A refinement and a generalization of the inequality $a^2+b^2+c^2 \ge ab+bc+ca$

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In this short paper, a refinement of a widely used inequality is established and demonstrated. Also, a sui-generis generalization of this refinement is demonstrated.

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The following elementary inequality is well-known and very frequently used ,

$$a^2+b^2+c^2 \ge ab+bc+ca \ , \qquad (1)$$

where a , b , c are arbitrary real numbers . It is equivalent to the obvious inequality, $(a-b)^2+(b-c)^2+(c-a)^2 \ge 0$. Equality is therefore achieved when $a=b=c$.

In [1] the following double inequality is presented as the proposed problem :

1. *Proposition* (*A refinement of the inequality* **(1)**)

If a, b, c are real numbers , then :

$$a^2+b^2+c^2 \ \ge \ |a|\cdot\sqrt{\frac{b^2+c^2}{2}} + |b|\cdot\sqrt{\frac{c^2+a^2}{2}} + |c|\cdot\sqrt{\frac{a^2+b^2}{2}} \ \ge \ ab+bc+ca \ . \qquad (2)$$

Proof

For the inequality on the left side. with the *CBS inequality*, we have :

$$\left(a^2+b^2+c^2\right)^2 = \left(|a|^2+|b|^2+|c|^2\right)\cdot\left(\frac{b^2+c^2}{2}+\frac{c^2+a^2}{2}+\frac{a^2+b^2}{2}\right) \overset{(CBS)}{\ge}$$

$$\overset{(CBS)}{\ge} \left(|a|\cdot\sqrt{\frac{b^2+c^2}{2}} + |b|\cdot\sqrt{\frac{c^2+a^2}{2}} + |c|\cdot\sqrt{\frac{a^2+b^2}{2}} \right)^2 \ ,$$

hence ,

$$a^2+b^2+c^2 \ge |a|\cdot\sqrt{\frac{b^2+c^2}{2}} + |b|\cdot\sqrt{\frac{c^2+a^2}{2}} + |c|\cdot\sqrt{\frac{a^2+b^2}{2}} \ ,$$

with equality iff :

$$\frac{|a|}{\sqrt{\frac{b^2+c^2}{2}}} = \frac{|b|}{\sqrt{\frac{c^2+a^2}{2}}} = \frac{|c|}{\sqrt{\frac{a^2+b^2}{2}}} \Leftrightarrow \frac{a^2}{\frac{b^2+c^2}{2}} = \frac{b^2}{\frac{c^2+a^2}{2}} = \frac{c^2}{\frac{a^2+b^2}{2}} \Leftrightarrow$$

$$\Leftrightarrow \frac{a^2}{b^2+c^2} = \frac{b^2}{c^2+a^2} = \frac{c^2}{a^2+b^2} = \frac{a^2+b^2+c^2}{2\cdot(a^2+b^2+c^2)} = \frac{1}{2} \ .$$

mean that , $2a^2 = b^2+c^2 \Leftrightarrow 3a^2 = a^2+b^2+c^2$ and analogous , meaning :

$a^2 = b^2 = c^2 \Leftrightarrow |a| = |b| = |c|$.

For the inequality on the right side, with the *QM-AM inequality* and then with an inequality of the moduli, we have successively :

$$|a|\cdot\sqrt{\frac{b^2+c^2}{2}}+|b|\cdot\sqrt{\frac{c^2+a^2}{2}}+|c|\cdot\sqrt{\frac{a^2+b^2}{2}}=$$

$$=|a|\cdot\sqrt{\frac{|b|^2+|c|^2}{2}}+|b|\cdot\sqrt{\frac{|c|^2+|a|^2}{2}}+|c|\cdot\sqrt{\frac{|a|^2+|b|^2}{2}}\overset{QM\text{-}AM\ inequality}{\geq}$$

$$\overset{QM\text{-}AM\ inequality}{\geq}|a|\cdot\frac{|b|+|c|}{2}+|b|\cdot\frac{|c|+|a|}{2}+|c|\cdot\frac{|a|+|b|}{2}=$$

$$=|a||b|+|b||c|+|c||a|\overset{|x|\geq x}{\geq}ab+bc+ca\ .$$

Equality occurs when $|a|=|b|=|c|$.

A natural generalization of the refined inequality (2) is described in the following,

2. Proposition (*A refinement of the inequality* **(2)**)

If $a_1, a_2, \ldots, a_n$ are real numbers , then holds the following inequalities :

$$a_1^2+a_2^2+\ldots+a_n^2\geq$$

$$\geq|a_1|\cdot\sqrt{\frac{a_2^2+a_3^2\ldots+a_n^2}{n-1}}+|a_2|\cdot\sqrt{\frac{a_1^2+a_3^2+\ldots+a_n^2}{n-1}}+\ldots+|a_n|\cdot\sqrt{\frac{a_1^2+a_2^2+\ldots+a_{n-1}^2}{n-1}}\geq \tag{3}$$

$$\geq\frac{2}{n-1}\cdot\left[(a_1a_2+a_1a_3+\ldots+a_1a_n)+(a_2a_3+\ldots+a_2a_n)+\ldots+(a_{n-2}a_{n-1}+a_{n-2}a_n)+a_{n-1}a_n\right]\cdot$$

