2x$$
$$\textit{as } x > 2 \textit{ then } \frac{1}{3}(x+6)^{-\frac{2}{3}} < \frac{1}{3}.8^{-\frac{2}{3}} = \frac{1}{12}, \frac{1}{2\sqrt{x-1}} < \frac{1}{2.1} = \frac{1}{2}, -2x < -4$$
$$\textit{for this } f'(x) < \frac{1}{12} + \frac{1}{2} - 4 = -\frac{41}{12} < 0$$
$$\textit{which indicate that } f(x) \textit{ is decreasing } \forall x > 2$$
$$\textit{that's why after hitting } 0 \textit{ at } x = 2$$
$$\textit{the function goes negative and never come back up}$$
$$\textit{so requested soution } x = 2$$

732. Solve for reals:

$$x\sqrt{x+7} + \sqrt{9-x} = 4\sqrt{x^2+1}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\textit{Set } p = \sqrt{x+7}, q = \sqrt{9-x}, r = \sqrt{x^2+1}$$
$$\textit{Clearly}, p^2 + q^2 = x + 7 + 9 - x = 16 = 4^2$$
$$\textit{Let } p = 4\cos t, q = 4\sin t, t \in \left[0, \frac{\pi}{2}\right] \textit{ and } p = \sqrt{x+7} \textit{ or}$$

$$x = p^2 - 7 = 16\cos^2 t - 7 \ (1)$$

Using above substitution we get:

$$x\sqrt{x+7} + \sqrt{9-x} = 4\sqrt{x^2+1} \ \ or$$
$$(16\cos^2 t - 7)\cos t + \sin t = \sqrt{(16\cos^2 t - 7)^2 + 1}$$
$$or\ A\cos t + \sin t \overset{16\cos^2 t - 7 = A}{=} \sqrt{A^2+1}$$
$$A\cos t + \sin t \overset{Cauchy\ Schwarz}{\leq} \sqrt{(A^2+1)(\cos^2 t + \sin^2 t)} = \sqrt{A^2+1}$$
$$for\ equality\ we\ take\ \cos t = \lambda A, \sin t = \lambda\ then \cos^2 t + \sin^2 t = 1 \Rightarrow \ \lambda^2 = \frac{1}{A^2+1}$$
$$\cos^2 t = \lambda^2 A^2 = \frac{A^2}{A^2+1}\ , from\ (1)\ we\ get:$$
$$x = p^2 - 7 = 16\cos^2 t - 7 \ \ or\ \ \cos^2 t = \frac{A+7}{16}$$
$$\frac{A^2}{A^2+1} = \frac{A+7}{16}\ or, A^3 - 9A^2 + A + 7 = 0\ or (A-1)(A^2 - 8A - 7) = 0$$
$$or\ A = 1\ and\ (A^2 - 8A - 7) = 0 \Rightarrow A = \frac{8 \pm \sqrt{64+28}}{2} or, A = 4 \pm \sqrt{23}$$
$$note\ that\ A = (16\cos^2 t - 7) = x\ by\ (1)$$

$$but\ A = x = \ 4 - \ \sqrt{23}\ not\ a\ valid\ solution\ as \sqrt{x+7}\ not\ valid\ at\ x = \ 4 - \ \sqrt{23}$$
$$so\ required\ solution\ x = 4, 4 + \sqrt{23}\ \ in\ real$$

733. Solve for integers:

$$\sqrt{x+y} + 2 = \sqrt{x} + \sqrt{y}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ a = \sqrt{x}, b = \sqrt{y}\ therefore\ \ x = a^2, y = b^2\ (as\ x, y \in Z^+, x, y > 0)$$
$$\sqrt{x+y} + 2 = \sqrt{x} + \sqrt{y}\ or\ \sqrt{a^2+b^2} + 2 = a + b\ or\ \sqrt{a^2+b^2} = a + b - 2$$
$$or \left(\sqrt{a^2+b^2}\right)^2 = (a+b-2)^2\ \left(as\ \sqrt{a^2+b^2} > 0\ then\ R.H.S\ part\ a+b-2 > 0\right)$$
$$a^2 + b^2 = a^2 + b^2 + 4 + 2ab - 4a - 4b\ or\ 2(ab - 2a - 2b + 2) = 0$$
$$ab - 2a - 2b = -2\ or, ab - 2a - 2b + 4 = -2 + 4\ or, (a-2)(b-2) = 2\ (1)$$

$$Integer\ factorization\ of\ 2\ are\ (1,2), (2,1), (-1,-2), (-2,-1)$$

$$From\ (1)\ if\ a - 2 = 1, b - 2 = 2\ then\ a = 3, b = 4$$
$$if\ a - 2 = 2, b - 2 = 1\ then\ a = 4, b = 3$$
$$if\ a - 2 = -1, b - 2 = -2\ then\ a = 1, b = 0$$
$$if\ a - 2 = -2, b - 2 = -1\ then\ a = 0, b = 1$$
$$Now, a = 3, b = 4 \Rightarrow \ x = 9, y = 16\ (as\ \ x = a^2, y = b^2)$$

$$a=4,b=3\Rightarrow\ x=16,y=19,$$
$$a=1,b=0\Rightarrow\ x=1,y=0\ \text{does not satisfy given equation}$$
$$a=0,b=1\Rightarrow\ x=0,y=1,\text{does not satisfy given equation}$$
$$\text{so the required solution } x=9,y=16\text{ or},x=16,y=9$$

734. Solve for integers:

$$\left(\sqrt{x}-\sqrt{y}\right)^4=3361-\sqrt{11296320}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\left(\sqrt{x}-\sqrt{y}\right)^4=\left(x+y-2\sqrt{xy}\right)^2=(x+y)^2+4xy-4\sqrt{xy}(x+y)$$
$$\text{now},(x+y)^2+4xy-4\sqrt{xy}(x+y)=3361-\sqrt{11296320}$$
$$\text{comparing we get },(x+y)^2+4xy=3361\quad(1)$$
$$4\sqrt{xy}(x+y)=\sqrt{11296320}\quad(2)$$
$$\text{Let } x+y=s \text{ and } xy=p \text{ then from}(1)\&(2)\text{we have}$$
$$s^2+4p=3361\ (3)$$
$$4s\sqrt{p}=\sqrt{11296320}\text{ or},s^2p=\frac{11296320}{16}=706020\ \ (4)$$

$$\text{From }(3)\&(4)\text{we have } s^2p=706020\text{ or } s^2\left(\frac{3361-s^2}{4}\right)\overset{(3)}{=}706020$$
$$s^2(3361-s^2)=2824080\text{ or } s^4-3361s^2+2824080=0$$
$$s^2=\frac{3361\pm\sqrt{(3361)^2-4.1.2824080}}{2}=\frac{3361\pm1}{2}=1681,1680$$
$$\text{as we solve the equation for } Z^+$$
$$\text{for this we take } s^2=(x+y)^2=1681=41^2$$
$$s=x+y=41\text{ and } p=xy\overset{from\ (3)}{=}\frac{3361-s^2}{4}=\frac{3361-1681}{4}=420$$
$$\text{We have } x+y=41\text{ and } xy=420$$
$$\text{To find }(x,y)\text{ we construct an equation with roots } x\ \&\ y,$$
$$\text{equation will be } t^2-(x+y)t+xy=0\text{ or}$$

$$t^2-41t+420=0\text{ or}(t-21)(t-20)=0\ \text{ or }\ t=21,20$$

$$\text{Required solution } x=21,y=20\text{ or } x=20,y=21.$$

735. $\lambda\in R$ *fixed. Solve for reals :*

$$3^{x+\lambda}+3x+3\lambda=1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ y = x + \lambda$$

$$The\ equation\ can\ be\ written\ as\ 3^y + 3(y - \lambda) + 3\lambda = 1\ or, 3^y + 3y = 1\ (1)$$

$$Let\ f(y) = 3^y + 3y\ \ then\ f'(y) = 3^y \ln 3 + 3 > 0\ \forall\ y \in R$$

$$f(x)\ is\ \ strictly\ increasing$$

$$For\ this\ equation\ (1) has\ at\ most\ one\ solution, clearly\ f(0) = 1$$

$$y = 0\ \ satisfy\ the\ equation\ (1)$$

$$y = 0\ is\ a\ solution\ or\ x + \lambda = 0\ or\ x = -\lambda$$

736. **Let be $\lambda \in R^+$ fixed. Solve for reals:**

$$\begin{cases} x^6 - y^6 = \lambda^6 - 1 \\ x^4 + x^2y^2 + y^4 = \lambda^4 + \lambda^2 + 1 \end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ a = x^2, b = y^2 (a, b \geq 0)$$

$$x^6 - y^6 = \lambda^6 - 1\ or, (x^2 - y^2)(\ x^4 + x^2y^2 + y^4) = (\lambda^2)^3 - (1)^3$$

$$(a - b)(a^2 + ab + b^2) = (\lambda^2 - 1\ )(\lambda^4 + \lambda^2 + 1\ )$$

$$Comparing\ we\ get:$$

$$a - b = \lambda^2 - 1\ (1), a^2 + ab + b^2 = \lambda^4 + \lambda^2 + 1\ \ (2)$$

$$Let\ a + b = s, d = a - b = \lambda^2 - 1\ \ then\ from\ (2)\ we\ get:$$

$$a^2 + ab + b^2 = \lambda^4 + \lambda^2 + 1$$

$$(a + b)^2 - 2ab + ab = \lambda^4 + \lambda^2 + 1\ \ \ or\ (a + b)^2 - ab = \lambda^4 + \lambda^2 + 1$$

$$(a + b)^2 - \frac{1}{4}((a + b)^2 - (a - b)^2) = \lambda^4 + \lambda^2 + 1$$

$$s^2 - \frac{1}{4}(s^2 - d^2) = \lambda^4 + \lambda^2 + 1$$

$$\frac{3s^2}{4} = \lambda^4 + \lambda^2 + 1\ - \frac{d^2}{4}\ or\ 3s^2 \overset{d=a-b=\lambda^2-1}{=} 4(\lambda^4 + \lambda^2 + 1\ ) - (\lambda^2 - 1\ )^2$$

$$s^2 = (\lambda^2 + 1\ )^2\ or\ s = a + b = \lambda^2 + 1\ \ (as\ a, b \geq 0\ )\ \ \ (3)$$

$$ab=\frac{1}{4}((a+b)^2-(a-b)^2)=\frac{1}{4}((\lambda^2+1)^2-(\lambda^2-1)^2)=4\frac{\lambda^2}{4}=\lambda^2\ (4)$$

We construct an equation with roots $a\ \&\ b$

$$t^2-(a+b)t+ab=0\ or\ t^2-(\lambda^2+1)t+\lambda^2=0\ or\ t=\frac{(\lambda^2+1)\pm(\lambda^2-1)}{2}$$

$$or\ t=\lambda^2,1\Rightarrow x^2=a=\lambda^2\ or\ x=\pm\lambda\ \&\ y^2=b=1\ or\ y=\pm1$$

$$Solutions: (x,y)\in\{(\lambda,1),(-\lambda,1),(-\lambda,-1),(\lambda,-1)\}$$

737. **Let be $\lambda\geq1$ fixed. Solve for reals:**

$$\begin{cases}(x+y)(x^2+y^2)=\lambda(4\lambda^2+1)\\ (x-y)(x^2-y^2)=2\lambda\end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ u=x+y, v=x-y\ then\ x=\frac{u+v}{2}, y=\frac{u-v}{2}, xy=\frac{u^2-v^2}{4}$$

$$(x+y)(x^2+y^2)=\lambda(4\lambda^2+1)\ or\ (x+y)((x+y)^2-2xy)=\lambda(4\lambda^2+1)$$

$$u\left(u^2-\frac{u^2-v^2}{2}\right)=\lambda(4\lambda^2+1)\ or, u(u^2+v^2)=2\lambda(4\lambda^2+1)\ (1)$$

$$(x-y)(x^2-y^2)=2\lambda\ or\ (x-y)^2(x+y)=2\lambda\ or\ uv^2=2\lambda\ or\ v^2=\frac{2\lambda}{u}\ (2)$$

$$Eliminating\ v^2 between\ (1)\&(2) we\ get:$$

$$u\left(u^2+\frac{2\lambda}{u}\right)=2\lambda(4\lambda^2+1)\ or, u^3+2\lambda=8\lambda^3+2\lambda\ or, u^3=8\lambda^3\ or$$

$$u=2\lambda\ (as\ \lambda\geq1)\ \&\ v^2=\frac{2\lambda}{u}=\frac{2\lambda}{2\lambda}=1\ or\ v=\pm1$$

$$x=\frac{u+v}{2}=\frac{2\lambda\pm1}{2}=\lambda\pm\frac{1}{2}, y=\frac{u-v}{2}=\frac{2\lambda\mp1}{2}=\lambda\mp\frac{1}{2}$$

$$Solutions: (x,y)=\left(\lambda+\frac{1}{2},\lambda-\frac{1}{2}\right),\left(\lambda-\frac{1}{2},\lambda+\frac{1}{2}\right)$$

738. Solve for reals:

$$\begin{cases}\sqrt{x+\frac{1}{5}}+\sqrt{y+\frac{1}{5}}=\sqrt{15}\\ x+y=3+8\sqrt{xy}\end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ a=\sqrt{x+\frac{1}{5}},b=\sqrt{x+\frac{1}{5}}\ then\ x=a^2-\frac{1}{5},y=b^2-\frac{1}{5}$$

$$According\ to\ the\ given\ condition, a+b=\sqrt{15}\ (1)$$

$$x+y=(a^2+b^2)-\frac{2}{5}=(a+b)^2-2ab-\frac{2}{5}\overset{(1)}{=}15-2ab-\frac{2}{5}=\frac{73}{5}-2ab\ (2)$$

$$again,\quad x+y=3+8\sqrt{xy}\ (3) eliminate\ (x+y)\ from\ (2)\ \&\ (3) we\ get,$$

$$3+8\sqrt{xy}=\frac{73}{5}-2ab\ or\ \sqrt{xy}=\frac{\frac{29}{5}-ab}{4}$$

$$or\ xy=\frac{1}{16}\left(\frac{29}{5}-ab\right)^2=\frac{1}{16}\left(\frac{841}{25}-\frac{58}{5}ab+a^2b^2\right)\ (4)$$

$$xy=\left(a^2-\frac{1}{5}\right)\left(b^2-\frac{1}{5}\right)=a^2b^2-\frac{a^2+b^2}{5}+\frac{1}{25}=a^2b^2-\frac{(a+b)^2-2ab}{5}+\frac{1}{25}\overset{(1)}{=}$$

$$=a^2b^2+\frac{2}{5}ab-\frac{74}{25}\ (5)\ from\ (4) and\ (5) eliminate\ xy\ we\ get:$$

$$\frac{1}{16}\left(\frac{841}{25}-\frac{58}{5}ab+a^2b^2\right)=a^2b^2+\frac{2}{5}ab-\frac{74}{25}$$

$$or\left(\frac{841}{25}-\frac{58}{5}ab+a^2b^2\right)=16a^2b^2+\frac{32}{5}ab-\frac{74}{25}or$$

$$841-290ab+25a^2b^2=400a^2b^2+16ab-1184$$

$$or\ 5a^2b^2+6ab-27=0\ or\ ab=\frac{-6\pm\sqrt{36+540}}{10}=\frac{9}{5},-3$$

$$as\ ab\ge 0\ so\ ab\ne -3, ab=\frac{9}{5}$$

$$Now\ xy\overset{(4)}{=}\frac{1}{16}\left(\frac{29}{5}-ab\right)^2\overset{ab=\frac{9}{5}}{=}1\ and\ x+y\overset{(3)}{=}3+8\sqrt{xy}\overset{xy=1}{=}11$$

$$We\ construct\ an\ equation\ with\ roots\ x,y:$$

$$m^2-(x+y)m+xy=0\ or, m^2-11m+1=0\ or, m=\frac{11\pm\sqrt{117}}{2}=\frac{11\pm 3\sqrt{13}}{2}$$

$$solution\ (x,y)=\left(\frac{11+3\sqrt{13}}{2},\frac{11-3\sqrt{13}}{2}\right)\ or\left(\frac{11-3\sqrt{13}}{2},\frac{11+3\sqrt{13}}{2}\right)$$

