

NEW IDENTITIES AND INEQUALITIES IN TRIANGLE-(III)

By Bogdan Fuștei-Romania

Lemma1:

For $x \geq y$, real numbers and also t is a real number and $x, y, t > 0, x, y > t$ then:

$$\frac{x}{x-t} \leq \frac{y}{y-t}.$$

Proof :

$$x(y-t) \leq y(x-t) \rightarrow xy - xt \leq xy - yt \rightarrow xt \geq yt \rightarrow x \geq y$$

Lemma2:

x, y, t, r_1 -real numbers and $x, y, t, r_1 > 0$; $x > t, y > r_1$. If $\frac{x}{y} = \frac{t}{r_1}$ then:

$$\frac{x}{y} = \frac{x-t}{y-r_1}.$$

Proof:

$$x(y-r_1) = y(x-t) \rightarrow xy - xr_1 = xy - yt \rightarrow -xr_1 = -yt \rightarrow xr_1 = yt$$

ABC triangle with usual notations:

$$\left(\frac{g_a}{r}\right)^2 = \left(\frac{n_a}{r_a}\right)^2 + 4\frac{b+c}{a} \text{ (and analogous)[1]}$$

$$\frac{n_a}{r_a} = \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{r} \text{ (and analogous)[2]}$$

$$\frac{b+c}{a} = \frac{g_a^2 - (n_a - \sqrt{4r^2 + (b-c)^2})^2}{4r^2} \text{ (and analogous)(1)}$$

$$\frac{b+c}{a} = \frac{(g_a + n_a - \sqrt{4r^2 + (b-c)^2})(g_a - n_a + \sqrt{4r^2 + (b-c)^2})}{4r^2} \text{ (and analogous) (2)}$$

From $2S = ah_a = 2sr = (a+b+c)r \rightarrow \frac{h_a}{r} = 1 + \frac{b+c}{a} \rightarrow \frac{b+c}{a} = \frac{h_a-r}{r}$ (and analogous)

$\frac{h_a-r}{r} = \frac{(g_a + n_a - \sqrt{4r^2 + (b-c)^2})(g_a - n_a + \sqrt{4r^2 + (b-c)^2})}{4r^2}$ obtain:

$$\frac{h_a-r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{g_a + n_a - \sqrt{4r^2 + (b-c)^2}}{4r} \text{ (and analogous)(3)}$$

$$\frac{4(h_a-r)}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{g_a}{r} + \frac{n_a}{r_a} \text{ (and analogous)(4)}$$

ROMANIAN MATHEMATICAL MAGAZINE

From (4) after summation:

$$4 \sum \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{g_a + g_b + g_c}{r} + \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} \quad (5)$$

From $g_a^2 = (s-a)^2 + 2rh_a$ (and analogous)[1] $\rightarrow g_a > s-a$ (and analogous)

$$s-a = \sqrt{g_a^2 - 2rh_a} \rightarrow s-a = \sqrt{g_a - \sqrt{2rh_a}} \sqrt{g_a + \sqrt{2rh_a}}$$

From $r_a = \frac{s}{s-a}$ (and analogous); $S = sr \rightarrow \frac{r_a}{r} = \frac{s}{s-a}$ (and analogous) we obtain:

$$\frac{r_a}{r} = \frac{s}{\sqrt{g_a - \sqrt{2rh_a}} \sqrt{g_a + \sqrt{2rh_a}}} \quad (\text{and analogous})$$

We know that: $\frac{r_a}{r} = \frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}}$ (and analogous)[2], we obtain:

$$\frac{n_a \sqrt{g_a + \sqrt{2rh_a}}}{n_a - \sqrt{4r^2 + (b-c)^2}} = \frac{s}{\sqrt{g_a - \sqrt{2rh_a}}} \quad (\text{and analogous}) \quad (6)$$

From (6) after summation :

$$\sum \frac{n_a \sqrt{g_a + \sqrt{2rh_a}}}{n_a - \sqrt{4r^2 + (b-c)^2}} = s \sum \frac{1}{\sqrt{g_a - \sqrt{2rh_a}}} \quad (7)$$

From $s^2 = n_a^2 + 2h_a r_a$ (and analogous)[3] $\rightarrow 2h_a r_a = s^2 - n_a^2 = (s-n_a)(s+n_a)$

$$s - n_a = \frac{2h_a r_a}{s+n_a} \rightarrow \frac{2h_a}{s+n_a} = \frac{s-n_a}{r_a} \quad (\text{and analogous}) \rightarrow \frac{s}{r_a} = \frac{n_a}{r_a} + \frac{2h_a}{s+n_a}$$

