Selected Problems in Geometry and Inequalities

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Abstract

This note presents a short collection of original problems in geometry and inequalities. The solutions use elementary but effective tools such as orthogonal coordinates, barycentric representations, the triangle inequality, Schur's inequality, Cauchy's inequality, and Jensen's inequality. The aim is to give clean statements and rigorous proofs suitable for olympiad-style mathematics.

1 A Geometry Problem in an Orthodiagonal Quadrilateral

Problem 1. *Let $ABCD$ be an orthodiagonal quadrilateral and let*

$$AC \cap BD = \{O\}.$$

Assume that

$$M \in \mathrm{Int}(\triangle AOB), \quad N \in \mathrm{Int}(\triangle BOC), \quad P \in \mathrm{Int}(\triangle COD), \quad Q \in \mathrm{Int}(\triangle DOA).$$

Let

$$\begin{aligned} MM_1 &\perp AO, & MM_2 &\perp BO, \\ NN_1 &\perp BO, & NN_2 &\perp CO, \\ PP_1 &\perp CO, & PP_2 &\perp DO, \\ QQ_1 &\perp DO, & QQ_2 &\perp AO. \end{aligned}$$

Prove that

$$\left|\overrightarrow{MN} + \overrightarrow{PQ}\right| \le M_1Q_2 + N_1M_2 + P_1N_2 + Q_1P_2.$$

Proof. Since $ABCD$ is orthodiagonal, the diagonals AC and BD are perpendicular. We take them as coordinate axes, with origin O. Denote by $\vec{r}_X$ the position vector of a point X.

For a point lying inside one of the four triangles determined by the two axes, its position vector can be expressed in terms of its perpendicular distances to the two axes. Thus, for $M \in \triangle AOB$, we have

$$\vec{r}_M = \frac{MM_2}{OA}\vec{r}_A + \frac{MM_1}{OB}\vec{r}_B.$$

Similarly,

$$\vec{r}_N = \frac{NN_2}{OB}\vec{r}_B + \frac{NN_1}{OC}\vec{r}_C,$$

$$\vec{r}_P = \frac{PP_2}{OC}\vec{r}_C + \frac{PP_1}{OD}\vec{r}_D,$$

and

$$\vec{r}_Q = \frac{QQ_1}{OA}\vec{r}_A + \frac{QQ_2}{OD}\vec{r}_D.$$

Therefore

$$\overrightarrow{MN} + \overrightarrow{PQ} = \vec{r}_N - \vec{r}_M + \vec{r}_Q - \vec{r}_P.$$

Using the expressions above, we get

$$\overrightarrow{MN} + \overrightarrow{PQ} = \frac{QQ_1 - MM_2}{OA}\vec{r}_A + \frac{NN_2 - MM_1}{OB}\vec{r}_B + \frac{NN_1 - PP_2}{OC}\vec{r}_C + \frac{QQ_2 - PP_1}{OD}\vec{r}_D.$$

Since

$$\left|\frac{\vec{r}_A}{OA}\right| = \left|\frac{\vec{r}_B}{OB}\right| = \left|\frac{\vec{r}_C}{OC}\right| = \left|\frac{\vec{r}_D}{OD}\right| = 1,$$

the triangle inequality gives

$$\left|\overrightarrow{MN} + \overrightarrow{PQ}\right| \le |QQ_1 - MM_2| + |NN_2 - MM_1| + |NN_1 - PP_2| + |QQ_2 - PP_1|.$$

The four absolute values are precisely the distances between the corresponding orthogonal projections:

$$|QQ_1 - MM_2| = M_1Q_2,$$

$$|NN_2 - MM_1| = N_1M_2,$$

$$|NN_1 - PP_2| = P_1N_2,$$

and

$$|QQ_2 - PP_1| = Q_1P_2.$$

Hence

$$\left|\overrightarrow{MN} + \overrightarrow{PQ}\right| \le M_1Q_2 + N_1M_2 + P_1N_2 + Q_1P_2.$$

