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Prove that:

$$\frac{1}{\sin^2\frac{4\pi}{9}}-\frac{1}{\sin^2\frac{\pi}{9}}=\frac{8\sqrt{3}}{3}\left(-2\sin\frac{4\pi}{9}+\sin\frac{\pi}{9}\right)$$

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$$Let\frac{\pi}{9}=A\ then\ \ 9A=\pi, sin5A=sin(9A-4A)=sin(\pi-4A)=sin4A\ (1)$$

$$L.H.S=\frac{1}{\sin^2\frac{4\pi}{9}}-\frac{1}{\sin^2\frac{\pi}{9}}=\frac{1}{\sin^2 4A}-\frac{1}{\sin^2 A}=\frac{\sin^2 A-\sin^2 4A}{.\sin^2 4A\sin^2 A}=-\frac{sin5A.sin3A}{.\sin^2 4A\sin^2 A}$$

$$\overset{(1)}{=}-\frac{.sin3A}{sin4A\sin^2 A}\overset{\frac{\pi}{9}=A}{=}-\frac{.\frac{\sqrt{3}}{2}}{sin4A\sin^2 A}=\frac{\sqrt{3}}{2sin4A\sin^2 A}$$

$$R.H.S=\frac{8\sqrt{3}}{3}\left(-2\sin\frac{4\pi}{9}+\sin\frac{\pi}{9}\right)=\frac{8\sqrt{3}}{3}(-2\sin 4A+\sin A)$$

$$We\ need\ to\ show:$$

$$\frac{\sqrt{3}}{2sin4A\sin^2 A}=\frac{8\sqrt{3}}{3}(-2\sin 4A+\sin A)\ \ or$$

$$-3=16\sin^3 A\,sin4A-32\sin^2 A\sin^2 4A\ \ (1)$$

$$16\sin^3 A\,sin4A=4(3sinA-sin3A)sin4A\overset{\frac{\pi}{9}=A}{=}4\left(3sinA-\frac{\sqrt{3}}{2}\right)sin4A=$$

$$=12sinAsin4A-2\sqrt{3}\,sin4A$$

$$=6(cos3A-cos5A)-2\sqrt{3}\,sin4A\overset{\frac{\pi}{9}=A}{=}3+6cos4A-2\sqrt{3}\,sin4A$$

$$32\sin^2 A\sin^2 4A=8(2sinAsin4A)^2=8(cos3A-cos5A)^2\overset{\frac{\pi}{9}=A}{=}8\left(\frac{1}{2}-cos5A\right)^2=$$

$$=2-8cos5A+8\cos^2 5A\overset{\frac{\pi}{9}=A}{=}2+8cos4A+8\cos^2 4A\ =6+8cos5A+4cos8A$$

$$16\sin^3 A\,sin4A-32\sin^2 A\sin^2 4A$$

$$=\left(3+6cos4A-2\sqrt{3}\,sin4A\right)-(6+8cos4A+4cos8A)$$

$$=-3-2cos4A-2\sqrt{3}sin4A-4cos8A$$

$$=-3-4\left(\frac{1}{2}cos4A+\frac{\sqrt{3}}{2}sin4A\right)-4cos8A$$

$$\overset{\frac{\pi}{9}=A}{=}-3-4\cos\left(\frac{4\pi}{9}-\frac{\pi}{3}\right)-4\cos\left(\frac{8\pi}{9}\right)=-3-4\cos\left(\frac{\pi}{9}\right)-4\cos\left(\pi-\frac{\pi}{9}\right)$$

$$=-3-4\cos\left(\frac{\pi}{9}\right)+4\cos\left(\frac{\pi}{9}\right)=-3$$

$$so\ relation\ (1)\ \ is\ \ true$$