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \rightarrow \frac{s}{r} = \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} + 2 \sum \frac{h_a}{s+n_a} \rightarrow \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} = \frac{s}{r} - 2 \sum \frac{h_a}{s+n_a} \quad \text{and (5) we}$$

obtain:

$$2 \sum \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \sum \frac{h_a}{s+n_a} = \frac{g_a + g_b + g_c + s}{2r} \quad (8)$$

$$\frac{r_a}{r} = \frac{s}{s-a} = \frac{s^2}{s(s-a)}; s(s-a) = r_b r_c \quad (\text{and analogous});$$

$s^2 = n_a^2 + 2h_a r_a$ (and analogous)

$$\frac{r_a}{r} = \frac{2s^2}{2r_b r_c} = \frac{n_b^2 + 2h_b r_b + n_c^2 + 2h_c r_c}{2r_b r_c} \geq \frac{2n_b n_c + 2h_b r_b + 2h_c r_c}{2r_b r_c} = \frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b}$$

$$\frac{r_a}{r} \geq \frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} \quad (\text{and analogous}) \quad (9)$$

ROMANIAN MATHEMATICAL MAGAZINE

From $\frac{r_a}{r} = \frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}}$ (and analogous) and (9):

$$\frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}} \geq \frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b} \text{ (and analogous)} \quad (10)$$

Is well-known that $r_a r_b r_c = Ss = s^2 r$ and (9) we obtain:

$$\frac{r_a r_b r_c}{r^3} = \left(\frac{s}{r}\right)^2 \geq \prod \left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b}\right) \quad (11)$$

From (8) and (11) :

$$2 \sum \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \sum \frac{h_a}{s + n_a} \geq \frac{g_a + g_b + g_c}{2r} + \frac{1}{2} \prod \left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b}\right)^{\frac{1}{2}} \quad (12)$$

We know that: $n_a \geq p_a \sqrt{\frac{l_a}{g_a}}$ (and analogous)[4] and using lemma1 :

$$\frac{r_a}{r} \leq \frac{p_a \sqrt{\frac{l_a}{g_a}}}{p_a \sqrt{\frac{l_a}{g_a} - \sqrt{4r^2 + (b-c)^2}}} \text{ (and analogous)} \quad (13)$$

From(13) :

$$\left(\frac{s}{r}\right)^2 \leq \sqrt{\frac{l_a l_b l_c}{g_a g_b g_c}} \prod \frac{p_a}{p_a \sqrt{\frac{l_a}{g_a} - \sqrt{4r^2 + (b-c)^2}}} \quad (14)$$

From(14) and (11):

$$\sqrt{\frac{l_a l_b l_c}{g_a g_b g_c}} \prod \frac{p_a}{p_a \sqrt{\frac{l_a}{g_a} - \sqrt{4r^2 + (b-c)^2}}} \geq \prod \left(\frac{n_b n_c}{r_b r_c} + \frac{h_b}{r_c} + \frac{h_c}{r_b}\right) \quad (15)$$

From (14) and (8):

$$\frac{g_a + g_b + g_c}{2r} + \frac{1}{2} \sqrt{\frac{l_a l_b l_c}{g_a g_b g_c}} \prod \left(\frac{p_a}{p_a \sqrt{\frac{l_a}{g_a} - \sqrt{4r^2 + (b-c)^2}}}\right)^{\frac{1}{2}} \geq 2 \sum \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \sum \frac{h_a}{s + n_a} \quad (16)$$

From $\frac{h_a - r}{r} = \frac{g_a^2 - (n_a - \sqrt{4r^2 + (b-c)^2})^2}{4r^2}$ (and analogous) we obtain:

$$g_a^2 = 4r(h_a - r) + \left(n_a - \sqrt{4r^2 + (b-c)^2}\right)^2 \text{ (and analogous)} \quad (17)$$

ROMANIAN MATHEMATICAL MAGAZINE

From (17) after summation:

$$g_a^2 + g_b^2 + g_c^2 = 4r(h_a + h_b + h_c - 3r) + \sum \left(n_a - \sqrt{4r^2 + (b-c)^2} \right)^2 \quad (18)$$

From C.B.S :

$$\sum \left(n_a - \sqrt{4r^2 + (b-c)^2} \right)^2 \geq \frac{1}{3} \left(n_a + n_a + n_a - \sum \sqrt{4r^2 + (b-c)^2} \right)^2 \text{ and (19):}$$

$$g_a^2 + g_b^2 + g_c^2 \geq 4r(h_a + h_a + h_a - 3r) + \frac{1}{3} \left(n_a + n_a + n_a - \sum \sqrt{4r^2 + (b-c)^2} \right)^2 \quad (20)$$