□

2 A Cyclic Inequality

Problem 2. *Let $a, b, c > 0$ with*

$$a + b + c = 3.$$

Prove that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge \frac{5b + 2c - a}{9 - 3a} + \frac{5c + 2a - b}{9 - 3b} + \frac{5a + 2b - c}{9 - 3c}.$$

Proof. Since $a + b + c = 3$, we have

$$9 - 3a = 3(b + c), \quad 9 - 3b = 3(c + a), \quad 9 - 3c = 3(a + b).$$

Thus the right-hand side becomes

$$\sum_{\text{cyc}} \frac{5b + 2c - a}{3(b + c)}.$$

A direct simplification gives

$$\sum_{\text{cyc}} \frac{5b + 2c - a}{3(b + c)} = 2\sum_{\text{cyc}} \frac{1}{a + b} + \sum_{\text{cyc}} \frac{b - a}{b + c}.$$

By the AM–HM inequality,

$$\frac{1}{a} + \frac{1}{b} \ge \frac{4}{a + b}.$$

Adding the three analogous inequalities, we obtain

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \ge 2\left(\frac{1}{a + b} + \frac{1}{b + c} + \frac{1}{c + a}\right).$$

Therefore it remains to prove that

$$\sum_{\text{cyc}} \frac{b-a}{b+c} \le 0.$$

Multiplying by the positive number $(a+b)(b+c)(c+a)$, this is equivalent to

$$a^3 + b^3 + c^3 \ge a^2b + b^2c + c^2a.$$

This follows from Schur's inequality:

$$a^3 + b^3 + c^3 + 3abc \ge a^2b + a^2c + b^2a + b^2c + c^2a + c^2b.$$

Hence

$$a^3 + b^3 + c^3 - (a^2b + b^2c + c^2a) \ge a^2c + b^2a + c^2b - 3abc.$$

By AM–GM,

$$a^2c + b^2a + c^2b \ge 3abc.$$

Thus

$$a^3 + b^3 + c^3 \ge a^2b + b^2c + c^2a.$$

Consequently,

$$\sum_{\text{cyc}} \frac{b-a}{b+c} \le 0,$$

and the desired inequality follows.

Equality holds for

$$a = b = c = 1.$$

□

3 A Bound Involving a Parameter

Problem 3. *Let $t > 1$ and let $x_1, \dots, x_n$ be real numbers such that*

$$\frac{1}{t} \le x_1 \le x_2 \le \cdots \le x_n \le t$$

and

$$\sum_{k=1}^{n} x_k = 1.$$

Prove that

$$t \ge n$$

and

$$t + \frac{1}{t} \ge n + \sum_{k=1}^{n} x_k^2.$$

Proof. Since

$$x_k \ge \frac{1}{t}$$

for every k, summing over $k = 1, \dots, n$ gives

$$1 = \sum_{k=1}^{n} x_k \ge \frac{n}{t}.$$

Therefore

$$t \ge n.$$

Now, for each $k \in \{1, \ldots, n\}$, we have

$$\frac{1}{t} \leq x_k \leq t.$$

Hence

$$(t - x_k)\left(x_k - \frac{1}{t}\right) \geq 0.$$

Expanding, we get

$$tx_k - x_k^2 - 1 + \frac{x_k}{t} \geq 0,$$

or equivalently,

$$x_k\left(t + \frac{1}{t}\right) \geq x_k^2 + 1.$$

Summing from $k = 1$ to n, we obtain

$$\left(t + \frac{1}{t}\right)\sum_{k=1}^{n} x_k \geq \sum_{k=1}^{n} x_k^2 + n.$$