739. $n \in N, fixed.$ **Solve for real numbers:**

$$5^x.2^{\frac{(n+1)(x-1)}{x}} = 5.10^n$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$5^x.2^{\frac{(n+1)(x-1)}{x}} = 5.10^n$$

$$or\ 5^x.2^{\frac{(n+1)(x-1)}{x}} = 5.(2^n.5^n) = 5^{n+1}.2^n$$

$$Comparing\ we\ get:$$

$$x = n+1\ \&\ \frac{(n+1)(x-1)}{x} = n$$

$$\frac{(n+1)(x\text{-}1)}{x} = n\ or\ nx = nx - n + x - 1\ or\ x = n+1$$

$$Solution:\ x = n+1$$

740. **Solve for reals:**

$$(x^6)^{x^6} = 6x - 5$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$As\ L.H.S > 0\ so\ R.H.S > 0\ or\ 6x - 5 > 0\ or\ x > \frac{5}{6}$$

$$Let\ f(x) = (x^6)^{x^6} - (6x - 5\,) = x^{6x^6} - (6x - 5)$$

$$f'(x) = 6x^5x^{6x^6}(6\ln x + 1) - 6 = 6\left(x^5x^{6x^6}(6\ln x + 1) - 1\right)$$

$$Case1: x \in \left(\frac{5}{6}, 1\right)\ then\ x^5x^{6x^6} < 1, 6\ln x < 1\ for\ this\ f'(x) < 0$$

$$so\ f(x) is\ decreasing\ on\ \left(\frac{5}{6}, 1\right)$$

$$Case2: x = 1\ then\ f(1) = 1 - (6 - 5) = 0\ so\ x = 1\ is\ root\ \ of f(x) = 0$$

$$Case\ 3: x > 1\ then\ \ln x > 0, x^5x^{6x^6} > 1\ for\ this\ f'(x) > 0$$

$$so\ f(x) is\ increasing\ for\ x > 1$$

$$\boldsymbol{Conclusion: f(x) \text{ is decreasing on } \left(\frac{5}{6}, 1\right)} \ \& \ increasing\ for\ x > 1\ and$$

$$\boldsymbol{at\ x = 1\ f(x) = 0}$$

$$\boldsymbol{so\ required\ solution\ x = 1.}$$

741. Solve for reals:

$$x(x+4) = 16 + 2x\sqrt{x(x-4)}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x(x+4) = 16 + 2x\sqrt{x(x-4)}\ \ clearly\ x(x-4) \ge 0\ or\ x \le 0\ or, x \ge 4$$

$$Let\ x = 2t\ then\ given\ equation\ can\ be\ written\ as:$$

$$2t(2t+4) = 16 + 2.2t\sqrt{2t(2t-4)}\ or\ 4t^2 + 8t = 16 + 8t\sqrt{t(t-2)}$$

$$t^2 + 2t = 4 + 2t\sqrt{t(t-2)}\ or\ \frac{t^2 + 2t - 4}{2t} = \sqrt{t(t-2)}$$

$$or\left(\frac{t^2 + 2t - 4}{2t}\right)^2 = \left(\sqrt{t(t-2)}\right)^2$$

$$(\boldsymbol{L.H.S} > 0\ as\ t \le 0\ or\ t \ge 2\ since\ x = 2\boldsymbol{t}\ \&\ \ x \le 0\ or\ x \ge 4)$$

$$(t^2 + 2t - 4)^2 \ge 4t^2(t^2 - 2t)$$

$$t^4 + 4t^3 - 4t^2 - 16t + 16 = 4t^4 - 8t^3$$

$$3t^4 - 12t^3 + 4t^2 + 16t - 16 = 0$$

$$t^4 - 4t^3 + \frac{4}{3}t^2 + \frac{16}{3}t - \frac{16}{3} = 0$$

$$Standard\ \ form\ t^4 + pt^3 + qt^2 + rt + s = 0$$

$$Let\ t = y = y - \frac{p}{4} = y - \frac{-4}{4} = y + 1\ then:$$

$$(y+1)^4 - 4(y+1)^3 + \frac{4}{3}(y+1)^2 + \frac{16}{3}(y+1) - \frac{16}{3} = 0$$

$$(y^4 + 4y^3 + 6y^2 + 4y + 1) - 4(y^3 + 3y^2 + 3y + 1) + \frac{4}{3}(y^2 + 2y + 1) + \frac{16}{3}(y+1) - \frac{16}{3} = 0$$

$$y^4 - \frac{14}{3}y^2 - \frac{5}{3} = 0\ or\ 3y^4 - 14y^2 - 5 \ge 0\ or\ (y^2 - 5)(3y^2 + 1) = 0$$

$$or\ y^2 = 5\ \left(as\ y^2 \ne -\frac{1}{3}\right)\ or\ y = \pm\sqrt{5}$$

$$Solutions\ \ x = 2t = 2(y+1) = 2\left(1 \pm \sqrt{5}\,\right)$$

742. Solve for reals:

$$\sqrt[3]{x-2024} + \sqrt[3]{x-2025} + \sqrt[3]{x-2026} = 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ y = x - 2025$$

$$The\ given\ equation\ can\ be\ written\ as:$$

$$\sqrt[3]{y+1} + \sqrt[3]{y} + \sqrt[3]{y-1} = 0$$

$$Let\ f(y) = \sqrt[3]{y+1} + \sqrt[3]{y} + \sqrt[3]{y-1}\ clearly\ f(0) = 0\ so\ y = 0\ is\ a\ solution\ of\ f(y) = 0$$

$$f'(y) = \frac{1}{3}\left(\frac{1}{(y+1)^{\frac{2}{3}}} + \frac{1}{y^{\frac{2}{3}}} + \frac{1}{(y-1)^{\frac{2}{3}}}\right) > 0\ \forall\, y \in R - \{0, 1, -1\}$$

$$So\ f(y) is\ \ strictly\ increasing\ continuous\ function\ can\ have\ at\ most\ one\ zero,$$

$$so\ required\ solution\ y = 0\ or\ \ x - 2025 = 0\ or\ x = 2025$$

743. Solve for real numbers and $m \in R$ fixed:

$$\begin{cases} x + y - z = m - 2 \\ x^2 + y^2 - z^2 = m^2 - 4m - 8 \\ x^3 + y^3 - z^3 = m^3 - 6m^2 - 24m - 26 \end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ a = x + y, p = xy$$

$$From\ 1st\ equation: x + y - z = m - 2\ or, z = a - (m-2)\ (1)$$

$$x^2 + y^2 = (x+y)^2 - 2xy = a^2 - 2p$$

$$From\ 2nd\ equation: x^2 + y^2 - z^2 = m^2 - 4m - 8$$

$$a^2 - 2p - (a - (m-2))^2 = m^2 - 4m - 8$$

$$-2p + 2a(m-2) - (m-2)^2 = m^2 - 4m - 8$$

$$-2p = m^2 - 4m - 8 + (m-2)^2 - 2a(m-2)$$

$$-2p = 2m^2 - 8m - 4 - 2a(m-2) or, p = a(m-2) - m^2 + 4m + 2\ (2)$$

$$x^3 + y^3 = (x+y)^3 - 3xy(x+y) = a^3 - 3ap$$

$$From\ 3rd\ equation: x^3 + y^3 - z^3 = m^3 - 6m^2 - 24m - 26$$

$$a^3-3ap-\left(a-(m-2)\right)^3\overset{(1)}{=}m^3-6m^2-24m-26$$
$$a^3-3ap-a^3+(m-2)^3+3a^2(m-2)-3a(m-2)^2=m^3-6m^2-24m-26$$
$$3a(-p+a(m-2)-(m-2)^2)=m^3-6m^2-24m-26-(m-2)^3$$
$$3a(-a(m-2)+m^2-4m-2+a(m-2)-(m-2)^2)\overset{(2)}{=}-36m-18$$
$$3a\times(-6)=-36m-18 \text{ or } a=2m+1$$

From (1):
$$z=a-(m-2)=2m+1-(m-2)=m+3\ (3)$$
From (2):
$$p=a(m-2)-m^2+4m+2=(2m+1)(m-2)-m^2+4m+2=m^2+m$$
now we have $a=x+y=2m+1$, *and* $p=xy=m^2+m$, *so* x,y *are roots of*
$$t^2-(2m+1)t+(m^2+m)=0 \text{ or}$$
$$t=\frac{2m+1\pm\sqrt{(2m+1)^2-4.1.(m^2+m)}}{2}=\frac{2m+1\pm1}{2}=m,m+1$$
so $(x,y)=(m,m+1)$ *or*, $(m+1,m)$

Required solution using (3): $(x,y,z)=(m,m+1,m+3),(m+1,m,m+3)$

744. Solve for real numbers:

$$2x^2-14x+30-5(x-4)\sqrt{x-1}=0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$2x^2-14x+30-5(x-4)\sqrt{x-1}=0$$

$$\text{Clearly } x-1\geq0 \text{ or } x\geq1$$

$$\text{Let } y=\sqrt{x-1}\Rightarrow x=y^2+1\quad(1)$$

Using (1) *given equation can be written as*:

$$2(y^2+1)^2-14(y^2+1)+30-5y(y^2-3)=0$$

$$2(y^4+2y^2+1)-(14y^2+14)+30-(5y^3-15y)=0$$

$$2y^4-5y^3-10y^2+15y+18=0$$

$$(y-2)(y+1)(y-3)(2y+3)=0$$

$$y-2=0\Rightarrow y=2\ then\ x=y^2+1=2^2+1=5$$

$$y+1=0\Rightarrow y=-1\ then:$$
$$x=y^2+1=(-1)^2+1=2\ ,not\ satisfied\ given\ equation$$

$$y-3=0\Rightarrow y=3\ then\ x=y^2+1=9^2+1=10$$
$$2y+3=0\Rightarrow y=-\frac{3}{2}\ then$$
$$x=\left(-\frac{3}{2}\right)^2+1=\frac{13}{4}=5,not\ satisfied\ given\ equation$$

$$So\ required\ solution\ x=5,10$$

745. Solve for real numbers:

$$x^5-\sqrt[5]{x+30}=30$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ f(x)=x^5-\sqrt[5]{x+30}-30$$

$$at\ x=2, f(2)=32-2-32=0\ so\ x=2\ is\ a\ solution\ of\ f(x)=0$$
$$now\ f'(x)=5x^4-\frac{1}{5}(x+30)^{-\frac{4}{5}}>0\ ,\forall\ x\in R$$
$$so\ f(x)\ is\ strictly\ increasing$$

$$For\ this\ f(x)=0\ has\ at\ most\ one\ zero\ so\ required\ solution\ x=2.$$

746. Solve for real numbers:

$$\sqrt[3]{x+2024}+\sqrt[3]{x+2028}=\sqrt[3]{x+2025}+\sqrt[3]{x+2027}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ a=\sqrt[3]{x+2024}, b=\sqrt[3]{x+2025}, c=\sqrt[3]{x+2027}, d=\sqrt[3]{x+2028}$$

$$then\ \sqrt[3]{x+2024}+\sqrt[3]{x+2028}=\sqrt[3]{x+2025}+\sqrt[3]{x+2027}\Rightarrow a+d=b+c$$

$$(a+d)^3=a^3+d^3+3ad(a+d)=x+2024+x+2028+3ad(a+d)=$$
$$=2x+4052+3ad(a+d)$$

$$(b+c)^3 = b^3 + c^3 + 3bc(b+c) = x + 2025 + x + 2027 + 3bc(b+c) =$$
$$= 2x + 4052 + 3bc(b+c)$$

$$Since\ a + d = b + c\ then\ (a+d)^3 = (b+c)^3$$
$$or\ 2x + 4052 + 3ad(a+d) = 2x + 4052 + 3bc(b+c)$$
$$or\ ad(a+d) = bc(b+c)\ or, \qquad ad(a+d) \overset{a+d=b+c}{=} bc(a+d)$$
$$or\ (a+d)(ad - bc) = 0$$

$$If a + d = 0 \Rightarrow a^3 = -d^3\ or\ x + 2024 = -(x + 2028)\ or$$
$$2x = -4052\ or\ x = -2026$$

$$Now\ ad - bc \neq 0$$
$$because\ a^3d^3 = (x+2024)(x+2028) = x^2 + x(4052) + 2028 \times 2024$$
$$b^3c^3 = (x+2025)(x+2027) = x^2 + x(4052) + 2025 \times 2027$$

$$Clearly\ a^3d^3 \neq b^3c^3\ as\ 2028 \times 2024 \neq 2025 \times 2027\ so\ ad \neq bc\ or\ ad - bc \neq 0$$
$$Required\ solution\ x = -2026$$

747. Solve for real numbers:

$$x^4 - x^2 - 16x + 2 = 2\sqrt{x^2 + 4x}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$x^4 - x^2 - 16x + 2 = 2\sqrt{x^2 + 4x}$$
$$Domain\ need\ x^2 + 4x \geq 0\ so\ x \in (-\infty, -4] \cup [0, \infty)$$
$$(x^4 - x^2 - 16x + 2)^2 = \left(2\sqrt{x^2 + 4x}\right)^2$$
$$x^8 + (x^2 + 16x - 2)^2 - 2x^4(x^2 + 16x - 2) = 4x^2 + 16x$$
$$x^8 - 2x^6 - 32x^5 + 5x^4 + 32x^3 + 248x^2 - 80x + 4 = 0\ (1)$$
$$Let\ x^8 - 2x^6 - 32x^5 + 5x^4 + 32x^3 + 248x^2 - 80x + 4 =$$
$$= (x^4 - 16x + p)(x^4 - 2x^2 - 16x + q)$$
$$= x^8 - 2x^6 - 3x^5 + x^4(p+q) + x^2(256 - 2p) - x(16q + 16p) + pq$$

$$Comparing\ coefficient\ \ we\ get\ coefficient\ of\ x^2 \rightarrow 256 - 2p = 248\ or, p = 4$$
$$Constant\ term \rightarrow\ pq = 4\ or\ q = 1\ as\ p = 4$$
$$x^8 - 2x^6 - 32x^5 + 5x^4 + 32x^3 + 248x^2 - 80x + 4 =$$
$$= (x^4 - 16x + 4)(x^4 - 2x^2 - 16x + 1)$$

$$\textbf{From } (1)\ (x^4-16x+4)(x^4-2x^2-16x+1)=0$$

$$x^4-2x^2-16x+1=0$$
$$\text{or } (x^2+3)^2-8(x+1)^2=0 \text{ or } x^2+3=\pm 2\sqrt{2}(x+1)$$

$$\text{Now } x^2+3=2\sqrt{2}(x+1)\left(\text{ as } x^2+3=-2\sqrt{2}(x+1) \text{ does not has real roots}\right)$$
$$\text{or } x^2-2\sqrt{2}x+\left(3-2\sqrt{2}\right)=0 \text{ or } x=\frac{2\sqrt{2}\pm\sqrt{8-4\left(3-2\sqrt{2}\right)}}{2}=\sqrt{2}\pm\sqrt{2\sqrt{2}-1}$$
$$\text{Clearly roots from } x^4-16x+4=0 \text{ does not satify given equation}$$
$$\text{since } x^4-16x+4=0 \Rightarrow x^4=16x-4$$

$$\text{If we put value of } x^4 \text{in given equation LHS part then}$$
$$LHS=x^4-x^2-16x+2=(16x-4)-x^2-16x+2=-2-x^2<0 \text{ but}$$
$$RHS=2\sqrt{x^2+4x}>0$$
$$\text{For this required solution } x=\sqrt{2}\pm\sqrt{2\sqrt{2}-1}$$