From (3) ,(4) and $g_a + n_a \geq 2m_a$ (and analogous)[5]:

$$\frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} \geq \frac{2m_a - \sqrt{4r^2 + (b-c)^2}}{4r} \text{ (and analogous)(21)}$$

$$\frac{g_a}{r} + \frac{n_a}{r_a} \geq \frac{2m_a - \sqrt{4r^2 + (b-c)^2}}{r} \text{ (and analogous)(22)}$$

From (21) and (22) after summation:

$$4 \sum \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} \geq \sum \frac{2m_a - \sqrt{4r^2 + (b-c)^2}}{r} \quad (23)$$

$$\frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} \geq \sum \frac{2m_a - g_a + \sqrt{4r^2 + (b-c)^2}}{r} \quad (24)$$

From (22) : $\frac{n_a}{r_a} \geq \frac{2m_a - g_a + \sqrt{4r^2 + (b-c)^2}}{r} \rightarrow \frac{r}{r_a} \geq \frac{2m_a - g_a + \sqrt{4r^2 + (b-c)^2}}{n_a}$ (and analogous);

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} \rightarrow 1 \geq \sum \frac{2m_a - g_a + \sqrt{4r^2 + (b-c)^2}}{n_a} \quad (25)$$

From $g_a^2 = (s-a)^2 + 2rh_a$ (and analogous) and (17) we obtain:

$$(s-a)^2 = \left(n_a - \sqrt{4r^2 + (b-c)^2} \right)^2 + 2r(h_a - 2r) \text{ (and analogous) (26)}$$

From (26) : $2r(h_a - 2r) = (s-a + n_a - \sqrt{4r^2 + (b-c)^2})(s-a - n_a + \sqrt{4r^2 + (b-c)^2})$

$$\frac{2(h_a - 2r)}{s-a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{s-a + n_a - \sqrt{4r^2 + (b-c)^2}}{r} \quad (*)$$

$$\frac{n_a}{r_a} = \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{r} \text{ (and analogous) } \rightarrow$$

$$\frac{2(h_a - 2r)}{s-a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{s-a}{r} + \frac{n_a}{r_a} \text{ (and analogous) (27)}$$

From (27) after summation :

$$2 \sum \frac{h_a - 2r}{s - a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{s}{r} + \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} \quad (28)$$

From (28) and $\frac{s}{r} = \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} + 2 \sum \frac{h_a}{s+n_a}$ we obtain:

$$\sum \frac{h_a - 2r}{s - a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{n_a}{r_a} + \frac{n_b}{r_b} + \frac{n_c}{r_c} + \sum \frac{h_a}{s+n_a} \quad (29)$$

From (*) $\rightarrow \frac{2r}{s - a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{s - a + n_a - \sqrt{4r^2 + (b-c)^2}}{h_a - 2r} = \frac{s - a}{h_a - 2r} + \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{h_a - 2r}$

From $\frac{s}{h_a} = \frac{a}{2r}$ (and analogous) and lemma2 : $\frac{s}{h_a} = \frac{a}{2r} = \frac{s-a}{h_a - 2r}$

From $\frac{n_a}{h_a} = \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{h_a - 2r}$ (and analogous) [2] and $\frac{s}{h_a} = \frac{a}{2r} = \frac{s-a}{h_a - 2r}$:

$$\frac{2r}{s - a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{a}{2r} + \frac{n_a}{h_a} = \frac{n_a + s}{h_a} \quad (\text{and analogous}) \quad (30)$$

From $s - n_a = \frac{2h_a r_a}{s+n_a} \rightarrow \frac{2r_a}{s+n_a} = \frac{s-n_a}{h_a} = \frac{a}{2r} - \frac{n_a}{h_a}$ and (30):

$$\frac{r}{s - a - n_a + \sqrt{4r^2 + (b-c)^2}} = \frac{n_a}{h_a} + \frac{r_a}{s+n_a} \quad (\text{and analogous}) \quad (31)$$

From $\frac{a}{2r} = \frac{n_a}{h_a} + \frac{2r_a}{s+n_a}$, $\frac{a}{2r} = \frac{s-a}{h_a - 2r}$ and $\frac{n_a}{h_a} = \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{h_a - 2r} \rightarrow \frac{s-a}{h_a - 2r} = \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{h_a - 2r} + \frac{2r_a}{s+n_a}$

$$\frac{2r_a}{s+n_a} = \frac{s - a - n_a + \sqrt{4r^2 + (b-c)^2}}{h_a - 2r} \quad (\text{and analogous}) \quad (32)$$

From (32) after summation :

$$2 \sum \frac{r_a}{s+n_a} = \sum \frac{s - a - n_a + \sqrt{4r^2 + (b-c)^2}}{h_a - 2r} \quad (33)$$