Since

$$\sum_{k=1}^{n} x_k = 1,$$

it follows that

$$t + \frac{1}{t} \geq n + \sum_{k=1}^{n} x_k^2.$$

□

4 A Four-Variable Quadratic Inequality

Problem 4. *Let* $a, b, c, d \in \mathbb{R}$ *with*

$$a + b + c + d = 0.$$

Prove that

$$5(a^2 + b^2 + c^2 + d^2) \geq 4(a + c)(b + d) + 4(bd - ac).$$

Proof. Since

$$a + b + c + d = 0,$$

we may write

$$d = -a - b - c.$$

The desired inequality is equivalent to

$$5(a^2 + b^2 + c^2 + d^2) - 4(a + c)(b + d) - 4(bd - ac) \geq 0.$$

Substituting $d = -a - b - c$, the left-hand side becomes

$$14a^2 + 14b^2 + 14c^2 + 14ab + 14bc + 22ac.$$

Let

$$u = a + c, \qquad v = a - c.$$

Then $a = (u + v)/2$ and $c = (u - v)/2$. Therefore

$$14a^2 + 14b^2 + 14c^2 + 14ab + 14bc + 22ac = \frac{1}{2}\left(28b^2 + 28bu + 25u^2 + 3v^2\right).$$

We rewrite this as

$$\frac{1}{2}\left(28\left(b+\frac{u}{2}\right)^2+18u^2+3v^2\right).$$

This quantity is nonnegative. Hence

$$5(a^2+b^2+c^2+d^2)\geq 4(a+c)(b+d)+4(bd-ac),$$

as required. ∎

5 A Jensen-Type Inequality

Problem 5. *Let $f : I \to \mathbb{R}$ be a convex function, and let*

$$0 \leq a,b,c \in I, \qquad a+b+c=1.$$

Assume also that

$$ab+bc+ca \in I$$

and

$$\frac{1-ab-bc-ca}{2} \in I.$$

Prove that

$$f(a)+f(b)+f(c) \geq 2f\left(\frac{1-ab-bc-ca}{2}\right)+f(ab+bc+ca).$$

Proof. Since $a,b,c \geq 0$ and

$$a+b+c=1,$$

the numbers

$$1+a, \quad 1+b, \quad 1+c$$

are positive and have sum 4. By Jensen's inequality,

$$(1+a)f(a)+(1+b)f(b)+(1+c)f(c) \geq 4f\left(\frac{(1+a)a+(1+b)b+(1+c)c}{4}\right).$$

Now

$$(1+a)a+(1+b)b+(1+c)c = a+b+c+a^2+b^2+c^2 = 1+a^2+b^2+c^2.$$

Since

$$(a+b+c)^2=1,$$

we have

$$a^2+b^2+c^2+2(ab+bc+ca)=1.$$

Thus

$$\frac{1+a^2+b^2+c^2}{4}=\frac{1-ab-bc-ca}{2}.$$

Therefore

$$(1+a)f(a)+(1+b)f(b)+(1+c)f(c) \geq 4f\left(\frac{1-ab-bc-ca}{2}\right).$$

Similarly, the numbers

$$1-a, \quad 1-b, \quad 1-c$$

are nonnegative and have sum 2. Applying Jensen's inequality again, we get

$$(1-a)f(a)+(1-b)f(b)+(1-c)f(c) \geq 2f\left(\frac{(1-a)a+(1-b)b+(1-c)c}{2}\right).$$

But

$$(1-a)a+(1-b)b+(1-c)c = a+b+c-(a^2+b^2+c^2) = 1-(a^2+b^2+c^2) = 2(ab+bc+ca).$$

Hence

$$(1-a)f(a)+(1-b)f(b)+(1-c)f(c) \geq 2f(ab+bc+ca).$$

Adding the two inequalities gives

$$2\big(f(a)+f(b)+f(c)\big) \geq 4f\left(\frac{1-ab-bc-ca}{2}\right)+2f(ab+bc+ca).$$

Dividing by 2, we conclude that

$$f(a)+f(b)+f(c) \geq 2f\left(\frac{1-ab-bc-ca}{2}\right)+f(ab+bc+ca).$$

□