748.Solve for integers:

$$x^3+2y^3+3x+4y=4x^2+4y^2$$

Proposed by Sakthi Vel-India

Solution by Tapas Das-India

$$x^3+2y^3+3x+4y=4x^2+4y^2$$
$$\Rightarrow x(x^2-4x+3)+2y(y^2-2y+2)=0$$
$$\Rightarrow x(x-1)(x-3)+2y((y-1)^2+1)\ =0\ (1)$$
$$\text{Clearly sign of entire y term } 2y((y-1)^2+1)$$
$$\text{is depend on y as } (y-1)^2+1\geq 1$$

$$\text{Case 1: } y=0 \text{ then } x(x-1)(x-3)=0 \quad \Rightarrow x=0,1,2$$
$$\text{so solutions are } (x,y)=(0,0),(1,0),(3,0)$$

$$\text{Case 2: } y>0 \text{ then } 2y((y-1)^2+1)>0 \text{ and from } (1)$$
$$x(x-1)(x-3) \text{must be } <0 \text{ so } x<0 \text{ or}, 1<x<3$$

$$\text{Sub case A: } 1<x<3 \text{ then only integer } x=2 \text{ then from}(1)\text{we get}$$
$$2(2-1)(2-3)+2y((y-1)^2+1)=0 \text{ or}, 2y^3-4y^2+4y-2=0$$
$$y^3-2y^2+2y-1=0 \text{ or}, (y-1)(y^2-y+1)=0 \text{ or}, y=1$$
$$\text{as } y^2-y+1=0 \text{ does not have real root since}$$
$$D=1-4=-3<0, \text{solution } (x,y)=(2,1)$$
$$\text{If } y>1 \text{ so } y\geq 2(\text{Integer}), \text{for } y=2 \text{ and } x=2 \quad \text{from}(1)\text{we get}$$

$$-2+2.2(1^2+1)=-2+8\neq 0\text{ , not satisfied ,}$$
$$\text{so their are not integer solution for } y>1$$

Sub case B: $x<0$, if $y=2$ from(1)we get
$$x(x-1)(x-3)+2.2((2-1)^2+1)=0 \text{ or, } x(x-1)(x-3)+8=0$$
$$\text{Clearly } x=-1 \text{ satisfy above relation ,}$$
$$\text{absolute value of } x(x-1)(x-3) \text{ increases as } x \text{ becomes more negative}$$
$$\text{so only solution } (x,y)=(-1,2)$$

$$\text{If } y=3 \text{ then from(1)we get } x(x-1)(x-3)+2.3((3-1)^2+1)=0$$
$$\text{or } x(x-1)(x-3)+30=0$$

$$\text{Clearly } x=-2 \text{ satisfy above relation , so solution } (x,y)=(-2,3)$$

$$\text{If } y=4 \quad \text{from(1)we get :}$$
$$x(x-1)(x-3)+2.4((4-1)^2+1)=0 \text{ or, } x(x-1)(x-3)+80=0$$
$$\text{If we take } x=-3 \text{ then } x(x-1)(x-3)=-72\neq -80$$
$$\text{If we take } x=-4 \text{ then } x(x-1)(x-3)=-140\neq -80$$
$$\text{so no solution exist since}$$
$$|x(x-1)(x-3)| \text{ is increases as } x \text{ becomes more negative}$$

Case 3 : $y<0$, so $2y((y-1)^2+1)>0$, from (1)we have
$$x(x-1)(x-3)>0 \text{ or, } 0<x<1 \text{ or, } x>3$$

$$\text{We must consider } x\geq 4 \text{ as no integer in } (0,1)$$
$$\text{sub case A , } x\geq 4 \text{ and } y=-1$$
$$\text{from (1)we have } x(x-1)(x-3)-10=0$$
$$\text{for } x=4 \text{ , } x(x-1)(x-3)=12\neq 10 \text{ and value of } x(x-1)(x-3) \text{ increases for}$$
$$x\geq 4 \text{ , so no integer solution exist}$$

$$\text{If } y=-2 \text{ then from (1)we have}$$
$$x(x-1)(x-3)-40=0$$
$$\text{Clearly } x=5 \text{ satisfy above expression and value of}$$
$$x(x-1)(x-3) \text{ increases for } x\geq 4$$
$$\text{so only solution } (x,y)=(5,-2)$$
$$\text{Combine all above result required solutions are}$$
$$(x,y)=(0,0),(1,0),(3,0),(-1,2),(-2,3),(5,-2),(2,1)$$

749. Solve for real numbers:

$$\sqrt[3]{2034-x}-\sqrt{x-2025}=x^2-2025x-2025$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Clearly\ x-2025\geq 0\ or\ x\geq 2025$$

$$Let\ \ g(x)=(x^2-2025x-2025)-\left(\sqrt[3]{2034-x}-\sqrt{x-2025}\right)$$
$$g'(x)=2x-2025+\frac{1}{3}(2024-x)^{-\frac{2}{3}}+\frac{1}{2}(x-2025)^{\frac{1}{2}}$$
$$g'(x)>0\ for\ x>2025\Rightarrow g(x)\ is\ strictly\ increasing\ ,therefore\ g(x)=0$$
$$has\ at\ most\ one\ solution\ .$$

$$Clearly\ \ g(2026)=2026(2026-2025)-2025-(2-1)=0$$
$$so\ x=2026\ is\ the\ only\ solution.$$

750. Solve for integers:

$$x^3+y^3+5(x+y)=3x^2+2y^2+5xy$$

Proposed by Sakthi Vel-India

Solution by Tapas Das-India

$$x^3+y^3+5(x+y)=3x^2+2y^2+5xy$$
$$(x+y)(x^2-xy+y^2)+5(x+y)=(x+y)(3x+2y)$$
$$(x+y)(x^2-xy+y^2+5-3x-2y)=0$$

$Case\ 1: x+y=0\ then\ x=-y,$

$$Solution\ (x,y)=(t,-t)\ for\ any\ integer\ t$$

$Case\ 2: (x^2-xy+y^2+5-3x-2y)=0$

$$x^2-x(y+3)+(y^2-2y+5)=0$$
$$x=\frac{y+3\pm\sqrt{(y+3)^2-4(y^2-2y+5)}}{2}\ (1)$$

The $discriminant$:

$$D=(y+3)^2-4(y^2-2y+5)=y^2+6y+9-4(y^2-2y+5)=-3y^2+14y-11$$

$$To\ get\ real\ roots\ D\geq 0\ as\ well\ as\ D\ must\ be\ perfect\ square$$
$$as\ we\ are\ find\ integer\ solutions.$$
$$For\ this\ \ -3y^2+14y-11\geq 0\ or, 3y^2-14y+11\leq 0$$
$$or\ (y-1)(3y-11)\leq 0\ or\ 1\leq y\leq\frac{11}{3}=3.66\ldots$$
$$So\ integer\ values\ of\ y=1,2,3$$
$$y=1\ gives\ from\ (1):\ x=\frac{1+3}{2}=2\ ,so\ (x,y)=(2,1)$$
$$y=3\ \ gives\ from\ (1):\ x=\frac{(3+3)\pm 2}{2}=4,2$$
$$so\ (x,y)=(2,3),(4,3)$$

$$For\ y = 2\ , the\ discriminant\ D = 5\ not\ a\ perfect\ square\ number\ , so\ y \neq 2$$

$$Required\ solutions\ (x, y) = (2, 1), (2, 3), (4, 3), (t, -t), t\ is\ an\ integer.$$

751. If $\lambda \in R$ then solve for reals:

$$\sqrt[3]{x - \lambda + 1} + \sqrt[3]{x - \lambda} + \sqrt[3]{x - \lambda - 1} = 0$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ y = x - \lambda\ then \sqrt[3]{x - \lambda + 1} + \sqrt[3]{x - \lambda} + \sqrt[3]{x - \lambda - 1} = 0$$
$$or\ \sqrt[3]{y + 1} + \sqrt[3]{y} + \sqrt[3]{y - 1} = 0$$
$$\left(\sqrt[3]{y + 1} + \sqrt[3]{y - 1}\right) = -\sqrt[3]{y} \quad (1)$$

$$\left(\sqrt[3]{y + 1} + \sqrt[3]{y - 1}\right)^3 = \left(-\sqrt[3]{y}\right)^3$$

$$y + 1 + y - 1 + 3\sqrt[3]{y + 1}.\sqrt[3]{y - 1}\left(\sqrt[3]{y + 1} + \sqrt[3]{y - 1}\right) = -y$$

$$2y - 3y^{\frac{1}{3}}\sqrt[3]{y + 1}.\sqrt[3]{y - 1} \overset{(1)}{=} -y$$
$$y = y^{\frac{1}{3}}\sqrt[3]{y + 1}.\sqrt[3]{y - 1}$$

$$y^3 = y(y + 1)(y - 1)$$

$$y(y^2 - y^2 + 1) = 0\ or\ y = 0$$

$$Solutions\ y = x - \lambda\ \ or\ x = \lambda\ (as\ y = 0)$$

752. Find all functions $f: \mathbb{R} \to \mathbb{R}$ such that:

$$f(f(x) + y^2 + 4) + 4y = x + f^2(y + 2)\ \forall\ x, y \in \mathbb{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$f(f(x) + y^2 + 4) + 4y = x + f^2(y + 2)\ \forall\ x, y \in R\ (1)$$

$$\text{We put } y=-2$$
$$\text{then } f(f(x)+8)-8=x+f^2(0) \quad (2)$$

$$\text{From } (2): f(u)=f(v) \Rightarrow f(f(u)+8)-8=f(f(v)+8)-8$$
$$\text{or } u+f^2(0)\overset{(2)}{=}v+f^2(0) \text{ or}, u=v, \quad \text{so } f \text{ is injective}$$

$$f(f(x)+8)=x+8+f^2(0)\text{, so every real number is in the image of } f,$$
$$\text{so } f \text{ is surjective}$$

$$\text{Again } f(f(x)+8) \text{ is a linear function of } x\text{, since from}(2) \text{ value of}$$
$$f(f(x)+8)=x+8+f^2(0)\text{. Clearly } f \text{ is linear.}$$

$$\text{Let } f(x)=ax+b \text{ then from } (1) \text{ we have:}$$
$$a(f(x)+y^2+4)+b+4y=x+(a(y+2)+b)^2$$
$$\text{or } a(ax+b+y^2+4)+b+4y=x+a^2y^2+4a^2+b^2+4a^2y+4ab+2aby$$
$$a^2x+ab+ay^2+4a+b+4y=x+a^2y^2+4a^2+b^2+4a^2y+4ab+2aby \quad (3)$$

$$\text{Equalizing coefficients of } x \,\&\, y^2 \text{ we get}$$
$$a^2=1 \text{ or}, a=\pm 1 \,\&\, a^2=a \text{ or}, a=0,1 \text{ so combine results is } a=1$$

$$\text{Putting } a=1 \text{ in } (3) \text{ we get}$$
$$x+b+y^2+4+b+4y=x+y^2+4+b^2+4y+4b+2by$$
$$\text{or } 2b=2by+4b+b^2$$

$$\text{Equalizing cofficients of } y \text{ we have } 2b=0 \text{ or } b=0$$
$$\text{so } f(x)=ax+b=x$$

753. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f^2(x)+2yf(x)+f(y)=f\big(y+f(x)\big)\ \forall\, x,y\in\mathbb{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$f^2(x)+2yf(x)+f(y)=f\big(y+f(x)\big)\ \forall\, x,y\in R \ (1)$$
$$\text{Suppose } f(x)=c \text{ is constant then the equation becomes}$$
$$c^2+2yc+c=c\ \forall\, y\text{, this can only be true if } c=0$$
$$\text{Hence one solution is } f(x)\equiv 0$$

$$\text{From } (1)\text{we have}$$
$$f\big(y+f(x)\big)-f(y)=f^2(x)+2yf(x)+f(y)$$

$$R.H.S\ part\ has\ the\ same\ form\ as\ the\ algebraic\ identity$$
$$(y+a)^2-y^2=2ay+a^2$$
$$this\ suggest\ that\ f(y)\ behave\ like\ y^2+some\ constant.$$

$$Let\ g(y)=\ f(y)-y^2\ or\ f(y)=g(y)+y^2\ (2)$$
$$where\ g(y)\ represent\ the\ non\ quadratic\ part\ of\ f(y).$$

$$From\ (1) we\ have\ f\big(y+f(x)\big)=f(y)+f^2(x)+2yf(x)+f(y)$$

$$Using\ (2) we\ \ have:$$
$$g\big(y+f(x)\big)+\big(y+f(x)\big)^2=g(y)+y^2+(g(x)+x^2)^2+2y(g(x)+x^2)$$
$$g\big(y+f(x)\big)+y^2+f^2(x)+2yf(x)=g(y)+y^2+(g(x)+x^2)^2+2y(g(x)+x^2)$$
$$g\big(y+f(x)\big)+(g(x)+x^2)^2+2y(g(x)+x^2)+2yf(x)\overset{f(x)=g(x)+x^2\ from\ (2)}{=}$$
$$=g(y)+(g(x)+x^2)^2+2y(g(x)+x^2)$$
$$g\big(y+f(x)\big)=g(y)$$

$$This\ equation\ says\ that\ g(y)\ does\ not\ change\ when\ its\ argument\ increase\ by\ f(x)$$
$$since\ f\ is\ non\ constant\ it\ can\ take\ infinitely\ many\ values\ so\ g(y)\ must$$
$$be\ constant\ g(y)=c\ hence\ f(y)=g(y)+y^2=c+y^2$$

$$So\ solutions\ are\ f(x)\equiv 0\ or\ f(x)=\ x^2+c.$$

754. If $\lambda>0$ then solve for reals:

$$\left(\frac{x+\lambda}{2}\right)^9+\left(\frac{x-\lambda}{2}\right)^9=\left(\frac{x^3+3\lambda^2x}{4}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ A=\frac{x+\lambda}{2},B=\frac{x-\lambda}{2}$$
$$A^3+B^3=\frac{1}{8}((x+\lambda)^3+(x-\lambda)^3)=\frac{1}{8}2x((x+\lambda)^2-(x+\lambda)(x-\lambda)+(x-\lambda)^2)=$$
$$=\frac{x}{4}(x^2+3\lambda^2)=\frac{(x^3+3\lambda^2x)}{4}\ (1)$$
$$Now:\left(\frac{x+\lambda}{2}\right)^9+\left(\frac{x-\lambda}{2}\right)^9=\left(\frac{x^3+3\lambda^2x}{4}\right)^3\Rightarrow$$
$$A^9+B^9=(A^3+B^3)^3\ or\ A^9+B^9-(A^3+B^3)^3=0\ (2)$$