From (32) because $r_a > 0$ and $s + n_a > 0 \rightarrow \frac{s - a - n_a + \sqrt{4r^2 + (b-c)^2}}{h_a - 2r} > 0$ and because $h_a - 2r > 0$

we obtain: $s - a - n_a + \sqrt{4r^2 + (b-c)^2} > 0$

$$s + \sqrt{4r^2 + (b-c)^2} > a + n_a \quad (\text{and analogous}) \quad (34)$$

From (34) : $s - a > n_a - \sqrt{4r^2 + (b-c)^2} \rightarrow \frac{s-a}{r} > \frac{n_a - \sqrt{4r^2 + (b-c)^2}}{r}$ and $\frac{r_a}{r} = \frac{n_a}{n_a - \sqrt{4r^2 + (b-c)^2}}$

(and analogous) we obtain :

$$\frac{s-a}{r} > \frac{n_a}{r_a} \text{ (and analogous) (35)}$$

From (27): $\frac{2(h_a-2r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s-a}{r} + \frac{n_a}{r_a}$ and (35) we obtain:

$$\frac{s-a}{r} > \frac{h_a-2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} > \frac{n_a}{r_a} \text{ (and analogous)(36)}$$

From $\frac{r_a}{r} = \frac{s}{s-a} \rightarrow \frac{s-a}{r} = \frac{s}{r_a}$ and (36) :

$$\frac{s}{r_a} > \frac{h_a-2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} > \frac{n_a}{r_a} \text{ (and analogous) (37)}$$

From (27) and $\frac{s-a}{r} = \frac{s}{r_a}$ (and analogous) :

$$\frac{2(h_a-2r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a+s}{r_a} \text{ (and analogous) (38)}$$

From (38) after summation :

$$2\sum \frac{h_a-2r}{n_a+s} = \sum \frac{s-a-n_a+\sqrt{4r^2+(b-c)^2}}{r_a} \text{ (39)}$$

From (38) after summation :

$$2(h_a + h_b + h_c - 6r) = \sum \frac{(n_a+s)(s-a-n_a+\sqrt{4r^2+(b-c)^2})}{r_a} \text{ (40)}$$

From (30) and (38) after summation:

$$\frac{2(h_a-r)}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = (n_a + s) \left(\frac{1}{r_a} + \frac{1}{h_a} \right) \text{ (and analogous)(41)}$$

From (30) : $\frac{2r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{a}{2r} + \frac{n_a}{h_a} = \frac{n_a+s}{h_a}$ (and analogous) we obtain :

$$2\sum \frac{r}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{s}{r} + \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \text{ (42)}$$

$\frac{h_a}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a+s}{2r}$ and after summation :

$$\sum \frac{h_a}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a+n_b+n_c+3s}{2r} \text{ (43)}$$

From (8) and (43):

$$2\sum \frac{h_a-r}{g_a-n_a+\sqrt{4r^2+(b-c)^2}} + \sum \frac{h_a}{s+n_a} + \sum \frac{h_a}{s-a-n_a+\sqrt{4r^2+(b-c)^2}} = \frac{n_a+n_b+n_c+g_a+g_b+g_c+4s}{2r} \text{ (44)}$$

From (44) and $g_a + n_a \geq 2m_a$ (and analogous) , we obtain:

ROMANIAN MATHEMATICAL MAGAZINE

$$2\sum \frac{h_a - r}{g_a - n_a + \sqrt{4r^2 + (b-c)^2}} + \sum \frac{h_a}{s + n_a} + \sum \frac{h_a}{s - a - n_a + \sqrt{4r^2 + (b-c)^2}} \geq \frac{2s + m_a + m_b + m_c}{r} \quad (45)$$

From $R \geq 2r$ (Euler) and (42):

$$R \sum \frac{1}{s - a - n_a + \sqrt{4r^2 + (b-c)^2}} \geq \frac{s}{r} + \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} \quad (46)$$

References:

- [1]. Bogdan Fuştei-NAGEL'S AND GERGONNE'S CEVIANS - APPLICATIONS AND RESULTS
<http://www.ssmrmh.ro>
- [2]. Bogdan Fuştei-NAGEL'S CEVIANS REVISITED <http://www.ssmrmh.ro>
- [3]. Bogdan Fuştei-ABOUT NAGEL AND GERGONNE'S CEVIANS (II) <http://www.ssmrmh.ro>
- [4]. Bogdan Fuştei-CONNECTIONS BETWEEN FAMOUS CEVIANS <http://www.ssmrmh.ro>
- [5]. Bogdan Fuştei-ABOUT NAGEL AND GERGONNE'S CEVIANS (III) <http://www.ssmrmh.ro>