Proof

For the first inequality, with the ***CBS inequality*** , we have :

$$a_1^2+a_2^2+\ldots+a_n^2=\sqrt{a_1^2+a_2^2+\ldots+a_n^2}\cdot\sqrt{a_1^2+a_2^2+\ldots+a_n^2}=$$

$$=\sqrt{a_1^2+a_2^2+\ldots+a_n^2}\cdot\sqrt{\frac{a_2^2+a_3^2\ldots+a_n^2}{n-1}+\frac{a_1^2+a_3^2+\ldots+a_n^2}{n-1}+\ldots+\frac{a_1^2+a_2^2+\ldots+a_{n-1}^2}{n-1}}\overset{(CBS)}{\geq}$$

$$\overset{(CBS)}{\geq}|a_1|\cdot\sqrt{\frac{a_2^2+a_3^2\ldots+a_n^2}{n-1}}+|a_2|\cdot\sqrt{\frac{a_1^2+a_3^2+\ldots+a_n^2}{n-1}}+\ldots+|a_n|\cdot\sqrt{\frac{a_1^2+a_2^2+\ldots+a_{n-1}^2}{n-1}}\ .$$

Equality is obtained if and only if :

$$|a_1|\Big/\sqrt{\frac{a_2^2+a_3^2\ldots+a_n^2}{n-1}}=|a_2|\Big/\sqrt{\frac{a_1^2+a_3^2+\ldots+a_n^2}{n-1}}=\ldots=|a_n|\Big/\sqrt{\frac{a_1^2+a_2^2+\ldots+a_{n-1}^2}{n-1}}\Leftrightarrow$$

$$\Leftrightarrow a_1^2\Big/\frac{a_2^2+a_3^2\ldots+a_n^2}{n-1}=a_2^2\Big/\frac{a_1^2+a_3^2+\ldots+a_n^2}{n-1}=\ldots=a_n^2\Big/\frac{a_1^2+a_2^2+\ldots+a_{n-1}^2}{n-1}\Leftrightarrow$$

$$\Leftrightarrow\frac{a_1^2}{a_2^2+a_3^2\ldots+a_n^2}=\frac{a_2^2}{a_1^2+a_3^2+\ldots+a_n^2}=\ldots=\frac{a_n^2}{a_1^2+a_2^2+\ldots+a_{n-1}^2}=\frac{a_1^2+a_2^2+\ldots+a_n^2}{(n-1)\left(a_1^2+a_2^2+\ldots+a_n^2\right)}=\frac{1}{n-1}\ ,$$

from which it follows, $(n-1)\cdot a_1^2=a_2^2+a_3^2\ldots+a_n^2\Rightarrow n\cdot a_1^2=a_2^2+a_3^2\ldots+a_n^2$, and analogous , meaning : $a_1^2=a_2^2=\ldots=a_n^2\Leftrightarrow|a_1|=|a_2|=\ldots=|a_n|\cdot$

For the second inequality , with the *QM-AM inequality* and then with an inequality of the moduli , we have successively :

$$|a_1|\cdot\sqrt{\frac{a_2^2+a_3^2+\ldots+a_n^2}{n-1}}+|a_2|\cdot\sqrt{\frac{a_1^2+a_3^2+\ldots+a_n^2}{n-1}}+\ldots+|a_n|\cdot\sqrt{\frac{a_1^2+a_2^2+\ldots+a_{n-1}^2}{n-1}}=$$

$$=|a_1|\cdot\sqrt{\frac{|a_2^2|+|a_3^2|+\ldots+|a_n^2|}{n-1}}+|a_2|\cdot\sqrt{\frac{|a_1^2|+|a_3^2|+\ldots+|a_n^2|}{n-1}}+\ldots+|a_n|\cdot\sqrt{\frac{|a_1^2|+|a_2^2|+\ldots+|a_{n-1}^2|}{n-1}}\overset{QM\text{-}AM}{\geq}$$

$$\overset{QM\text{-}AM}{\geq}|a_1|\cdot\frac{|a_2|+|a_3|+\ldots+|a_n|}{n-1}+|a_2|\cdot\frac{|a_1|+|a_3|+\ldots+|a_n|}{n-1}+\ldots+|a_n|\cdot\frac{|a_1|+|a_2|+\ldots+|a_{n-1}|}{n-1}\overset{|x|\geq x}{\geq}$$

$$\overset{|x|\geq x}{\geq}a_1\cdot\frac{a_2+a_3+\ldots+a_n}{n-1}+a_2\cdot\frac{a_1+a_3+\ldots+a_n}{n-1}+\ldots+a_n\cdot\frac{a_1+a_2+\ldots+a_{n-1}}{n-1}=$$

$$=\frac{2}{n-1}\cdot\left[(a_1a_2+a_1a_3+\ldots+a_1a_n)+(a_2a_3+\ldots+a_2a_n)+\ldots+(a_{n-2}a_{n-1}+a_{n-2}a_n)+a_{n-1}a_n\right]\cdot$$

Equality occurs if $|a_1|=|a_2|=\ldots=|a_n|\cdot$

For $n=3$, the inequality from *Proposition* 1 is obtained **.**

References

[1.] **Mărghidanu Dorin** , ***Proposed problem*** , in « **Mathematical Inequalities** » , **24 April , 2026** , ***on line*** :
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