$$We\ put\ u=A^3,v=B^3\ then\ A^9+B^9-(A^3+B^3)^3=u^3+v^3-(u+v)^3=$$

$$= u^3 + v^3 - \left(u^3 + v^3 + 3uv(u+v)\right) = -3uv(u+v) = -3A^3B^3(A^3+B^3)\ (3)$$

$$AB = \frac{x+\lambda}{2} \times \frac{x-\lambda}{2} = \frac{x^2-\lambda^2}{4} \Rightarrow A^3B^3 = \frac{1}{64}(x^2-\lambda^2)^3\ (4)$$

$$\text{From } (2) \text{we have } A^9 + B^9 - (A^3+B^3)^3 = 0$$

$$-A^3B^3(A^3+B^3) \overset{(3)}{=} 0 \text{ or } -\frac{3}{64}(x^2-\lambda^2)^3 \frac{(x^3+3\lambda^2 x)}{4} \overset{(1)\&(4)}{=} 0$$

$$x^2 - \lambda^2 = 0 \Rightarrow x = \pm\lambda \text{ and } \frac{(x^3+3\lambda^2 x)}{4} = 0 \text{ gives } x(x^2+3\lambda^2) = 0 \text{ or } x = 0$$

$$\text{Required solution } x = \pm\lambda, 0.$$

755. If $\lambda \geq 0$ then solve for reals:

$$x^2 + 2(x-\lambda)\sqrt{x-\lambda} + (1-2\lambda)x + \lambda(\lambda-1) = \lambda^2(\lambda+1)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\text{Domain}: x - \lambda \geq 0 \text{ or}, x \geq \lambda$$

$$\text{Let } t = \sqrt{x-\lambda}\ (t \geq 0) \Rightarrow x = t^2 + \lambda\ (1)$$

$$x^2 + 2(x-\lambda)\sqrt{x-\lambda} + (1-2\lambda)x + \lambda(\lambda-1) \overset{(1)}{=}$$
$$= (t^2+\lambda)^2 + 2t^3 + (1-2\lambda)(t^2+\lambda) + \lambda(\lambda-1) =$$
$$= (t^4 + 2t^2\lambda + \lambda^2) + 2t^3 + (t^2 + \lambda - 2\lambda t^2 - 2\lambda^2) + \lambda^2 - \lambda =$$
$$= t^4 + 2t^3 + t^2 = t^2(t+1)^2$$

$$x^2 + 2(x-\lambda)\sqrt{x-\lambda} + (1-2\lambda)x + \lambda(\lambda-1) = \lambda^2(\lambda+1)^2$$
$$t^2(t+1)^2 = \lambda^2(\lambda+1)^2 \text{ or}, t(t+1) = \lambda(\lambda+1), (t, \lambda \geq 0)$$
$$t^2 + t - (\lambda^2+\lambda) = 0 \text{ or } (t-\lambda)(t+\lambda+1) = 0 \Rightarrow t - \lambda = 0 \text{ or } t = \lambda$$
$$\text{Since}: t + \lambda + 1 = 0 \Rightarrow t = -1 - \lambda \text{ not possible as } t > 0 \text{ and } \lambda > 0$$
$$\text{Required solution } x \overset{(1)}{=} \lambda + t^2 = \lambda + \lambda^2.$$

756. If $\lambda \geq 0$ fixed, then solve for reals:

$$x^2 - 3x + 1 = -\frac{1}{\sqrt{\lambda+2}}\sqrt{x^4 + \lambda x^2 + 1}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{\sqrt{\lambda+2}}\sqrt{x^4+\lambda x^2+1} > 0\ \forall\, x \in R \text{ then } R.H.S\ \frac{-1}{\sqrt{\lambda+2}}\sqrt{x^4+\lambda x^2+1} < 0$$

$$so\ L.H.S\ must\ be < 0\ or\ x^2-3x+1<0\ (1)$$

$$x^2-3x+1=-\frac{1}{\sqrt{\lambda+2}}\sqrt{x^4+\lambda x^2+1}$$

$$(x^2-3x+1)^2=\frac{1}{\lambda+2}(x^4+\lambda x^2+1)$$

$$(\lambda+2)(x^2-3x+1)^2-(x^4+\lambda x^2+1)=0$$

$$(\lambda+2)(x^4+9x^2+1-6x^3-6x+2x^2)-(x^4+\lambda x^2+1)=0$$

$$(\lambda+1)x^4-6(\lambda+2)x^3+(10\lambda+22)x^2-6(\lambda+2)x+(\lambda+1)=0$$

$$(x-1)^2\left((\lambda+1)x^2-(4\lambda+10)x+(\lambda+1)\right)=0$$

$$now, x-1=0\Rightarrow x=1$$

$$and\ \left((\lambda+1)x^2-(4\lambda+10)x+(\lambda+1)\right)=0$$

$$\Rightarrow\ x^2-\frac{4\lambda+10}{\lambda+1}x+1=0\ \ (2)$$

$$Clearly\ product\ of\ the\ two\ roots=1,$$

$$which\ indicate\ that\ roots\ are\ reiprocal\ to\ each\ other\ let\ roots\ are\ r,\frac{1}{r}$$

$$then\ from\ (2), r+\frac{1}{r}=\frac{4\lambda+10}{\lambda+1}=4+\frac{6}{\lambda+1}>4\ as\ \lambda\ge 0$$

$$but\ we\ should\ also\ remember\ that\ from(1)\ that\ roots\ of\ eqution\ (2) must\ satisfy$$

$$x^2-3x+1<0$$

$$x^2-3x-1<0\ or\ \left(x-\frac{3+\sqrt{5}}{2}\right)\left(x-\frac{3-\sqrt{5}}{2}\right)<0\ so:$$

$$\frac{3-\sqrt{5}}{2}<x<\frac{3+\sqrt{5}}{2}\ which\ indicate\ x>0$$

$$x+\frac{1}{x}-3=\frac{x^2-3x+1}{x}<0\ as\ \ x^2-3x+1<0\ and\ x>0$$

$$or, x+\frac{1}{x}<3\ but\ from(2) we\ get$$

$$r+\frac{1}{r}>4, their\ does\ not\ exist\ any\ x\ in\ equation(2)$$

$$so\ the\ only\ solution: x=1$$

757. **For $a>1$fixed then solve for real numbers:**

$$\left(\frac{1}{a}\right)^{x+1}+\left(\frac{1}{a(a+1)}\right)^x-\sqrt{a}\left(\frac{\sqrt{a}}{a(a+1)}\right)^x=1$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\text{Let } s=a^{\frac{x+1}{2}}>0, y=\left(a(a+1)\right)^{-x}>0$$

$$\left(\frac{1}{a}\right)^{x+1}=a^{-(x+1)}=s^{-2},\left(\frac{1}{a(a+1)}\right)^{x}=y,$$

$$\sqrt{a}\left(\frac{\sqrt{a}}{a(a+1)}\right)^{x}=a^{\frac{1}{2}}.a^{\frac{x}{2}}.\left(a(a+1)\right)^{-x}=a^{\frac{x+1}{2}}.\left(a(a+1)\right)^{-x}=sy$$

$$\left(\frac{1}{a}\right)^{x+1}+\left(\frac{1}{a(a+1)}\right)^{x}-\sqrt{a}\left(\frac{\sqrt{a}}{a(a+1)}\right)^{x}=1$$

$$s^{-2}+y-sy=1\ or\ 1+ys^2-s^3y=s^2$$

$$1-s^2+ys^2(1-s)=0\ \ or(1-s)(1+s+s^2y)=0$$

$$as\ s=a^{\frac{x+1}{2}}>0\ since\ a>1, y=\left(a(a+1)\right)^{-x}>0\ so\ (1+s+s^2y)\ can\ not\ be\ 0$$

$$so\ 1-s=0\ or\ s=1\Rightarrow a^{\frac{x+1}{2}}=1\Rightarrow\frac{x+1}{2}=0\ or\ x=-1.$$

758. Solve for reals:

$$x^3-x+1=\sqrt[3]{2x-1}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ y=\sqrt[3]{2x-1}\ \ then\ y^3=2x-1\ (1)$$

$$From\ given\ equation\ we\ have\ x^3-x+1=y\ (2)$$

$$Let\ we\ put\ y=x\ in\ equation\ (1)\ then\ we\ get\ x^3-x+1=x\ or$$
$$x^3-2x+1=0\ or\ (x-1)(x^2+x-1)=0\Rightarrow$$
$$x=1\ and\ x^2+x-1=0\Rightarrow x=\frac{-1\pm\sqrt{5}}{2}$$
$$Now\ if\ we\ put\ y=x\ in\ equation\ (2) we\ get\ x^3=2x-1\ or\ x^3-2x+1=0$$
$$which\ is\ same\ as\ previous\ equation\ , so\ required\ solution:$$

$$x=1,\frac{-1\pm\sqrt{5}}{2}$$

759. Solve for reals:

$$x^2-x-1=\sqrt{2x+1}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Domain\ 2x+1\geq 0\ or\ x\geq-\frac{1}{2}$$

$$x^2-x-1=\sqrt{2x+1} \text{ or } (x^2-x-1)^2=2x+1$$
$$\text{or } x^4+x^2+1-2x^3-2x^2+2x=2x+1$$
$$\text{or } x^2(x^2-2x-1)=0 \Rightarrow\ x=0 \text{ or } x^2-2x-1=0 \Rightarrow x=1\pm\sqrt{2}.$$

$$Now\ for\ x=0\ given\ equation\ not\ satisfied$$
$$for\ x=1-\sqrt{2}, L.H.S=\left(1-\sqrt{2}\right)^2-\left(1-\sqrt{2}\right)-1=1-\sqrt{2}<0$$
$$R.H.S=\sqrt{2\left(1-\sqrt{2}\right)+1}>0\ so\ not\ satisfied$$
$$so\ required\ solution\ x=1+\sqrt{2}.$$

760. Solve for real numbers:

$$x^2-x+1=\sqrt[3]{3x-2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ t=\sqrt[3]{3x-2}\ \ or, t^3=3x-2\ or, x=\frac{t^3+2}{3}\ (1)$$

$$x^2-x+1=\sqrt[3]{3x-2}$$
$$\left(\frac{t^3+2}{3}\right)^2-\frac{t^3+2}{3}+1=t$$
$$(t^3+2)^2-3(t^3+2)+9=9t$$
$$t^6+4t^3+4-3t^3-6+9=9t$$
$$t^6+t^3-9t+7=0$$
$$(t-1)^2(t^4+2t^3+3t^2+5t+7)=0\ (2)$$

$$(t^4+2t^3+3t^2+5t+7)=t^2(1+t)^2+(2t^2+5t+7)$$
$$2t^2+5t+7>0\ as\ a=2>0\ and\ D=5^2-4.2.7=55-56<0$$
$$and\ \ t^2(1+t)^2\geq 0\ \forall t\in R, so\ t^2(1+t)^2+(2t^2+5t+7)>0$$
$$For\ this\ from (2) we\ get\ (t-1)^2=0\ or\ t=1$$
$$so\ required\ solution\ x=\frac{t^3+2}{3}=\frac{1^3+2}{3}=1$$

761. If $\lambda>1$ then solve for real numbers:

$$(\lambda^{2x}+\lambda^{2y}+\lambda^{2z})\left(\frac{1}{\lambda^x}+\frac{1}{\lambda^y}+\frac{1}{\lambda^z}\right)=3$$

$$3(\lambda^{x+y}+\lambda^{y+z}+\lambda^{z+x})=\lambda^x+\lambda^y+\lambda^z$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\text{Let } \lambda^x = a, \lambda^y = b, \lambda^z = c \text{ clearly } a,b,c > 0 \text{ as } \lambda > 1$$
$$\text{then given equation can be written as:}$$
$$(a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 3 \ \ (1)$$

$$3(ab+bc+ca) = a+b+c \ (2)$$

$$\text{Let } S = a+b+c, P = ab+bc+ca, R = abc \text{ from } (2) \text{ we get } 3P = S$$

$$\text{From } (1) \text{ we get } (a^2+b^2+c^2)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 3$$

$$\left((a+b+c)^2 - 2(ab+bc+ca)\right)\left(\frac{ab+bc+ca}{abc}\right) = 3$$

$$(S^2-2P)\frac{P}{R} = 3 \text{ or}, R = \frac{P}{3}(S^2-2P) \overset{S=3P}{=} \frac{P(9P^2-2P)}{3} \ \ (3)$$

$$\text{Using A.M} - \text{G.M inequality we get:}$$

$$\left(\frac{a+b+c}{3}\right)^3 \ge abc \text{ or } S^3 \ge 27R \text{ or}, R \overset{S=3P}{\le} P^3$$

$$\frac{P(9P^2-2P)}{3} \overset{(3)}{\le} P^3 \text{ or } 6P^2 \overset{P=abc>0}{\le} 2P \text{ or } P \le \frac{1}{3} \text{ so } S = 3P \le 3\times\frac{1}{3} \le 1 \ (4)$$

$$\text{Well known inequality } (a+b+c)^2 \ge 3(ab+bc+ca)$$

$$\text{or}, S^2 \ge 3P \text{or}, S^2 \overset{S=3P}{\ge} 3.\frac{S}{3} \text{ or}, S \overset{S=a+b+c>0}{\ge} 1 \ (5) \text{ now from}(4) \ \&\ (5)$$

$$\text{We get } S \ge 1 \ \&\ S \le 1,$$
$$\text{so } S = 1 \text{ or } a+b+c = 1 \text{ and } ab+bc+ca = P = \frac{S}{3} = \frac{1}{3}$$

$$(a-b)^2+(b-c)^2+(c-a)^2 = 2(a^2+b^2+c^2) - 2(ab+bc+ca)$$
$$= 2(S^2-2P) - 2P = 2(s^2-3P) \overset{S=1,P=\frac{1}{3}}{=} 2(1-1) = 0$$

$$a-b = 0 \Rightarrow a = b, (b-c) = 0 \Rightarrow b = c, (c-a) = 0 \Rightarrow c = a$$

$$a=b=c=\frac{1}{3}(as\ S=a+b+c=1)$$

$$\lambda^x=a=\frac{1}{3}\ or, x=\log_\lambda\left(\frac{1}{3}\right), \lambda^y=b=\frac{1}{3}\ or\ y=\log_\lambda\left(\frac{1}{3}\right),$$

$$\lambda^z=c=\frac{1}{3}\ or\ z=\log_\lambda\left(\frac{1}{3}\right)$$

$$Solution: x=y=z=\log_\lambda\left(\frac{1}{3}\right)$$

762. If $n,k\in N$, $n\geq k$ fixed then solve for natural numbers:

$$3^a+3^k=3^b+3^{n+k}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$3^a+3^k=3^b+3^{n+k}$$
$$3^a=3^b+3^{n+k}-3^k$$
$$3^a=3^b+3^k(3^n-1)$$
$$3^{a-k}=3^{b-k}+(3^n-1)$$

$$3^A=3^B+(3^n-1)\ \ (let\ \ A=a-k, B=b-k)$$
$$3^A-3^B=(3^n-1)$$

$$3^B(3^{A-B}-1)=3^n-1\ (1)$$

$$R.H.S\ part\ is\ not\ divisible\ by\ 3\ so, L.H.S\ part\ is\ divisible\ by\ 3\ unless\ B=0$$
$$so\ B=0\Rightarrow b-k=0\ or\ \ b=k$$

$$From(1):\ 3^A-1=3^n-1\ or\ \ 3^A=3^n\ or, A=n\ or, a-k=n\ or\ a=k+n$$
$$So\ required\ solution\ a=k+n, b=k$$

763. If $\lambda\in R$, $n\geq 0$ fixed then solve for reals:

$$x^2+2(x-\lambda)\sqrt{x-\lambda}+(1-2\lambda)x+\lambda(\lambda-1)=n^2(n+1)^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ t=\sqrt{x-\lambda}, t\geq 0\ so, x=t^2+\lambda$$

$$Domain: \ x \geq \lambda$$

$$x^2+2(x-\lambda)\sqrt{x-\lambda}+(1-2\lambda)x+\lambda(\lambda-1)=$$
$$=(t^2+\lambda)^2+2t^2.t+(1-2\lambda)(t^2+\lambda)+\lambda(\lambda-1)=$$
$$= t^4+2t^3+t^2=t^2(t+1)^2$$

$$x^2+2(x-\lambda)\sqrt{x-\lambda}+(1-2\lambda)x+\lambda(\lambda-1)=n^2(n+1)^2$$

$$t^2(t+1)^2=n^2(n+1)^2 \ or \ t(t+1)=n(n+1)(as\ n,t>0) \ or \ t=n$$

$$t=\sqrt{x-\lambda}=n \ or \ x=n^2+\lambda$$

764. Solve for naturals:

$$\left(1+\sqrt{6}\right)^3=\sqrt{p}+\sqrt{p-125}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ x=\sqrt{p}, \quad y=\sqrt{p-125}$$
$$\left(1+\sqrt{6}\right)^3=\sqrt{p}+\sqrt{p-125}$$

$$x+y=\left(1+\sqrt{6}\right)^3=1+6\sqrt{6}+3\sqrt{6}+18=19+9\sqrt{6}\ (1)$$

$$x-y=\frac{x^2-y^2}{x+y}=\frac{p-(p-125)}{19+9\sqrt{6}}=\frac{125\left(19-9\sqrt{6}\right)}{361-486}=9\sqrt{6}-9\ (2)$$

$$Adding\ (1)\ \&\ (2) we\ get\ 2x=18\sqrt{6}\ or\ \ x=9\sqrt{6}\ or\ x^2=486$$

$$p=x^2=486$$

765. If $a>1,\lambda>1$ fixed. Solve for reals:

$$x+\sqrt{\lambda^x\log_a(x+2a)}=\sqrt{\log_a(x+2a)}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Let\ y=\sqrt{\log_a(x+2a)}\geq 0\ then:$$
$$x+\sqrt{\lambda^x\log_a(x+2a)}=\sqrt{\log_a(x+2a)}$$

$$\Rightarrow x+y\lambda^{\frac{x}{2}}=y\ \Rightarrow x=y\left(1-\lambda^{\frac{x}{2}}\right)\ (1)$$

$$Case\ 1. If\ x>0\ then\ \lambda^{\frac{x}{2}}>1\ as\ \lambda>1\ so\ R.H.S\ part\ of\ (1)<0$$
$$but\ L.H.S>0\ contradiction.$$
$$Case\ 2. If\ x<0\ then\ \lambda^{\frac{x}{2}}<1\ as\ \lambda>1\ so\ R.H.S\ part\ of\ (1)\geq 0$$
$$but\ L.H.S<0\ contradiction.$$
$$Clearly, x=0\ satisfy\ equation\ (1)$$
$$so, required\ solution\ x=0$$

766. **If $\lambda>0$ fixed then solve for reals:**

$$(x-2y)^2+5\lambda^2=6x(\lambda-2x)+6y(2\lambda-y)$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$(x-2y)^2+5\lambda^2=6x(\lambda-2x)+6y(2\lambda-y)$$

$$Expanding\ \ we\ get:$$
$$13x^2-(4y+6\lambda)x+(10y^2-12\lambda y+5\lambda^2)=0$$

$$or\ x=\frac{4y+6\lambda\pm\sqrt{(4y+6\lambda)^2-52(10y^2-12\lambda y+5\lambda^2)}}{26}\ (1)$$
$$The\ discriminant\ D=(4y+6\lambda)^2-52(10y^2-12\lambda y+5\lambda^2)=$$
$$=-504y^2+672\lambda y-224\lambda^2=-56(3y-2\lambda)^2\ clearly\ D<0\,.$$

$$So\ real\ solution\ exist\ for\ fixed\ \lambda\ only\ when\ (3y-2\lambda)^2=0\ or\ y=\frac{2\lambda}{3}$$
$$From\ (1):\ x=\frac{\frac{4.2\lambda}{3}+6\lambda}{26}=\frac{\lambda}{3}$$

$$So\ required\ solution\ x=\frac{\lambda}{3}, y=\frac{2\lambda}{3}.$$

767. **Solve for integers:**

$$169x^2-13xy+12y=119$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$169x^2-13xy+12y=119\ \ or, y=\frac{119-169x^2}{12-13x}=\frac{169x^2-119}{13x-12}$$
$$=13x+12+\frac{25}{13x-12}\ \ (1)$$

$$for\ integer\ solution\ 25\ is\ divisible\ by\ 13x-12\ ,$$
$$for\ this\ 13x-12\in\{\pm1,\pm5,\pm25\}$$

$$Let\ 13x-12=d\ then\ x=\frac{d+12}{13}$$
$$If\ d=1\ then\ x=1, y\overset{(1)}{=}13\times1+12+\frac{25}{1}=50$$
$$d=-1\ then\ x=\frac{11}{13}, y\overset{(1)}{=}-2\ , so\ x\ not\in Z$$
$$d=5\ then\ x=\frac{17}{13}, so\ x\ not\in Z$$
$$d=-5\ then\ x=\frac{7}{13}, so\ x\ not\in Z$$
$$d=25\ then\ x=\frac{11}{13}, so\ x\ not\in Z$$
$$d=-25\ then\ x=-1, y=-2$$

$$So\ required\ solutions\ (x,y)=(1,50),(-1,-2)$$

768. Solve for real numbers:

$$\begin{cases} x(y+z)=35 \\ y(x+z)=35 \\ z(x+y)=27 \end{cases}$$

Proposed by Netai Chandra Bhar-India

Solution by Tapas Das-India

$$x(y+z)=35\ \ (1),\qquad y(x+z)=35\ \ (2),\qquad z(x+y)=27\ \ (3)$$
$$From (1)\&(2)\ we\ get\ x(y+z)=y(x+z)\ or\ z(x-y)=0\Rightarrow z=0\ or\ x=y$$
$$If\ z=0\ then\ from\ (3) we\ get\ 0=27\ , not\ possible\ so\ z\neq0$$
$$If\ x=y=m(say)\ then\ from (3) we\ have\ z.(2m)=27\ or, z=\frac{27}{2m}$$
$$From (1):\ m(m+z)=35\ or, m\left(m+\frac{27}{2m}\right)=35\ or\ m^2=\frac{43}{2}\ \ or$$

$$m = \pm\sqrt{\frac{43}{2}}, z = \frac{27}{2m} = \pm\frac{27}{\sqrt{86}}$$

$$So\ required\ solutions: x = \sqrt{\frac{43}{2}}, y = \sqrt{\frac{43}{2}}, z = \frac{27}{\sqrt{86}} and$$

$$x = -\sqrt{\frac{43}{2}}, y = -\sqrt{\frac{43}{2}}, z = -\frac{27}{\sqrt{86}}$$

769. If λ ∈ Z fixed then prove that the equation:

$$x^3 + y^3 - z^3 = \lambda^2(x + y - z)$$

has an infinitely solutions in integers.

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$We\ are\ asked\ to\ prove\ existence\ of\ infinitely\ many\ integer\ solution,$$
$$not\ to\ find\ all\ solution, so\ it\ is\ enough\ to\ find\ of\ solution\ one\ infinite\ family$$
$$for\ this\ we\ put\ x + y - z = 0 \Rightarrow z = x + y\ then\ the\ equation\ becomes$$
$$x^3 + y^3 - (x + y)^3 = 0\ or - 3xy(x + y) = 0 \ \Rightarrow\ x = 0\ or\ y = 0\ or\ x + y = 0$$
$$Case\ 1)\ x = 0\ then\ z = y\ as\ z = x + y\ , solution\ (0, y, y), y \in z$$
$$Case\ 2)\ y = 0\ then\ z = x\ as\ z = x\ , solution\ (x, 0, x), x \in z$$
$$Case\ 3) x + y = 0\ then\ y = -x, z = 0\ as\ z = x + y\ , solution\ (x, -x, 0), x \in \mathbb{Z}$$
$$For\ all\ these\ solutions\ x + y - z = 0 \Rightarrow\ \lambda^2(x + y - z) = 0\ then\ x^3 + y^3 - z^3 = 0$$
$$then\ orginal\ equation\ is\ satisfied\ for\ every\ integer\ \lambda\ , (0, y, y), (x, 0, x), (x, -x, 0)$$
$$x, y, z \in\ Z\ infinitely\ many\ integer\ solutions\ .$$

770.

$$Solve\ for\ x \in \left(0, \frac{\pi}{2}\right):$$

$$\frac{1}{\cos^2 x} + \frac{1}{\cos^2\left(\frac{\pi}{3} - x\right)} + \frac{1}{\cos^2\left(\frac{\pi}{3} + x\right)} = 18$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\frac{1}{\cos^2 x} + \frac{1}{\cos^2\left(\frac{\pi}{3} - x\right)} + \frac{1}{\cos^2\left(\frac{\pi}{3} + x\right)} = 18$$

$$\sec^2 x + \sec^2\left(\frac{\pi}{3} - x\right) + \sec^2\left(\frac{\pi}{3} + x\right) = 18$$

$$3 + \tan^2 x + \tan^2\left(\frac{\pi}{3} - x\right) + \tan^2\left(\frac{\pi}{3} + x\right) = 18$$

$$3+\tan^2 x+\left(\frac{\sqrt{3}-tanx}{1+\sqrt{3}tanx}\right)^2+\left(\frac{\sqrt{3}+tanx}{1-\sqrt{3}tanx}\right)^2=18$$
$$t^2+\left(\frac{\sqrt{3}-t}{1+\sqrt{3}t}\right)^2+\left(\frac{\sqrt{3}+t}{1-\sqrt{3}t}\right)^2\overset{t=tanx}{=}15$$
$$t^2+\left(\frac{6+20t^2+6t^4}{(1-3t^2)^2}\right)=15\ or\ \frac{6+21t^2+9t^6}{(1-3t^2)^2}=15$$
$$6+21t^2+9t^6=15(1-3t^2)^2\ or\ 3t^6-45t^4+37t^2-3=0$$
$$3x^3-45x^2+37x-3\overset{x=t^2}{=}0\ or\ (x-1)(3x^2-42x+3)=0$$
$$t^2=x=1\ and\ (3x^2-42x+3)=0\ or\ x=7\pm 4\sqrt{3}$$
$$t^2=1\ or\ t=1\ or\ tanx=1\ or\ x=\frac{\pi}{4}\ and\ t^2=7\pm 4\sqrt{3}\ \ or, t=\sqrt{7\pm 4\sqrt{3}}\ or,$$
$$tanx=2\pm\sqrt{3}\ \ when\ tanx=2+\sqrt{3}\ or\ \ x=\frac{5\pi}{12}, tanx=2-\sqrt{3}\ or\ x=\frac{\pi}{12}$$
$$So\ solutions:\ x=\frac{5\pi}{12},\frac{\pi}{12},\frac{\pi}{4}$$

771.

Find the conditions on a and b so that following equation has 3 distinct solutions

$$x^4-2(a^2+b^2-1)x^2+(a^2-b^2+1)^2-4a^2=0$$

Proposed by Netai Chandra Bhar-India

Solution by Tapas Das-India

$$x^4-2(a^2+b^2-1)x^2+(a^2-b^2+1)^2-4a^2=0\ \ (1)$$
$$Let\ y=x^2\ then\ from(1)we\ get\ y^2-2(a^2+b^2-1)y+(a^2-b^2+1)^2-4a^2=0\ (2)$$
$$since\ y=x^2\ if\ both\ roots\ y_1,y_2>0\ then\ eqution\ (1)has\ 4\ distinct\ root.$$
$$If\ one\ root\ is\ 0\ then\ \ y=0\ and\ other\ positive\ root$$
$$y_1>0\ and\ equation\ (1)has\ 3\ distinct\ roots\ as\ 0,\sqrt{y_1},-\sqrt{y_1}$$
$$We\ put\ x=0\ in\ (2)we\ get$$
$$(a^2-b^2+1)^2-4a^2=0\ or,(a^2-b^2+1+2a)(a^2-b^2+1-2a)=0$$
$$or\ (a^2-b^2+1+2a)=0\Rightarrow(a+1)^2=b^2\ or, a=b-1$$
$$(a^2-b^2+1-2a)=0\ \Rightarrow(a-1)^2=b^2\ or\ \ b=a-1$$
$$From(2)sum\ of\ the\ two\ roots$$
$$2(a^2+b^2-1)>0\ (as\ roots\ are\ \ 0,y_1>0)\ or\ a^2+b^2>1$$
$$So\ required\ conditions: a=b-1\ ,b=a-1\ ,a^2+b^2>1$$

772. Solve for natural numbers:

$$5^a-5^b=620$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$5^a - 5^b = 620 \ \ or \ \ 5^b(5^{a-b} - 1) = 5 \times 124 = 5(5^3 - 1)$$

$$Comparing\ we\ get\ b = 1\ and\ a - b = 3\ or$$
$$a = b + 3 = 3 + 1 = 4$$

$$Solution\ a = 4, b = 1.$$

773. **If $a > 1, \lambda \in R$ then solve for real numbers:**

$$a^{\sqrt{x-\lambda-1}} = \frac{1}{\sqrt{x-\lambda}}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Domain\ x - \lambda - 1 \geq 0\ or\ x \geq \lambda + 1\ let\ t = \sqrt{x - \lambda - 1} \Rightarrow x - \lambda = t^2 + 1$$
$$a^{\sqrt{x-\lambda-1}} = \frac{1}{\sqrt{x-\lambda}} \ \ or \ \ a^t = (t^2+1)^{-\frac{1}{2}}$$
$$t \log a = -\frac{1}{2}\log(t^2+1) \ \ or \ \ \log(t^2+1) + 2t\log a = 0$$
$$Let\ f(t) = \log(t^2+1) + 2t\log a\ \ then\ f'(t) = \frac{2t}{t^2+1} + 2log a > 0\ as$$
$$a > 1\ and \geq 0.\ So\ f(t)\ is\ increasing.$$

$$The\ only\ one\ solution\ is\ t = 0\ as\ f(0) = 0,$$

$$x - \lambda = t^2 + 1 or\ x = \lambda + 1\ as\ t = 0$$

774. **Solve for real numbers:**

$$16x^2 - 56x + 51 = \sqrt{4x-6} + \sqrt{8-4x}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Domain: 4x-6\geq 0\ or, x\geq\frac{3}{2}\ and\ 8-4x\geq 0\ or, x\leq 2\ so\ \frac{3}{2}\leq x\leq 2$$

$$We\ know\ that\ upward\ opening\ parabola\ y=ax^2+bx+c$$

$$\min at\ x=-\frac{b}{2a}, y_{min.}=c-\frac{b^2}{4a}$$

$$so\ y=16x^2-56x+51 \min.\ at\ x=-\frac{-56}{2\times 16}=\frac{7}{4}\ and\ y_{Min}=51-\frac{56\times 56}{4\times 16}=2$$

$$then\ 16x^2-56x+51\geq 2\ and\ \sqrt{4x-6}+\sqrt{8-4x}\overset{CBS}{\leq}\sqrt{2(4x-6+8-4x)}$$

$$=2\ in\left[\frac{3}{2},2\right]\ so, L.H.S\geq 2\ and$$

$$R.H.S\leq 2\ \ equality\ possible\ only\ when\ both\ equal\ to\ 2\ or\ x=\frac{7}{4}$$

775. Solve for natural numbers:

$$7^a-7^b=294$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$7^a-7^b=294\ or, 7^b\left(7^{a-b}-1\right)=7^2\times 6=7^2(7-1)$$

$$Comparing: b=2, a-b=1\ then\ a=3.$$

776. Solve for real numbers:

$$\begin{cases} x+\frac{1}{z}=3 \\ y+\frac{1}{x}=2 \\ xyz=1 \end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$x+\frac{1}{z}=3\ then\ z=\frac{1}{3-x}, y+\frac{1}{x}=2\ then\ x=\frac{1}{2-y}$$

$$xyz=1\ or\ \frac{y}{(3-x)(2-y)}=1\ or\ \frac{y}{\left(3-\frac{1}{2-y}\right)(2-y)}=1$$

$$\frac{y}{5-3y}=1\ or\ y=\frac{5}{4}$$

$$x=\frac{1}{2-y}=\frac{1}{2-\frac{5}{4}}=\frac{4}{3}\ and\ z=\frac{1}{xy}=\frac{1}{\frac{4}{3}\cdot\frac{5}{4}}=\frac{3}{5}$$

777. If $\begin{cases} x^2+y^2+2x+4y-20=0 \\ z^2+w^2-2w-143=0 \\ xw+yz-x+w+2z-61\geq 0 \end{cases}$ $S=\frac{y^2}{25}+\frac{w^2}{144}$ then find:

$$S_{\min}+S_{\max}=?$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

The first two equations define the a couple of circles.

$$Equation\ 1.\ \ x^2+y^2+2x+4y-20=0\to\ (x+1)^2+(y+2)^2=5^2$$

$$M_1(-1;-2)\quad R_1=5$$

$$Equation\ 2.\ \ z^2+w^2-2w-143=0\to z^2+(w-1)^2=12^2$$

$$M_2(0;1)\quad R_2=12$$

$$(x-a)^2+(y-b)^2=R^2\ \to\ \begin{cases} x=a+Rcos(\alpha) \\ y=b+Rsin(\alpha) \end{cases}$$

$$\begin{cases} x+1=5\cos(\alpha) \\ y+2=5\sin(\alpha) \end{cases}\to\ \begin{cases} x=5\cos(\alpha)-1 \\ y=5\sin(\alpha)-2 \end{cases}$$

$$\begin{cases} z=12\cos(\beta) \\ w-1=12\sin(\beta) \end{cases}\ \to\ \ w=12\sin(\beta)+1$$

$$xw+yz-x+w+2z-61\geq 0\to\ x(w-1)+z(y+2)+w-1\geq 60$$

$$(5\cos(\alpha)-1)12\sin(\beta)+12\cos(\beta).5\sin(\alpha)+12\sin(\beta)\geq 60$$

$$60\sin(\alpha)\cos(\beta)+60\sin(\beta)\cos(\alpha)\geq 60$$

$$\sin(\alpha+\beta)\geq 1\to(\alpha+\beta)\in\left[\frac{\pi}{2}+\pi k\right]\ \ \sin(\beta)=\cos(\alpha)$$

$$S=\frac{y^2}{25}+\frac{w^2}{144}=\left(\frac{y}{5}\right)^2+\left(\frac{w}{12}\right)^2=\left(\frac{5\sin(\alpha)-2}{5}\right)^2+\left(\frac{12\cos(\alpha)-1}{12}\right)^2=$$

$$=\sin^2(\alpha)-\frac{4}{5}\sin(\alpha)+\frac{4}{25}+\cos^2(\alpha)-\frac{1}{6}\cos(\alpha)+\frac{1}{144}$$
$$=\frac{4201}{3600}+\frac{1}{6}\cos(\alpha)-\frac{4}{5}\sin(\alpha)$$
$$S=C+\frac{1}{6}\cos(\alpha)-\frac{4}{5}\sin(\alpha)\quad Amplitude\rightarrow\ A=\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{4}{5}\right)^2}=\frac{\sqrt{601}}{30}$$
$$S_{\max}=C+A\quad S_{\min}=C-A\quad S_{\max}+S_{\min}=2C=2.\frac{4201}{3600}=\frac{4201}{1800}$$

778. Solve for natural numbers:

$$(x+2)^{\frac{1}{x}}+(x+1)^x=2^{2x}-5$$

Proposed by Sakthi Vel-India

Solution by Tapas Das-India

$for\ x=1$

$$L.H.S=(1+2)^1+2^1=5, R.H.S=2^2-5=-1\ so\ L.H.S\neq R.H.S$$

$for\ x=2$

$$L.H.S=(2+2)^{\frac{1}{2}}+3^2=11, R.H.S=2^4-5=11\ so\ L.H.S=R.H.S$$
$$so\ x=2\ is\ a\ solution.$$

$for\ x\geq 3\ , (x+2)^{\frac{1}{x}}>1\ and\ (x+1)^x>4^x=2^{2x}$

$$then (x+2)^{\frac{1}{x}}+(x+1)^x>2^{2x}+1>2^{2x}-5$$

$$so\ for\ n\geq 3\ equality\ impossible.$$

$$The\ only\ solution\ x=2$$

779. Solve for real numbers:

$$\sqrt{2x^2-5x-42}\sqrt[4]{(2x^2+x-3)}+1=\left(\left\lfloor sin\left(\frac{\pi x}{7}\right)\right\rfloor+\left\lfloor cos\left(\frac{2\pi x}{3}\right)\right\rfloor\right)^2$$
$$\lfloor * \rfloor\rightarrow\ floor\ function$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$LHS=\sqrt{2x^2-5x-42}\sqrt[4]{(2x^2+x-3)}+1>0$$
$$RHS=\left(\left\lfloor sin\left(\frac{\pi x}{7}\right)\right\rfloor+\left\lfloor cos\left(\frac{2\pi x}{3}\right)\right\rfloor\right)^2\geq 0$$

$$\begin{cases}\sqrt{2x^2-5x-42}\geq 0\\ \sqrt[4]{(2x^2+x-3)}\geq 0\end{cases}\to\begin{cases}(2x+7)(x-6)\geq 0\\ (2x+3)(x-1)\geq 0\end{cases}\begin{matrix}x\geq 6, x\leq -\frac{7}{2}\\ x\geq 1, x\leq -\frac{3}{2}\end{matrix}$$

$$x\in\left(-\infty;-\frac{7}{2}\right]\cup[6;+\infty)\quad LHS=1\to\quad x_1=-\frac{7}{2}\ x_2=6$$

$$-1\leq sin(\alpha)\leq 1\quad -1\leq cos(\alpha)\leq 1$$

$$RHS=\left(\left|sin\left(\frac{\pi x}{7}\right)\right|+\left|cos\left(\frac{2\pi x}{3}\right)\right|\right)^2=\{0;1;4\}$$

$$RHS=LHS=1\to\quad RHS\neq\{0;4\}$$

$$x_1=-\frac{7}{2}\to\ RHS=\left(\left|sin\left(-\frac{\pi}{2}\right)\right|+\left|cos\left(-\frac{7\pi}{3}\right)\right|\right)^2=(-1+0)^2=1$$

$$x_2=6\to\ RHS=\left(\left|sin\left(\frac{6\pi}{7}\right)\right|+\lfloor cos(4\pi)\rfloor\right)^2=(0+1)^2=1$$

$$x\to\left(-\frac{7}{2};6\right)$$

780. Find $x,y,z\in R^+$ such that:

$$\begin{cases}\sqrt{3}(x-y)\leq 1+xy\\ \sqrt{3}(y-z)\leq 1+yz\\ \sqrt{3}(1+xz)\leq x-z\end{cases}$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\begin{cases}\frac{x-y}{1+xy}\leq\frac{1}{\sqrt{3}}\\ \frac{y-z}{1+yz}\leq\frac{1}{\sqrt{3}}\\ \frac{x-z}{1+xz}\geq\sqrt{3}\end{cases}\to\tan(a-b)=\frac{\tan(a)-\tan(b)}{1+\tan(a)\tan(b)}.\to\begin{cases}x=\tan(\alpha)\\ y=\tan(\beta)\\ z=\tan(\gamma)\end{cases}$$

$$x,y,z\in R^+\to\alpha,\beta,\gamma\in\left(0;\frac{\pi}{2}\right)$$

$$\begin{cases}\tan(\alpha-\beta)\leq\frac{1}{\sqrt{3}}\\ \tan(\beta-\gamma)\leq\frac{1}{\sqrt{3}}\\ \tan(\alpha-\gamma)\geq\sqrt{3}\end{cases}\to\begin{cases}\alpha-\beta\leq\frac{\pi}{6}\\ \beta-\gamma\leq\frac{\pi}{6}\\ \alpha-\gamma\geq\frac{\pi}{3}\end{cases}\to\begin{cases}\alpha-\beta+\beta-\gamma\leq\frac{\pi}{3}\\ \alpha-\gamma\geq\frac{\pi}{3}\end{cases}\to\begin{cases}\alpha-\gamma\leq\frac{\pi}{3}\\ \alpha-\gamma\geq\frac{\pi}{3}\end{cases}$$

$$\alpha-\gamma=\frac{\pi}{3},\qquad\alpha-\beta=\frac{\pi}{6},\qquad\beta-\gamma=\frac{\pi}{6}$$

1. $\sqrt{3}x-\sqrt{3}y-xy=1\quad y=\frac{x\sqrt{3}-1}{x+\sqrt{3}}\quad y\in R^+\ \ x>\frac{1}{\sqrt{3}}$

$$2.\ \ x-z=\sqrt{3}(1+xz)\ \ z=\frac{x-\sqrt{3}}{x\sqrt{3}+1}\ \ z\in R^+\ \ x>\sqrt{3}$$

$$x>\sqrt{3}$$

$$\textit{The solution set of the system: } x,y,z\in R^+\to\left(x>\sqrt{3};\ \frac{x\sqrt{3}-1}{x+\sqrt{3}};\frac{x-\sqrt{3}}{x\sqrt{3}+1}\right)$$

781. **Solve for positive numbers:**

$$\begin{cases}x\sqrt[3]{x}+y\sqrt[3]{y}=162\\ x\sqrt[3]{y}+y\sqrt[3]{x}=162\end{cases}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\textit{Let } a^3=x,b^3=y \textit{ then above equation can be written as}$$

$$a^4+b^4=162\ (1)$$

$$ab(a^2+b^2)=162\ (2)$$

$$a^4+b^4=162 \text{ or } (a^2+b^2)^2-2a^2b^2=162 \text{ or } s^2-2p^2\overset{s=a^2+b^2,p=ab}{=}162$$

$$\textit{From } (2)\ ps=162 \text{ or } s=\frac{162}{p}$$

$$s^2-2p^2=162 \text{ or } \left(\frac{162}{p}\right)^2-2p^2=162 \text{ or } 2p^4+162p^2-162^2=0$$

$$p^4+81p^2-13122=0 \text{ or } t^2+81t-13122\overset{t=p^2}{=}0$$

$$t=p^2=\frac{-81\pm 243}{2} \text{ or } t=p^2=81\ (\textit{taking positive value}) \text{ or}$$

$$p=9 \text{ and } s=\frac{162}{p}=18$$

$$p=a^2+b^2=18,s=ab=9$$

$$(a-b)^2=a^2+b^2-2ab=18-18=0 \text{ or } a=b$$

$$\textit{then } a=b=\sqrt{9}=3 \textit{ and } x=a^3=27,y=b^3=27$$

782. **If $a>1$ fixed then solve fror real numbers:**

$$\sqrt{a^x-1}+\sqrt{a^x+1}=\sqrt{2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$Domain\ a^x - 1 \ge 0\ or, a^x \ge 1\ or, x \ge 0\ as\ a > 1$$

$$Let\ y = a^x\ then\ \sqrt{a^x - 1} + \sqrt{a^x + 1} = \sqrt{2}$$
$$\sqrt{y-1} + \sqrt{y+1} = \sqrt{2}\ \ or\ \left(\sqrt{y-1} + \sqrt{y+1}\right)^2 = \left(\sqrt{2}\right)^2$$

$$y + 1 + y - 1 + 2\sqrt{y^2 - 1} = 2$$

$$y + \sqrt{y^2 - 1} = 1\ or\ \sqrt{y^2 - 1} = 1 - y$$

$$L.H.S \ge 0 \Rightarrow R.H.S \ge 0\ or\ \ 1 - y \ge 0\ or\ y \le 1$$

$$But\ y = a^x\ (a > 1) \ge 1\ so\ we\ can\ say\ y = 1\ or\ a^x = 1\ or\ x = 0\ (as\ a > 1).$$

783. Solve for real numbers:

$$\cot^2(x) = \lfloor \cos^2(x) \rfloor + \{\sin^2(x)\}$$
$$\lfloor * \rfloor - floor\ function\ ,\quad \{*\} - \ fractional\ part\ function$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\lfloor \cos^2(x) \rfloor \to 0 \le \cos^2(x) \le 1; \{\sin^2(x)\} \to 0 \le \sin^2(x) \le 1$$
$$\{\sin^2(x)\} = \sin^2(x) - \lfloor \sin^2(x) \rfloor,\qquad 0 \le \sin^2(x) < 1 \longrightarrow \lfloor \sin^2(x) \rfloor = 0$$
$$\{\sin^2(x)\} = \sin^2(x) - 0 = \sin^2(x)$$
$$\cot(x) = \frac{\cos(x)}{\sin(x)} \to \sin(x) \ne 0\ \ x \ne \pi k \to \lfloor \cos^2(x) \rfloor = 0$$
$$\cot^2(x) = \lfloor \cos^2(x) \rfloor + \{\sin^2(x)\}, \cot^2(x) = \sin^2(x)$$
$$\cos^2(x) = \sin^4(x) \to\ \sin^4(x) + \sin^2(x) - 1 = 0$$
$$\sin^2(x) = t \to\ t^2 + t - 1 = 0, t_{1;2} = \frac{-1 \pm \sqrt{5}}{2}$$
$$t \ne \frac{-1 - \sqrt{5}}{2} < 0,\qquad t = \frac{-1 + \sqrt{5}}{2} > 0$$
$$t = \frac{\sqrt{5} - 1}{2}\quad \sin^2(x) = t \to \sin(x) = \pm\sqrt{\frac{\sqrt{5} - 1}{2}}$$
$$x_1 = \pm \arcsin\sqrt{\frac{\sqrt{5} - 1}{2}} + \pi k$$
$$\sin^2(x) = 1 \to \sin(x) = \pm 1\ \ \cos(x) = 0$$

$$\frac{\cos^2(x)}{\sin^2(x)} = \lfloor\cos^2(x)\rfloor + \{\sin^2(x)\}, \qquad 0 = \lfloor 0\rfloor + \{1\} = 0$$

$$x_1 = \pm\arcsin\sqrt{\frac{1}{\varphi}} + \pi k, \qquad x_2 = \frac{\pi}{2} + \pi k; \quad k \in Z$$

784. Let be $f, g, p: \mathbb{R} \to \mathbb{R}$, bijectifs functions such that:

$$p(x) = 13x - 15, f^{-1}(-1) = 1.5,$$

$$f\left(g^2(x) - 2g(x)\right) = \frac{3(x+1)}{3x+2}, x \in \mathbb{R} - \{1, -\frac{2}{3}\}$$

Find: $p\left(g^{-1}(1)\right)$.

Proposed by Amin Hajiyev-Azerbaijan

Solution by Shirvan Tahirov-Azerbaijan

$$f\left(g^2(x) - 2g(x)\right) = \frac{3(x+1)}{3x+2} = t$$

$$x = \frac{3t^{-1} + 3}{3t^{-1} + 2} \to \begin{cases} 3xt^{-1} - 3t^{-1} + 2x = 3 \\ 3xt^{-1} - 3t^{-1} = 3 - 2x \\ t^{-1}(3x - 3) = 3 - 2x \\ t^{-1} = \dfrac{3 - 2x}{3x - 3} \end{cases}$$

$$f^{-1}\left(g^2(x) - 2g(x)\right) = \frac{3 - 2x}{3x - 3}$$

$$p\left(g^{-1}(1)\right) \to g^{-1}(1) = x, \qquad g(x) = 1$$

$$\frac{3 - 2x}{3x - 3} = 1.5 \to \begin{cases} 4.5x - 4.5 = 3 - 2x \\ x = \dfrac{15}{13} \end{cases}$$

$$g\left(\frac{15}{13}\right) = 1, \; g^{-1}(1) = \frac{15}{13}$$

$$p\left(g^{-1}(1)\right) = p\left(\frac{15}{13}\right) = 13.\frac{15}{13} - 15 = 0$$

785. Find $x, y, z \in Z$ such that:

$$x^4 + y^4 = 21 - x^2y^2$$

Proposed by Bui Hong Suc-Vietnam

Solution by Tapas Das-India

$$x^4 + y^4 = 21 - x^2y^2 \;\; or, \qquad x^4 + y^4 + x^2y^2 = 21$$

$$or, a^2 + b^2 + ab \overset{a=x^2\geq 0, b=y^2\geq 0}{=} 21 \; (1)$$

$$for\ a=0\ \ from\ (1) we\ get\ b^2=21\neq Z\ no\ integer\ solution$$

$$for\ a=1\ from\ (1) we\ get\ 1+b^2+b=21\ or, (b-4)(b+5)=0\ or, b=4$$
$$(as\ b=-5\neq Z)$$

$$for\ a=2\ from\ (1) we\ get\ 4+b^2+2b=21\ or, b^2+2b-17=0$$
$$Discriminant=D=2^2-4.1.(-17)=72\ not\ a\ perfect\ square\ ,$$
$$no\ integer\ solution$$

$$for\ a=3\ from\ (1) we\ get\ 9+b^2+3b=21\ or, b^2+3b-12=0$$
$$Discriminant=D=3^2-4.1.(-12)=57\ not\ a\ perfect\ square\ ,$$
$$no\ integer\ solution$$

$$for\ a=4\ from\ (1) we\ get\ b^2+4b-5=0\ \ or, (b-1)(b+5)=0\ or, b=1$$
$$(as\ b=-5\neq Z)$$
$$for\ a\geq 5\ L.H.S\ of\ (1)\geq 21\ not\ possible$$

$$now\ a=1, b=4\Rightarrow\ x^2=1, y^2=4\ or, x=\pm 1, y=\pm 2$$
$$solution: (1,2),(1,-2),(-1,2),(-1,-2)$$

$$now\ a=4, b=1\Rightarrow\ x^2=4, y^2=1\ or, x=\pm 2, y=\pm 1$$
$$solution: (2,1),(2,-1),(-2,1),(-2,-1)$$

786. **Solve for integers:**

$$\begin{cases}(x+y+z)^2+(x-y+z)^2-18(z-1)(5-z)=2y^2\\ x+y-z=4\end{cases}$$

Proposed by Bui Hong Suc-Vietnam

Solution by Tapas Das-India

$$(x+y+z)^2+(x-y+z)^2-18(z-1)(5-z)=2y^2$$
$$or, 2((x+z)^2-y^2)-18(z-1)(5-z)=2y^2$$
$$or, (x+z)^2=9(z-1)(5-z)$$
$$or, (4+2z-y)^2\overset{x+y-z=4}{=}9(z-1)(5-z)\ \ (1)$$

$$now\ L.H.S\geq 0\ so\ R.H.S\geq 0\ for\ this\ (z-1)(5-z)\geq 0\ or, 1\leq z\leq 5$$

$$when\ z=1, R.H.S=0\ so\ from\ (1) we\ get\ (6-y)^2=0\ or, y=6$$
$$and\ x=4+z-y=4+1-6=-1\ , solution\ (x,y,z)=(-1,6,1)$$

$$when\ z=2, R.H.S=27\ so\ from\ (1) we\ get\ (8-y)^2=27$$
$$no\ integer\ solution\ exist$$

$$when\ z=3, R.H.S=36\ so\ from\ (1) we\ get\ (10-y)^2=36\ or, y=4,16$$
$$and\ x=4+z-y\overset{y=4}{=}4+3-4=3\ , solution\ (x,y,z)=(3,4,3)$$
$$and\ x=4+z-y\overset{y=16}{=}4+3-16=-9\ , solution\ (x,y,z)=(-9,16,3)$$

$$when\ z=4, R.H.S=27\ so\ from\ (1) we\ get\ (12-y)^2=27$$
$$no\ integer\ solution\ exist$$

$$when\ z=5, R.H.S=0\ so\ from\ (1) we\ get\ (14-y)^2=0\ or, y=14$$
$$and\ x=4+z-y=4+5-14=-5$$
$$solution\ (x,y,z)=(-5,14,5)$$

787. **Solve for integers:**

$$(x+1)(x+2)(x+3)(x+4)=y^2-12$$

Proposed by Bui Hong Suc-Vietnam

Solution by Tapas Das-India

$$(x+1)(x+4)=x^2+5x+4$$
$$(x+2)(x+3)=x^2+5x+6\ ,$$
$$Let\ u=x^2+5x+5\ \ then\ (x+1)(x+4)=u-1\ \&\ (x+2)(x+3)=u+1$$
$$(x+1)(x+2)(x+3)(x+4)=y^2-12$$
$$or, (u+1)(u-1)=y^2-12$$
$$or, u^2-1=y^2-12\ or, y^2-u^2=11$$

$$11\ is\ a\ prime\ number\ , so\ possible\ pairs\ are\ :$$

$$(y-u,y+u)=(1,11),(-1,-11),(11,1),(-11,-1)$$
$$y-u=1, y+u=11\Rightarrow y=6, u=5$$
$$y-u=-1, y+u=-11\Rightarrow y=-6, u=5$$
$$y-u=11, y+u=1\Rightarrow y=6, u=-5$$
$$y-u=-11, y+u=-1\Rightarrow y=-6, u=5$$

$$for\ u=5\ \&\ y=6\ , y=-6\ we\ get\ x^2+5x+5=5\ or, x=0,-5$$
$$solution\ (x,y)=(0,6),(0,-6),(-5,6),(-5,-6)$$

$$for\ u=-5\ \&\ y=6\ we\ get\ x^2+5x+5=-5\ or, x^2+5x+10=0$$
$$Discriminant=-15\ no\ integer\ solution\ exist$$

$$for\ u = 5\ \&\ y = 6\ we\ get\ x^2 + 5x + 5 = 5\ or, x = 0, -5$$

788. Solve for integers:

$$x^4 + 6x^3 + 13x^2 - 4y^2 + 12x + 8y + 12 = 0$$

Proposed by Bui Hong Suc-Vietnam

Solution by Tapas Das-India

$$x^4 + 6x^3 + 13x^2 - 4y^2 + 12x + 8y + 12 = 0$$
$$or, (x^2 + 3x + 2)^2 - 4(y-1)^2 = -12$$
$$or, 4(y-1)^2 - (x^2 + 3x + 2)^2 = 12$$
$$or, 4(y-1)^2 - \left((x+1)(x+2)\right)^2 = 12\ (1)$$

$$clearly\ (x+1), (x+2)\ are\ two\ consecutive\ numbers$$
$$so\ (x+1)(x+2) is\ multiple\ of\ 2\ , let\ (x+1)(x+2) = 2m\ , m \in Z$$
$$from\ (1) we\ get\ 4(y-1)^2 - 4m^2 = 12\ \ or, (y-1)^2 - m^2 = 3$$
$$or, (y-1+m)(y-1-m) = 3$$
$$possible\ pairs\ are:$$
$$(y-1+m)(y-1-m) = (3,1), (-1,-3), (-3,-1), (1,3)$$

$$y-1+m = 3, y-1-m = 1 \Rightarrow y-1 = 2, m = 1\ or, y = 3, m = 1$$
$$m = 1 \Rightarrow (x+1)(x+2) = 2m\ or, x^2 + 3x + 2 = 2\ or, x = -3, 0$$
$$solution = (x,y) = (-3,3), (0,3)$$

$$y-1+m = -1, y-1-m = -3 \Rightarrow y-1 = -2, m = 1\ or, y = -1, m = 1$$
$$m = 1 \Rightarrow (x+1)(x+2) = 2m\ or, x^2 + 3x + 2 = 2\ or, x = -3, 0$$
$$solution = (x,y) = (-3,-1), (0,-1)$$

$$y-1+m = -3, y-1-m = -1 \Rightarrow y-1 = -2, m = -1\ or, y = 3, m = -1$$
$$m = -1 \Rightarrow (x+1)(x+2) = 2m\ or, x^2 + 3x + 2 = -2\ or, x^2 + 3x + 4 = 0$$
$$Discriminant = 9 - 16 < 0\ no\ solution\ and\ similar\ for\ the\ pair\ (3,1)$$
$$solution = (x,y) = (-3,3), (0,3), (-3,-1), (0,-1)$$

789. If $\lambda \in \mathbb{N}, \lambda \geq 2$ then solve for naturals:

$$\frac{1}{1+2} + \frac{1}{1+2+3} + \cdots + \frac{1}{1+2+3+\cdots+x} = \frac{2\lambda}{2\lambda+1}$$

Proposed by Marin Chirciu-Romania

Solution by Amin Hajiyev-Azerbaijan

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+\cdots+\frac{1}{1+2+3+\cdots+x}=\frac{4\lambda+1}{2\lambda+1}$$

$$\sum_{n=1}^{x}\frac{1}{\sum_{k=1}^{n}k}=\frac{4\lambda+1}{2\lambda+1}$$

$$Note:\rightarrow\begin{cases}1+2+3+\cdots+x=S\\x+(x-1)+(x-2)+\cdots+1=S\end{cases}$$

$$2S=\underbrace{(x+1)+(x+1)+(x+1)+\cdots+(x+1)}_{x}\rightarrow S=\frac{x(x+1)}{2}$$

$$\sum_{k=1}^{n}k=\frac{n(n+1)}{2}\rightarrow 2\sum_{n=1}^{x}\frac{1}{n(n+1)}=\frac{4\lambda+1}{2\lambda+1}$$

$$\underbrace{2\left(\sum_{n=1}^{x}\frac{1}{n}-\sum_{n=1}^{x}\frac{1}{n+1}\right)}_{Teleskopik\ sum}=\frac{4\lambda+1}{2\lambda+1}\rightarrow 2\left(1-\frac{1}{x+1}\right)=\frac{4\lambda+1}{2\lambda+1}$$

$$\frac{x-1}{x+1}=\frac{2\lambda}{2\lambda+1}\rightarrow\frac{x+1}{x-1}=1+\frac{1}{2\lambda}\rightarrow\frac{2}{x-1}=\frac{1}{2\lambda}$$

$$x=4\lambda+1$$

790. Find $x,y,z\in Z$ such that:

$$2x^2+3y^2=z^2-37$$
$$x-y+z=3$$

Proposed by Bui Hong Suc-Vietnam

Solution by Tapas Das-India

$$2x^2+3y^2=z^2-37\ or, 2x^2+3y^2\overset{x-y+z=3}{=}(3-x+y)^2-37$$
$$or,\quad 2x^2+3y^2=x^2+y^2+9-6x+6y-2xy-37$$
$$or, x^2+(2y+6)x+2y^2-6y+28=0\ (1)$$

$$Discriminant=D=(2y+6)^2-4(2y^2-6y+28)=4(17-(y-6)^2)$$
$$for\ integer\ solution\ (17-(y-6)^2)\geq 0\ and\ must\ be\ perfect\ square.$$
$$for\ this\ (y-6)^2=1,16\ so, y=2,5,7,10\ , z=3-x+y$$

$$for\ y=7:\ from(1)\ x^2+20x+84=0\ or, (x+6)(x+14)=0\ or, x=-6,-14$$
$$and\ z=3-x+y=3-(-6)+7=16\ and\ z=3-(-14)+7=24$$
$$solution=(x,y,z)=\{(-6,7,16),(-14,7,24)\}$$

$$for\ y=5: from(1)\ x^2+16x+48=0\ or, (x+4)(x+12)=0\ or, x=-4,-12$$
$$and\ z=3-x+y=3-(-4)+5=12\ and\ z=3-(-12)+5=20$$
$$solution=(x,y,z)=\{(-4,5,12),(-4,5,20)\}$$

$$for\ y = 2:\ from(1)\ x^2 + 10x + 24 = 0\ or, (x+6)(x+4) = 0\ or, x = -6, -4$$
$$and\ z = 3 - x + y = 3 - (-4) + 2 = 9\ and\ z = 3 - (-6) + 2 = 11$$
$$solution = (x, y, z) = \{(-6, 2, 9), (-4, 2, 11)\}$$

$$for\ y = 10\ from(1)\ x^2 + 26x + 168 = 0\ or, (x+12)(x+14) = 0\ or, x = -14, -12$$
$$and\ z = 3 - x + y = 3 - (-14) + 10 = 27\ and\ z = 3 - (-12) + 10 = 25$$
$$solution = (x, y, z) = \{(-14, 10, 10), (-12, 10, 25)\}$$

791.

$$(1 + x + x^2 + x^3 + \cdots + x^{100})^3 = 1 + \cdots + kx^{100} + \cdots + x^{300}$$
$$k = ?$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$f(x) = \Big(\underbrace{1 + x + x^2 + \cdots + x^{100}}_{101}\Big)$$
$$g(x) = \Big(\underbrace{1 + x + x^2 + x^3 + \cdots + x^{100} + \cdots}_{\infty}\Big)$$
$$f(x)(coefficient\ x^{100}) = g(x)(coefficient\ x^{100})$$
$$g(x) = (1 + x + x^2 + \cdots)^3 = \left(\sum_{n=0}^{\infty} x^n\right)^3 = \left(\frac{1}{1-x}\right)^3$$
$$Note:\ \ p(x) = (1 + x + x^2 + \cdots)^n = \left(\frac{1}{1-x}\right)^n = \sum_{m=0}^{\infty} \binom{m+n-1}{m} x^m$$
$$g(x) = \sum_{m=0}^{\infty} \binom{m-1+3}{m} x^m = \sum_{m=0}^{\infty} \binom{m+2}{m} x^m$$
$$m = 100 \to\ g(x)(coefficient\ x^{100}) = x^{100}\binom{102}{100} = \frac{102!}{100!\,2!}x^{100} = 5151x^{100}$$
$$k = 5151$$

792.

$$If\ x, y \in Z\ \ solve: (x+1)^3 + (y-1)^3 + 1 = (x+y+1)^2$$

Proposed by Sakthi Vel-India

Solution by Amin Hajiyev-Azerbaijan

$$(x+1)^3 + (y-1)^3 + 1 = (x+y+1)^2$$
$$\begin{cases} x + 1 = a \\ y - 1 = b \end{cases} \to (x+1)^3 + (y-1)^3 + 1 = (x + 1 + y - 1 + 1)^2$$
$$a^3 + b^3 + 1 = (a+b+1)^2$$

$$(a+b)(a^2-ab+b^2)+1=a^2+b^2+2ab+2a+2b+1$$
$$(a+b)(a^2-ab+b^2)=(a+b)^2+2(a+b)$$
$$(a+b)(a^2-ab+b^2-a-b-2)=0\to a+b=0$$
$$2a^2-2ab+2b^2-2a-2b-4=0$$
$$a^2-2ab+b^2+a^2-2a+1+b^2-2b+1-6=0$$
$$(a-b)^2+(a-1)^2+(b-1)^2=6=1^2+1^2+2^2$$
$$a-b=\pm1, a-1=\pm1, b-1=\pm2\to a,b\ (2;3)(0;-1)$$
$$a-b=\pm2, a-1=\pm1, b-1=\pm1\to (2;0)(0;2)$$
$$a-b=\pm1, a-1=\pm2, b-1=\pm1\to\ a,b\ (3;2)(-1;0)$$
$$a,b\to(2;3)(0;-1)(2;0)(0;2)(3;2)(-1;0)$$
$$\begin{cases}a=x+1\\ b=y-1\end{cases}\to\ x,y\in\{(2;3),(-2;1),(1;4),(-1;0),(1;1),(-1;3)\}$$
$$a+b=0\to\ x+y=0\,, x=-y$$
$$General\ solution: x=-y,\qquad where\ y\in Z$$

793. **Solve for integers:**

$$\lfloor\sqrt[3]{1}\rfloor+\lfloor\sqrt[3]{2}\rfloor+\lfloor\sqrt[3]{3}\rfloor+\cdots+\left\lfloor\sqrt[3]{x^3-1}\right\rfloor=400$$
$$where\ \lfloor*\rfloor-\ floor\ function$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\lfloor\sqrt[3]{1}\rfloor+\lfloor\sqrt[3]{2}\rfloor+\lfloor\sqrt[3]{3}\rfloor+\cdots+\left\lfloor\sqrt[3]{x^3-1}\right\rfloor=400\to\sum_{k=1}^{x^3-1}\lfloor\sqrt[3]{k}\rfloor=400$$
$$Floor\ function\ \lfloor\sqrt[3]{k}\rfloor=n\to\ n\le\sqrt[3]{k}<n+1\to\ n^3\le k<(n+1)^3$$
$$k\in[n^3;(n+1)^3)\to\ k\in[n^3;(n+1)^3]$$

The number of integers in this interval $P_n=(n+1)^3-n^3=3n^2+3n+1$

$\sqrt[3]{x^3-1}<x\to\left\lfloor\sqrt[3]{x^3-1}\right\rfloor=x-1$. **Thus,** n **ranges from 1 to** $x-1$

$$\sum_{n=1}^{x-1}n\cdot P_n=400\to\sum_{n=1}^{x-1}n\cdot(3n^2+3n+1)=400$$
$$3\sum_{n=1}^{x-1}n^3+3\sum_{n=1}^{x-1}n^2+\sum_{n=1}^{x-1}n=400$$
$$3\cdot\frac{(x-1)^2x^2}{4}+3\cdot\frac{x(x-1)(2x-1)}{6}+\frac{x(x-1)}{2}=400$$
$$(x-1)\left(\frac{3x^3-3x^2}{4}+\frac{2x^2-x}{2}+\frac{x}{2}\right)=400\to(x-1)(3x^3+x^2)=1600$$
$$3x^4-2x^3-x^2-1600=0$$
$$3x^4-15x^3+13x^3-65x^2+64x^2-1600=0$$
$$3x^3(x-5)+13x^2(x-5)+64(x-5)(x+5)=0$$

$$(x-5)(3x^3+13x^2+64x+320)=0 \rightarrow \boxed{x_1=5}$$

$$3x^3+13x^2+64x+320 \neq 0 \text{ as } x \in N, \quad \text{There is no other natural solution.}$$

794. If $\lambda \in \mathbb{N}^*$ fixed then solve for natural numbers:

$$\frac{(x+y+z)^5-x^5-y^5-z^5}{(x+y+z)^3-x^3-y^3-z^3}=10\lambda^2$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$(x+y+z)^5=x^5+y^5+z^5+5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)$$

$$(x+y+z)^5-x^5-y^5-z^5=$$

$$=5(x+y)(y+z)(z+x)(x^2+y^2+z^2+xy+yz+zx)(1)$$

$$(x+y+z)^3=x^3+y^3+z^3+3(x+y)(y+z)(z+x)$$

$$(x+y+z)^3-x^3-y^3-z^3=3(x+y)(y+z)(z+x)\ (2)$$

$$\frac{(x+y+z)^5-x^5-y^5-z^5}{(x+y+z)^3-x^3-y^3-z^3}=10\lambda^2$$

$$\frac{5}{3}(x^2+y^2+z^2+xy+yz+zx) \overset{(1)\&(2)}{=} 10\lambda^2(3)$$

The equation is symmetric in x,y,z and' homogeneus of degree 2
hence we examine symmetric case $x=y=z=\ t(say)$
From (3)we get $6t^2=6\lambda^2$ or $t=\lambda$
Solution $x=y=z=\lambda$

795. ***Solve the equation:***

$$\lfloor x\rfloor\cdot x+\lfloor x\rfloor\cdot\{x\}+\{x\}\cdot x=2000$$

$$\lfloor * \rfloor - \text{ floor function}, \{*\} - \text{ fractional part function}$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$x=\lfloor x\rfloor+\{x\}=n+m \begin{cases} \lfloor x\rfloor=n \\ \{x\}=m,\ \ 0\le m<1 \end{cases}$$

$$\lfloor x\rfloor\cdot x+\lfloor x\rfloor\cdot\{x\}+\{x\}\cdot x=2000$$

$$\rightarrow\ \ n\cdot(n+m)+m\cdot n+m\cdot(n+m)=2000$$

$$(n+m)^2+m\cdot n=2000 \rightarrow\ n^2+m^2+3mn=2000$$

$$0 \le m < 1 \to m = 0 \quad n^2 = 2000 \quad n = \sqrt{2000} = 20\sqrt{5}$$
$$n \le \lfloor 20\sqrt{5} \rfloor = 44$$
$$m = 1 \to n^2 + 3n - 1999 = 0$$
$$n = \frac{-3 + \sqrt{9 + 7996}}{2} = \frac{\sqrt{8005}}{2} - 1,5 \approx \frac{89,5 - 3}{2} \approx 43.25$$
$$n \ge 44 \to 44 \le n \le 44 \quad n = 44$$
$$m^2 + 132m + 1936 - 2000 = 0 \to m^2 + 132m - 64 = 0$$
$$m = \frac{-132 + \sqrt{132^2 + 256}}{2} = \frac{\sqrt{17680} - 132}{2} = \frac{4\sqrt{1105}}{2} - 66 = 2\sqrt{1105} - 66$$
$$x = \lfloor x \rfloor + \{x\} = n + m = 44 + 2\sqrt{1105} - 66 = 2\sqrt{1105} - 22$$
$$x = 2\sqrt{1105} - 22$$

796. ***Prove or disprove that:***

The roots of the equation $x^3 + ax^2 + bx + c = 0$ are in A.P.
if $4a^3 = 9ab + 27c$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$Let\ p - d, p, p + d\ are\ roots\ of\ the\ equation\ \ x^3 + ax^2 + bx + c = 0$$
$$then\ \ p - d + p + p + d = -a\ or, d = -\frac{a}{3}\ (1)$$
$$(p - d)p(p + d) = -c\ or, p(p^2 - d^2) = -c\ (2)$$
$$(p - d)p + p(p + d) + (p - d)(p + d) = b\ or, 3p^2 - d^2 = b\ (3)$$
$$from\ (1)\&\ (3) we\ get\ d^2 = 3\left(-\frac{a}{3}\right)^2 - b = \frac{a^2}{3} - b\ (4)$$
$$From\ (2)\&\ (4) we\ get, -\frac{a}{3}\left(\frac{a^2}{9} - \left(\frac{a^2}{3} - b\right)\right) = -c\ or, -\frac{a}{3}\left(b - \frac{2a^2}{9}\right) = -c$$
$$or\ \ 2a^3 = 9ab - 27c\ \ , so\ the\ given\ condition\ is\ not\ true.$$

797. ***If $x \in R$ then solve the equation:***

$$\lfloor x^2 - 2 \rfloor + 2\lfloor x \rfloor = \lfloor x \rfloor^2\ , \qquad \lfloor * \rfloor -\ floor\ function$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$We\ have \begin{cases} x \in R \\ k \in Z \end{cases} \Rightarrow \lfloor x+k \rfloor = \lfloor x \rfloor + k$$

$$RHS = \lfloor x^2 - 2 \rfloor = \lfloor x^2 \rfloor - 2$$

$$n \in Z \to \lfloor x \rfloor = n$$

$$n \le x < n+1 \Rightarrow n^2 \le x^2 < (n+1)^2$$

$$n^2 \le x^2 < n^2 + 2n + 1$$

$$n^2 \le x^2 \le n^2 + 2n$$

$$\lfloor x^2 \rfloor = 2 - 2\lfloor x \rfloor + \lfloor x \rfloor^2 \Rightarrow \lfloor x^2 \rfloor = 2 - 2n + n^2$$

$$n^2 - 2n + 2 \le x^2 < n^2 - 2n + 3$$

$$n^2 \le n^2 - 2n + 2 \le n^2 + 2n$$

$$n^2 \le n^2 - 2n + 2 < n^2 + 2n \Rightarrow \begin{cases} n \le 1 \\ n \ge \dfrac{1}{2} \end{cases} \to n \in Z$$

$$n = 1 \Rightarrow \lfloor x \rfloor = 1, \quad \lfloor x^2 \rfloor = 1$$

$$\begin{cases} n^2 \le x^2 < n^2 + 2n + 1 \\ n^2 - 2n + 2 \le x^2 < n^2 - 2n + 3 \end{cases} \Rightarrow \begin{cases} 1 \le x < 2 \\ 1 \le x < \sqrt{2} \end{cases} \quad x \in [1; \sqrt{2})$$

798.

$$\begin{cases} a = \dfrac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \\ b = \dfrac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \end{cases} \quad Express\ x\ and\ y\ in\ terms\ of\ a\ and\ b$$

Proposed by Mais Hasanov-Azerbaijan

Solution by Amin Hajiyev-Azerbaijan

$$\begin{cases} a = \dfrac{x - y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \\ b = \dfrac{y - x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \end{cases} \Rightarrow \begin{cases} a = \dfrac{x}{\sqrt{1 - x^2 + y^2}} - \dfrac{y\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \\ b = \dfrac{y}{\sqrt{1 - x^2 + y^2}} - \dfrac{x\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}} \end{cases}$$

$$k = \frac{1}{\sqrt{1 - x^2 + y^2}}, \quad m = \frac{\sqrt{x^2 - y^2}}{\sqrt{1 - x^2 + y^2}}$$

$$\begin{cases} kx - my = a \\ -mx + ky = b \end{cases} \Rightarrow \begin{pmatrix} k & -m \\ -m & k \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow A \cdot B = C$$

$$A = \begin{pmatrix} k & -m \\ -m & k \end{pmatrix} \to Det(A) = k^2 - m^2 = \frac{1}{1 - x^2 + y^2} - \frac{x^2 - y^2}{1 - x^2 + y^2} = 1 \ne 0$$

$$A^{-1} \cdot C = B \Rightarrow A^{-1} = \frac{1}{Det(A)} \begin{pmatrix} k & m \\ m & k \end{pmatrix} \Rightarrow \begin{pmatrix} k & m \\ m & k \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{cases} ka+mb=x \\ ma+kb=y \end{cases} \Rightarrow \ x^2-y^2=(ka+mb)^2-(ma+kb)^2$$
$$=a^2(k^2-m^2)-b^2(k^2-m^2)$$

$$x^2-y^2=Det(A)\cdot(a^2-b^2)=a^2-b^2$$

$$k=\frac{1}{\sqrt{1-x^2+y^2}}=\frac{1}{\sqrt{1-a^2+b^2}},\ \ m=\frac{\sqrt{x^2-y^2}}{\sqrt{1-x^2+y^2}}=\frac{\sqrt{a^2-b^2}}{\sqrt{1-a^2+b^2}}$$

$$Therefore\ \ \begin{cases} ka+mb=x \\ ma+kb=y \end{cases} \Rightarrow \begin{cases} x=\dfrac{a+b\sqrt{a^2-b^2}}{\sqrt{1-a^2+b^2}} \\ y=\dfrac{b+a\sqrt{a^2-b^2}}{\sqrt{1-a^2+b^2}} \end{cases}$$

799. Solve for real numbers:

$$\sqrt[5]{1+x}+\sqrt[5]{1-x}=2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

Denote:

$$a=\sqrt[5]{1+x},\qquad b=\sqrt[5]{1-x},\qquad S=a+b,\qquad P=ab$$
$$a+b=2\Longrightarrow S=2$$
$$a^5+b^5=1+x+1-x=2$$
$$(a+b)(a^4-a^3b+a^2b^2-ab^3+b^4)=2$$
$$2(a^4+b^4-ab(a^2+b^2)+a^2b^2)=2$$

$$a^4+b^4-ab(a^2+b^2)+a^2b^2=1$$

$$(a^2+b^2)^2-2a^2b^2-ab(a^2+b^2)+a^2b^2=1$$

$$(S^2-2P)^2-P(S^2-2P)-P^2=1$$

$$(2^2-2P)^2-P(2^2-2P)-P^2=1$$

$$16-16P+4P^2-4P+2P^2-P^2=1$$

$$5P^2-20P+15=0$$

$$P^2-4P+3=0\Longrightarrow(P-1)(P-3)=0$$

$$P=3,S=2\Longrightarrow a,b\notin\mathbb{R}$$

$$P=1,S=2\Longrightarrow a=b=1\Longrightarrow x=0$$

800. **Find all positive integers k such that $(k+2)^2+3^k$ is a perfect square.**

Proposed by Stewart Ming-Vietnam

Solution by Jenish Rijal-Nepal

$$Let\ (k+2)^2+3^k = n^2 \qquad (where\ n \in \mathbb{N})$$

$$\Rightarrow n^2-(k+2)^2 = 3^k \Rightarrow (n+k+2)(n-k-2) = 3^k$$

$$We\ know\ that\ since\ both\ (n+k+2)\ and\ (n-k-2)$$
$$are\ powers\ of\ 3, and\ are\ distinct.$$

$$Therefore\ they\ differ\ by\ at\ least\ a\ factor\ of\ 3.$$

$$\Rightarrow n+k+2 \geq 3(n-k-2) \Rightarrow 3n-n \leq k+2+3(k+2) \Rightarrow n \leq 2(k+2)$$
$$Now: (k+2)^2+3^k = n^2 \leq [2(k+2)]^2$$
$$\Rightarrow 3^k \leq 3(k+2)^2 \Rightarrow k \leq 4$$
$$and\ only\ k=2\ works.$$

$$\therefore k=2\ is\ the\ only\ solution!$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru