RMM - Triangle Marathon 3801 - 3900

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ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

Available online
www.ssmrmh.ro

ISSN-L 2501-0099

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www.ssmrmh.ro

Proposed by

Nguyen Hung Cuong-Vietnam, Zaza Mzhavanadze-Georgia

Sarkhan Adgozalov-Georgia, Mohamed Amine Ben Ajiba-Morocco

Bogdan Fuștei-Romania, Marin Chirciu-Romania

Nguyen Van Canh-Vietnam, Ertan Yildirim-Turkiye

Neculai Stanciu-Romania, Vasile Mircea Popa-Romania

Daniel Sitaru-Romania, Kostantinos Geronikolas-Greece

Gheorghe Crăciun-Romania, Dang Ngoc Minh-Vietnam

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Solutions by

Daniel Sitaru-Romania, Soumava Chakraborty-India

Qurban Muellim-Azerbaijan, Mirsadix Muzefferov-Azerbaijan

Ertan Yildirim-Turkiye, Tapas Das-India

Chew Cheong-Malaysia, Amin Hajiyev-Azerbaijan

3801. In ΔABC the following relationship holds:

$$\frac{b}{r_a}+\frac{c}{r_b}+\frac{a}{r_c}\geq\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{b}{r_a}+\frac{c}{r_b}+\frac{a}{r_c}\geq\frac{b}{h_a}+\frac{c}{h_b}+\frac{a}{h_c}\Leftrightarrow\sum_{cyc}\frac{b}{r_a}\geq\sum_{cyc}\frac{b}{h_a}\Leftrightarrow$$

$$\sum_{cyc}\frac{b}{\frac{F}{s-a}}\geq\sum_{cyc}\frac{b}{\frac{2F}{a}}\Leftrightarrow 2\sum_{cyc}b(s-a)\geq\sum_{cyc}ab$$

$$2s\sum_{cyc}b-2\sum_{cyc}ab\geq\sum_{cyc}ab\Leftrightarrow 4s^2\geq 3\sum_{cyc}ab$$

$$4s^2\geq 3(s^2+r^2+4Rr)\Leftrightarrow s^2\geq 3r^2+12Rr\text{ (to prove)}$$

$$s^2\overset{GERRETSEN}{\geq}16Rr-5r^2\geq 3r^2+12Rr\Leftrightarrow 4Rr\geq 8r^2\Leftrightarrow R\geq 2r\text{(Euler)}$$

Equality holds for $a=b=c$.

3802. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\csc^4\frac{B}{2}\csc^4\frac{C}{2}\left(\csc^6\frac{B}{2}+\csc^6\frac{C}{2}\right)}{\csc^5\frac{B}{2}+\csc^5\frac{C}{2}}\geq 3\cdot 8^3$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\text{LHS}\overset{\text{Chebyshev}}{\geq}\sum_{cyc}\frac{\csc^4\frac{B}{2}\csc^4\frac{C}{2}\left(\csc^5\frac{B}{2}+\csc^5\frac{C}{2}\right)\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)}{2\left(\csc^5\frac{B}{2}+\csc^5\frac{C}{2}\right)}\overset{\text{AM-GM}}{\geq}$$

$$\frac{3}{2}\cdot\sqrt[3]{\left(\prod_{cyc}\csc^8\frac{A}{2}\right)\cdot\prod_{cyc}\left(\csc\frac{B}{2}+\csc\frac{C}{2}\right)}\overset{Cesaro}{\geq}\frac{3}{2}\cdot\sqrt[3]{\left(\prod_{cyc}\csc^8\frac{A}{2}\right).8\prod_{cyc}\csc\frac{A}{2}}$$

$$=3.\prod_{cyc}\csc^3\frac{A}{2}=3.\left(\frac{4R}{r}\right)^3\overset{Euler}{\geq}3.8^3$$

$$\therefore\sum_{cyc}\frac{\csc^4\frac{B}{2}\csc^4\frac{C}{2}\left(\csc^6\frac{B}{2}+\csc^6\frac{C}{2}\right)}{\csc^5\frac{B}{2}+\csc^5\frac{C}{2}}\geq 3.8^3$$

$\forall\,\Delta\,ABC,''=''$ iff $\Delta\,ABC$ is equilateral (QED)

3803. In any Δ ABC the following relationship holds :

$$m_a+m_b+m_c\leq g_a+g_b+g_c+2(R-2r)$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}g_ag_b\geq 4Rr+r^2+\frac{r(s^2+4Rr+r^2)}{R}$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba – 70; Relation (m); published at www.ssmrmh.ro)

$$\therefore\left(g_a+g_b+g_c+2(R-2r)\right)^2=$$

$$\sum_{cyc}g_a^2+2\sum_{cyc}g_ag_b+4(R-2r)^2+4(R-2r)\left(\sum_{cyc}g_a\right)\overset{Bogdan\ Fustei}{=}$$

$$\sum_{cyc}(s-a)^2+2r\sum_{cyc}h_a+2\sum_{cyc}g_ag_b+4(R-2r)^2+4(R-2r)\left(\sum_{cyc}g_a\right)\geq$$

$$s^2-8Rr-2r^2+\frac{r(s^2+4Rr+r^2)}{R}+2\left(4Rr+r^2+\frac{r(s^2+4Rr+r^2)}{R}\right)+$$

$$4(R-2r)^2+4(R-2r)\left(\frac{s^2+4Rr+r^2}{2R}\right)\overset{?}{\geq}4s^2-16Rr+5r^2$$

$$\Leftrightarrow 4R^3+8R^2r+9Rr^2-r^3\underset{(*)}{\overset{?}{\geq}}(R+r)s^2\text{ and indeed, }(R+r)s^2\overset{Gerretsen}{\leq}$$

$$(R+r)(4R^2+4Rr+3r^2)\overset{?}{\leq}4R^3+8R^2r+9Rr^2-r^3\Leftrightarrow 2r^2(R-2r)\overset{?}{\geq}0$$

$\rightarrow$ true via Euler $\Rightarrow(*)$ is true $\therefore g_a+g_b+g_c+2(R-2r)\geq\sqrt{4s^2-16Rr+5r^2}$

$$\overset{\text{Chu-Yang}}{\geq} m_a + m_b + m_c \Rightarrow m_a + m_b + m_c \leq g_a + g_b + g_c + 2(R - 2r)\ \forall\, \Delta\, ABC,$$
$$'' = ''\text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3804. In any Δ ABC the following relationship holds :

$$\frac{h_a - h_b}{b + c} + \frac{h_b - h_c}{c + a} + \frac{h_c - h_a}{a + b} \leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{h_a - h_b}{b + c} = \frac{2rs}{(b + c)(c + a)(a + b).4Rrs} \cdot \sum_{cyc}\left(c(b - a)\left(a^2 + \sum_{cyc} ab\right)\right)$$

$$= \frac{1}{(b + c)(c + a)(a + b).2R} \cdot \left(\begin{array}{c}\left(\sum_{cyc} ab\right)(c(b - a) + a(c - b) + b(a - c)) \\ +abc\sum_{cyc} a - \sum_{cyc} ca^3\end{array}\right)$$

$$= \frac{1}{(b + c)(c + a)(a + b).2R} \cdot \left(abc\sum_{cyc} a - abc.\sum_{cyc}\frac{a^2}{b}\right) \overset{\text{Bergstrom}}{\leq}$$

$$\frac{1}{(b + c)(c + a)(a + b).2R} \cdot \left(abc\sum_{cyc} a - abc.\frac{\left(\sum_{cyc} a\right)^2}{\sum_{cyc} a}\right) = 0$$

$$\therefore \frac{h_a - h_b}{b + c} + \frac{h_b - h_c}{c + a} + \frac{h_c - h_a}{a + b} \leq 0\ \ \forall\, \Delta\, ABC, '' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3805. In any Δ ABC the following relationship holds :

$$\frac{r_a - h_a}{b + c} + \frac{r_b - h_b}{c + a} + \frac{r_c - h_c}{a + b} \geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{r_a - h_a}{b + c} = \frac{1}{(b + c)(c + a)(a + b)} \cdot \sum_{cyc}\left((r_a - h_a)\left(a^2 + \sum_{cyc} ab\right)\right) =$$

$$\frac{1}{(b+c)(c+a)(a+b)}\cdot\left(\left(\sum_{cyc}ab\right)\left(4R+r-\sum_{cyc}h_a\right)+\sum_{cyc}a^2r_a-\sum_{cyc}\left(a^2\cdot\frac{2rs}{a}\right)\right)$$

$$\overset{\text{Chebyshev}}{\geq}\frac{1}{(b+c)(c+a)(a+b)}\cdot\left(\begin{array}{c}\left(\sum_{cyc}ab\right)\left(4R+r-\sum_{cyc}m_a\right)+\\ \frac{1}{3}\left(\sum_{cyc}a^2\right)(4R+r)-4s^2r\end{array}\right)$$

$$(\because \text{WLOG assuming } a\geq b\geq c\Rightarrow a^2\geq b^2\geq c^2 \text{ and } r_a\geq r_b\geq r_c)\overset{\text{Bager and Euler}}{\geq}$$

$$\frac{1}{(b+c)(c+a)(a+b)}\cdot\left(\begin{array}{c}\left(\sum_{cyc}ab\right)\left(\sum_{cyc}m_a-\sum_{cyc}m_a\right)+\\ \frac{2}{3}(s^2-4Rr-r^2)(9r)-4s^2r\end{array}\right)$$

$$=\frac{1}{(b+c)(c+a)(a+b)}\cdot 2r(s^2-12Rr-3r^2)$$

$$=\frac{1}{(b+c)(c+a)(a+b)}\cdot 2r\left(s^2-16Rr+5r^2+4r(R-2r)\right)\geq 0$$

$$\left(\because s^2-16Rr+5r^2\overset{\text{Gerretsen}}{\geq}0 \text{ and } R-2r\overset{\text{Euler}}{\geq}0\right)$$

$$\therefore \frac{r_a-h_a}{b+c}+\frac{r_b-h_b}{c+a}+\frac{r_c-h_c}{a+b}\geq 0\ \forall\ \Delta\ ABC,''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3806. In acute ΔABC the following relationship holds:

$$\sum_{cyc}\frac{secA}{sinA}\geq 4\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\sum_{cyc}\frac{secA}{sinA}=\sum_{cyc}\frac{1}{sinAcosA}=2\sum_{cyc}\frac{1}{sin2A}\overset{BERGSTROM}{\geq}$$

$$\geq 2\cdot\frac{(1+1+1)^2}{sin2A+sin2B+sin2C}=\frac{2\cdot 9}{4sinAsinBsinC}=$$

$$=\frac{9}{2\cdot\frac{a}{2R}\cdot\frac{b}{2R}\cdot\frac{c}{2R}}=\frac{36R^3}{abc}=\frac{36R^3}{4RF}=\frac{9R^2}{rs}\overset{EULER}{\geq}$$

$$\geq\frac{9R^2}{\frac{R}{2}\cdot s}=\frac{18R}{s}\overset{MITRINOVICI}{\geq}\frac{18\cdot\frac{2s}{3\sqrt{3}}}{s}=\frac{12}{\sqrt{3}}=4\sqrt{3}$$

Equality holds for $a=b=c$.

3807. In acute ΔABC the following relationship holds:

$$\frac{1+sinA}{cosA}+\frac{1+sinB}{cosB}+\frac{1+sinC}{cosC}\geq 6+3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru

$$\frac{1+sinA}{cosA}+\frac{1+sinB}{cosB}+\frac{1+sinC}{cosC}=\sum_{cyc}\frac{1+sinA}{cosA}=$$

$$=\sum_{cyc}\frac{1}{cosA}+\sum_{cyc}tanA\overset{JENSEN}{\geq}\sum_{cyc}\frac{1}{cosA}+3tan\left(\frac{A+B+C}{3}\right)\geq$$

$$\overset{BERGSTROM}{\geq}\frac{(1+1+1)^2}{cosA+cosB+cosC}+3tan\left(\frac{\pi}{3}\right)=\frac{9}{1+\frac{r}{R}}+3\sqrt{3}=$$

$$=\frac{9R}{R+r}+3\sqrt{3}\overset{EULER}{\geq}\frac{9R}{R+\frac{R}{2}}+3\sqrt{3}=\frac{9\cdot 2}{3}+3\sqrt{3}=6+3\sqrt{3}$$

Equality holds for $a=b=c$.

3808.

If in $\Delta ABC, I-incenter, O_A, O_B, O_C$ *circumcenters of* $\Delta BIC, \Delta AIC, \Delta AIB$ *then*:

$$\frac{s}{2r^2}\geq\frac{b+c}{AO_A.AI}+\frac{c+a}{BO_B.BI}+\frac{a+b}{CO_C.CI}\geq\frac{3\sqrt{3}}{R}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$Lemma\ 1$:

$$AO_A = AI + R_A\ (Let\ R_A\ circumradii\ of\ \Delta BIC)$$

$Proof: 180 - \frac{A+B}{2} + 90 - \frac{C}{2} = 270 - \frac{A+B+C}{2} = 180 \Rightarrow A - I - O_A\ are\ collinear$

$Lemma2$:

$$R_A = \frac{a}{2\cos\left(\frac{A}{2}\right)}$$

$Proof: \frac{a}{\sin\left(180 - \frac{B+C}{2}\right)} = 2\mathrm{R_A} \Rightarrow \mathrm{R_A} = \frac{\mathrm{a}}{2\cos\left(\frac{A}{2}\right)}$

$Lemma\ 3$:

$$\cos^2\left(\frac{\mathrm{A}}{2}\right) = \frac{s(s-a)}{bc}$$

$$\sum\frac{b+c}{AO_A.AI} = \sum\frac{b+c}{(AI+R_A)AI} = \sum\frac{b+c}{\frac{r^2}{\sin\left(\frac{A}{2}\right)} + \frac{ar}{2\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)}} = \sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right)}{r^2 + \frac{ar}{2}.\tan\left(\frac{A}{2}\right)} =$$

$$= \sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right)}{r^2 + \frac{ar^2}{2}.\frac{1}{s-a}} = \sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right)}{r^2\left(1 + \frac{a}{2(s-a)}\right)} = \sum\frac{(b+c)\sin^2\left(\frac{A}{2}\right).2(s-a)}{r^2(2s-2a+a)} =$$

$$= \sum\frac{2\mathrm{s}(\mathrm{s}-\mathrm{a})\sin^2\left(\frac{A}{2}\right)}{sr^2} = \sum\frac{2bc\cos^2\left(\frac{A}{2}\right)\sin^2\left(\frac{A}{2}\right)}{sr^2} = \sum\frac{bc\left(2\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)\right)^2}{2sr^2}$$

$$= \sum\frac{bc\sin^2(A)}{2sr^2} = \sum\frac{bca^2}{8sR^2r^2} = \frac{abc}{8sR^2r^2}.\sum a = \frac{s}{rR} \geq \frac{3\sqrt{3}}{R}.$$

$$Again,\qquad \frac{s}{rR} \leq \frac{s}{2r^2}.$$

3809.

$\dot{I}n\ \Delta ABC, I\ and\ O\ incenter\ and\ circumcenter.$
$\angle IAO = \alpha, \angle IBO = \beta, \angle IOC = \phi. Prove\ that:$

$$cos(\alpha).sin\left(\frac{A}{2}\right) + cos(\beta).sin\left(\frac{B}{2}\right) + cos(\phi).sin\left(\frac{C}{2}\right) = 1 + \frac{r}{R}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$Lemma$ 1:

$$AI=\frac{r}{\sin\left(\frac{A}{2}\right)}$$

$Lemma$ 2:

$$OI^2=R(R-2r)\ (Euler)$$

$Lemma$ 3:

$$\sum\cos A=1+\frac{r}{R},1-\cos A=2\sin^2\left(\frac{A}{2}\right)$$

$$\text{Let } \Delta AIO\Rightarrow\cos(\alpha)=\frac{AI^2+R^2-IO^2}{2R.AI}$$

$$Let\ \Delta BIO\Rightarrow\cos(\beta)=\frac{BI^2+R^2-IO^2}{2R.BI}$$

$$Let\ \Delta CIO\Rightarrow\cos(\phi)=\frac{CI^2+R^2-IO^2}{2R.CI}$$

$$LHS=\sum\cos(\alpha).\sin\left(\frac{A}{2}\right)=\sum\frac{AI^2+R^2-IO^2}{2R.AI}.\sin\left(\frac{A}{2}\right)=\sum\frac{\frac{r^2}{\sin^2\left(\frac{A}{2}\right)}+R^2-R^2+2Rr}{2R.\frac{r}{\sin\left(\frac{A}{2}\right)}}$$

$$=$$

$$=\sum\frac{r^2+2Rr.\sin^2\left(\frac{A}{2}\right)}{2Rr}=\frac{3r^2+Rr\sum2\sin^2\left(\frac{A}{2}\right)}{2Rr}=\frac{3r^2+Rr.\left(3-\sum\cos A\right)}{2Rr}=$$

$$=\frac{3r^2+Rr\left(3-1-\frac{r}{R}\right)}{2Rr}=\frac{(3r^2+2Rr-r^2)}{2Rr}=1+\frac{r}{R}$$

3810. In ΔABC the following relationship holds:

$$\sum_{cyc}\csc\left(\frac{A}{2}\right)\left(\sqrt{\left(ctg^3\left(\frac{B}{2}\right)+ctg^3\left(\frac{C}{2}\right)\right)}+\sqrt{\left(ctg^3\left(\frac{A}{2}\right)+ctg^3\left(\frac{C}{2}\right)\right)}\right)\geq12\sqrt{6\sqrt{3}}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Mirsadix Muzefferov-Azerbaijan

$$Let\ :\ LHS=LHS_1+LHS_2+LHS_3$$

$$LHS_1=\csc\left(\frac{A}{2}\right)\left(\sqrt{\left(ctg^3\left(\frac{B}{2}\right)+ctg^3\left(\frac{C}{2}\right)\right)}+\sqrt{\left(ctg^3\left(\frac{B}{2}\right)+ctg^3\left(\frac{C}{2}\right)\right)}\right)\overset{AM-GM}{\geq}$$

$$\csc\left(\frac{A}{2}\right)\left(\sqrt{2\left(ctg\left(\frac{B}{2}\right).ctg\left(\frac{C}{2}\right)\right)^{\frac{3}{2}}}+\sqrt{2\left(ctg\left(\frac{B}{2}\right).ctg\left(\frac{C}{2}\right)\right)^{\frac{3}{2}}}\right)\overset{AM-GM}{\geq}$$

$$\csc\left(\frac{A}{2}\right)\left(\sqrt[4]{4\left(ctg\left(\frac{A}{2}\right).ctg\left(\frac{B}{2}\right).ctg\left(\frac{C}{2}\right)\right)^{\frac{3}{2}}.\left(ctg\left(\frac{C}{2}\right)\right)^{\frac{3}{2}}}\right)$$

Analogously, it is calculated for LHS_2 *and* LHS_3

$$LHS\overset{A-G}{\geq}3\sqrt[3]{LHS_1.LHS_2.LHS_3}=$$

$$=3.2\left(64\left(ctg\left(\frac{A}{2}\right).ctg\left(\frac{B}{2}\right).ctg\left(\frac{C}{2}\right)\right)^{\frac{9}{2}}.\left(ctg\left(\frac{A}{2}\right).ctg\left(\frac{B}{2}\right).ctg\left(\frac{C}{2}\right)\right)^{\frac{3}{2}}\right)^{\frac{1}{12}}.$$

$$\left(csc\left(\frac{A}{2}\right).csc\left(\frac{B}{2}\right).csc\left(\frac{C}{2}\right)\right)^{\frac{1}{3}}=$$

$$6\left(2\left(ctg\left(\frac{A}{2}\right).ctg\left(\frac{B}{2}\right).ctg\left(\frac{C}{2}\right)\right)\right)^{\frac{1}{2}}.\left(csc\left(\frac{A}{2}\right).csc\left(\frac{B}{2}\right).csc\left(\frac{C}{2}\right)\right)^{\frac{1}{3}}=$$

$$6\left(2\left(\frac{p}{r}\right)\right)^{\frac{1}{2}}.\left(\frac{4R}{r}\right)^{\frac{1}{3}}\overset{Mitrinovic,\ Euler}{\geq}6(2.3\sqrt{3})^{\frac{1}{2}}.(8)^{\frac{1}{3}}=12\sqrt{6\sqrt{3}}$$

Equality holds if $: A=B=C$

3811. In ΔABC the following relationship holds:

$$\frac{sinA+sinB-sinC}{cosA+cosB-cosC+1}=tan\frac{A}{2}tan\frac{B}{2}cot\frac{C}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{sinA+sinB-sinC}{cosA+cosB-cosC+1}=\frac{2sin\frac{A+B}{2}cos\frac{A-B}{2}-sin\left(2\cdot\frac{C}{2}\right)}{2cos\frac{A+B}{2}cos\frac{A-B}{2}-cos\left(2\cdot\frac{C}{2}\right)+1}=$$

$$=\frac{2sin\frac{\pi-C}{2}cos\frac{A-B}{2}-2sin\frac{C}{2}cos\frac{C}{2}}{2cos\frac{\pi-C}{2}cos\frac{A-B}{2}+2sin^2\frac{C}{2}}=\frac{2cos\frac{C}{2}cos\frac{A-B}{2}-2sin\frac{C}{2}cos\frac{C}{2}}{2sin\frac{C}{2}cos\frac{A-B}{2}+2sin^2\frac{C}{2}}=$$

$$=\frac{\left(cos\frac{A-B}{2}-sin\frac{C}{2}\right)}{\left(cos\frac{A-B}{2}+sin\frac{C}{2}\right)}\cdot\frac{cos\frac{C}{2}}{sin\frac{C}{2}}=\frac{\left(cos\frac{A-B}{2}-cos\frac{\pi-C}{2}\right)}{\left(cos\frac{A-B}{2}+cos\frac{\pi-C}{2}\right)}\cdot cot\frac{C}{2}=$$

$$=\frac{2sin\frac{\pi-C+A-B}{4}sin\frac{\pi-C-A+B}{4}}{2cos\frac{\pi-C+A-B}{4}cos\frac{A-B-\pi+C}{4}}\cdot cot\frac{C}{2}=$$

$$=\frac{2sin\frac{A+B+C-C+A-B}{4}sin\frac{A+B+C-C-A+B}{4}}{2cos\frac{A+B+C-C+A-B}{4}cos\frac{A-B-A-B-C+C}{4}}\cdot cot\frac{C}{2}=$$

$$=\frac{sin\frac{2A}{4}sin\frac{2B}{4}}{cos\frac{2A}{4}cos\frac{2B}{4}}\cdot cot\frac{C}{2}=\frac{sin\frac{A}{2}sin\frac{B}{2}}{cos\frac{A}{2}cos\frac{B}{2}}\cdot cot\frac{C}{2}=tan\frac{A}{2}tan\frac{B}{2}cot\frac{C}{2}$$

3812. In any Δ ABC the following relationship holds :

$$\frac{n_a^2+g_a^2+r_ar-r_br_c}{n_a-\sqrt{4r^2+(b-c)^2}}\le\frac{2Rr_a}{r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{n_a^2+g_a^2+r_ar-r_br_c}{n_a-\sqrt{4r^2+(b-c)^2}}-\frac{2Rr_a}{r}\overset{\text{Bogdan Fustei}}{=}$$

$$\frac{r\left((b-c)^2+2s(s-a)+(s-b)(s-c)-s(s-a)\right)-2R.\frac{rs}{s-a}.\left(n_a-\frac{an_a}{s}\right)}{r\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}$$

$$=\frac{r\left((b-c)^2+s(s-a)+(s-b)(s-c)\right)-r.2Rn_a}{r\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}\le$$

$$\frac{r\left((b-c)^2+s(s-a)+(s-b)(s-c)\right)-r.2R\left(\frac{b^2-bc+c^2}{2R}\right)}{r\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}$$

$$\left(\text{Reference : Inequality in Triangle by Mohamed Amine Ben Ajiba} - 28; \text{ published at www.ssmrmh.ro}\right)$$

$$= \frac{-bc + s(s-a) - s^2 + sa + bc}{n_a - \sqrt{4r^2 + (b-c)^2}} = 0 \therefore \frac{n_a^2 + g_a^2 + r_a r - r_b r_c}{n_a - \sqrt{4r^2 + (b-c)^2}} \le \frac{2Rr_a}{r}$$

$$\forall \Delta ABC, " = " \text{ iff } b = c \text{ (QED)}$$

3813. In any Δ ABC the following relationship holds :

$$18\sum_{cyc}\frac{r^3}{h_a^3} \ge 1 + 3\sum_{cyc}\frac{r^2}{h_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = \frac{3r}{h_a}, y = \frac{3r}{h_b}, z = \frac{3r}{h_c} \text{ and then : } 18\sum_{cyc}\frac{r^3}{h_a^3} \overset{?}{\ge} 1 + 3\sum_{cyc}\frac{r^2}{h_a^2}$$

$$\Leftrightarrow \frac{18}{27}.\sum_{cyc}x^3 \overset{?}{\ge} 1 + \frac{3}{9}.\sum_{cyc}x^2$$

$$\Leftrightarrow 18\sum_{cyc}x^3 \overset{?}{\ge} \left(\sum_{cyc}x\right)^3 + 3\left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)\left(\because \sum_{cyc}x = \sum_{cyc}\frac{3r}{h_a} = 3\right)$$

$$\to \text{true} \because 9\sum_{cyc}x^3 \overset{\text{Holder}}{\ge} \left(\sum_{cyc}x\right)^3 \text{ and } 3\sum_{cyc}x^3 \overset{\text{Chebyshev}}{\ge} \left(\sum_{cyc}x^2\right)\left(\sum_{cyc}x\right)$$

$$\therefore 18\sum_{cyc}\frac{r^3}{h_a^3} \ge 1 + 3\sum_{cyc}\frac{r^2}{h_a^2} \;\forall \Delta ABC, " = " \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3814. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{h_a^2 - r^2} \ge \frac{3}{8r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x = \frac{3r}{h_a}, y = \frac{3r}{h_b}, z = \frac{3r}{h_c} \text{ and then : } \sum_{cyc}\frac{1}{h_a^2 - r^2} \overset{?}{\ge} \frac{3}{8r^2} \Leftrightarrow \sum_{cyc}\frac{1}{\frac{9}{x^2} - 1} \underset{(*)}{\overset{?}{\ge}} \frac{3}{8}$$

$$\text{Let } f(t)=\frac{t^2}{9-t^2}\ \forall\, t\in(0,3)\ \text{and then } f''(t)=\frac{54(t^2+3)}{(9-t^2)^3}>0 \text{ and so,}$$

$$\text{as}\sum_{\text{cyc}} x=\sum_{\text{cyc}}\frac{3r}{h_a}=3\Rightarrow x,y,z<3\ \therefore\ \sum_{\text{cyc}}\frac{1}{\frac{9}{x^2}-1}\overset{\text{Jensen}}{\geq}\frac{3}{\frac{9}{\left(\frac{\sum_{\text{cyc}} x}{3}\right)^2}-1}=\frac{3}{9-1}=\frac{3}{8}$$

$$\Rightarrow(*)\text{ is true}\ \therefore\ \sum_{\text{cyc}}\frac{1}{h_a^2-r^2}\geq\frac{3}{8r^2}\ \forall\,\Delta\, ABC,''=''\ \text{iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3815. **In any $\Delta\, ABC$ the following relationship holds :**

$$3\leq\sum_{\text{cyc}}\frac{m_a^2}{h_b h_c}\leq 3\left(\frac{R}{2r}\right)^3$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{\text{cyc}}\frac{m_a^2}{h_b h_c}=\frac{1}{16r^2s^2}\cdot\sum_{\text{cyc}}\left(bc\left(2\sum_{\text{cyc}} a^2-3a^2\right)\right)$$

$$=\frac{4(s^4-(4Rr+r)^2)-3(4Rrs)(2s)}{16r^2s^2}\overset{?}{\leq}3\left(\frac{R}{2r}\right)^3$$

$$\Leftrightarrow(3R^3+12Rr^2)s^2+2r^3(4R+r)^2\underbrace{\overset{?}{\geq}}_{(*)}2rs^4$$

$$\text{Now, } 2rs^4\overset{\text{Gerretsen}}{\leq}2rs^2(4R^2+4Rr+3r^2)\overset{?}{\leq}\text{LHS of }(*)\Leftrightarrow$$

$$(3R^3-8R^2r+4Rr^2-6r^3)s^2+2r^3(4R+r)^2\underbrace{\overset{?}{\geq}}_{(**)}0 \text{ and it's trivially true when :}$$

$$3R^3-8R^2r+4Rr^2-6r^3\geq 0 \text{ and when : } 3R^3-8R^2r+4Rr^2-6r^3<0,$$

$$\text{LHS of }(**)\overset{\text{Gerretsen}}{\geq}(3R^3-8R^2r+4Rr^2-6r^3)(4R^2+4Rr+3r^2)+$$

$$2r^3(4R+r)^2\overset{?}{\geq}0\Leftrightarrow 12t^5-20t^4-7t^3+4t-16\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(12t^4+4t^3+t^2+2t+8)\overset{?}{\geq}0\rightarrow\text{true}\ \because\ t\overset{\text{Euler}}{\geq}2\Rightarrow(**)\Rightarrow(*)$$

$$\text{is true } \forall \Delta ABC \therefore \sum_{cyc}\frac{m_a^2}{h_bh_c} \le 3\left(\frac{R}{2r}\right)^3 \text{ and again, } \sum_{cyc}\frac{m_a^2}{h_bh_c} \ge \sum_{cyc}\frac{m_a^2}{w_bw_c}$$

$$\overset{Lascu+AM-GM}{\ge} \sum_{cyc}\frac{s(s-a)}{\sqrt{s(s-b)}.\sqrt{s(s-c)}} \overset{AM-GM}{\ge} 3\sqrt[3]{\frac{s(s-a).s(s-b).s(s-c)}{s(s-a).s(s-b).s(s-c)}} = 3$$

$$\text{and so, } 3 \le \sum_{cyc}\frac{m_a^2}{h_bh_c} \le 3\left(\frac{R}{2r}\right)^3 \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3816. In any Δ ABC the following relationship holds :

$$3 \le \sum_{cyc}\frac{m_a^2}{r_br_c} \le \frac{R^2+2r^2}{Rr}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^2}{r_br_c} = \sum_{cyc}\frac{4m_a^2(-s^2+sa+bc)}{4r^2s^2}$$

$$= \frac{1}{4r^2s^2}\left(\begin{array}{c}-6s^2(s^2-4Rr-r^2) + s\sum_{cyc}\left(a\left(2\sum_{cyc}a^2-3a^2\right)\right) + \\ \sum_{cyc}\left(bc\left(2\sum_{cyc}a^2-3a^2\right)\right)\end{array}\right)$$

$$= \frac{\begin{array}{c}-6s^2(s^2-4Rr-r^2)+8s^2(s^2-4Rr-r^2)-6s^2(s^2-6Rr-3r^2)+ \\ 4(s^4-(4Rr+r)^2)-24Rrs^2\end{array}}{4r^2s^2}$$

$$= \frac{(R+4r)s^2-r(4R+r)^2}{rs^2} \overset{?}{\le} \frac{R^2+2r^2}{Rr} \Leftrightarrow R(4R+r)^2 \overset{?}{\ge} (4R-2r)s^2$$

$$\to \text{true via Blundon} - \text{Gerretsen} \therefore \sum_{cyc}\frac{m_a^2}{r_br_c} \le \frac{R^2+2r^2}{Rr} \text{ and again,}$$

$$\sum_{cyc}\frac{m_a^2}{r_br_c} \overset{Lascu+AM-GM}{\ge} \sum_{cyc}\frac{s(s-a)(s-b)(s-c)}{r^2s^2} = \frac{3r^2s^2}{r^2s^2} = 3 \text{ and so,}$$

$$3 \le \sum_{cyc}\frac{m_a^2}{r_br_c} \le \frac{R^2+2r^2}{Rr} \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3817. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{r_a^2+r^2}{r_a^2-r^2}\ge\frac{15}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{3r}{r_a}, y=\frac{3r}{r_b}, z=\frac{3r}{r_c} \text{ and then : } \sum_{cyc}\frac{r_a^2+r^2}{r_a^2-r^2}\overset{?}{\ge}\frac{15}{4}\Leftrightarrow\sum_{cyc}\frac{\frac{9}{x^2}+1}{\frac{9}{x^2}-1}\underset{(*)}{\overset{?}{\ge}}\frac{15}{4}$$

$$\text{Let } f(t)=\frac{9+t^2}{9-t^2}\ \forall t\in(0,3) \text{ and then } f''(t)=\frac{108(t^2+3)}{(9-t^2)^3}>0\Rightarrow f(t) \text{ is convex}$$

$$\text{and so, as } \sum_{cyc}x=\sum_{cyc}\frac{3r}{r_a}=3\Rightarrow 0<x,y,z<3\ \therefore \sum_{cyc}\frac{\frac{9}{x^2}+1}{\frac{9}{x^2}-1}\overset{\text{Jensen}}{\ge}3.\frac{9+\left(\frac{\sum_{cyc}x}{3}\right)^2}{9-\left(\frac{\sum_{cyc}x}{3}\right)^2}$$

$$=3.\frac{10}{8}=\frac{15}{4}\Rightarrow(*) \text{ is true }\therefore\sum_{cyc}\frac{r_a^2+r^2}{r_a^2-r^2}\ge\frac{15}{4}\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3818. In ΔABC the following relationship holds:

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\ge\frac{3\sqrt{3}}{s}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c}\overset{AM-GM}{\ge}\frac{3}{\sqrt[3]{w_aw_bw_c}}\ge\frac{3}{\sqrt[3]{r_ar_br_c}}=$$

$$=\frac{3}{\sqrt[3]{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}}=\frac{3}{\sqrt[3]{\frac{F^3}{(s-a)(s-b)(s-c)}}}=\frac{3}{\sqrt[3]{\frac{F^3s}{s(s-a)(s-b)(s-c)}}}=$$

$$=\frac{3}{\sqrt[3]{\frac{F^3s}{F^2}}}=\frac{3}{\sqrt[3]{Fs}}=\frac{3}{\sqrt[3]{rs^2}}$$

$$\frac{3}{\sqrt[3]{rs^2}} \ge \frac{3\sqrt{3}}{s} \Leftrightarrow s \ge \sqrt{3}\cdot\sqrt[3]{rs^2} \Leftrightarrow s^3 \ge 3\sqrt{3}\cdot rs^2 \Leftrightarrow$$

$$\Leftrightarrow s \ge 3\sqrt{3} \text{ (Mitrinovici)}$$

Equality holds for $a = b = c$.

3819. In ΔABC the following relationship holds:

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c} \ge \frac{2}{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1}{w_a}+\frac{1}{w_b}+\frac{1}{w_c} \overset{AM-GM}{\ge} \frac{3}{\sqrt[3]{w_aw_bw_c}} \ge \frac{3}{\sqrt[3]{r_ar_br_c}} =$$

$$= \frac{3}{\sqrt[3]{\frac{F}{s-a}\cdot\frac{F}{s-b}\cdot\frac{F}{s-c}}} = \frac{3}{\sqrt[3]{\frac{F^3}{(s-a)(s-b)(s-c)}}} = \frac{3}{\sqrt[3]{\frac{F^3s}{s(s-a)(s-b)(s-c)}}} = \frac{3}{\sqrt[3]{\frac{F^3s}{F^2}}}$$

$$= \frac{3}{\sqrt[3]{Fs}} = \frac{3}{\sqrt[3]{rs^2}}$$

$$\frac{3}{\sqrt[3]{rs^2}} \ge \frac{2}{R} \Leftrightarrow \frac{27}{rs^2} \ge \frac{8}{R^3} \Leftrightarrow 7R^3 \ge 8rs^2 \text{ (to prove)}$$

$$8rs^2 \overset{EULER}{\le} 4Rs^2 \overset{MITRINOVICI}{\le} 4R\cdot\left(\frac{3\sqrt{3}R}{2}\right)^2 = 27R^3$$

Equality holds for $a = b = c$.

3820. If I –incenter in ΔABC then the following relationship holds:

$$\frac{1}{IA}+\frac{1}{IB}+\frac{1}{IC} \ge \frac{3}{R}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1}{IA}+\frac{1}{IB}+\frac{1}{IC}=\sum_{cyc}\frac{1}{IA}=\sum_{cyc}\frac{\sin\frac{A}{2}}{r}=\frac{1}{r}\sum_{cyc}\sin\frac{A}{2}\overset{JENSEN}{\underset{\frown}{\geq}}$$

$$\geq\frac{1}{r}\cdot 3\sin\left(\frac{A+B+C}{6}\right)=\frac{6}{2r}\cdot\sin\frac{\pi}{3}\overset{EULER}{\underset{\frown}{\geq}}\frac{6}{R}\cdot\frac{1}{2}=\frac{3}{R}$$

Equality holds for: $a=b=c$.

3821.

In any Δ ABC the following relationship holds :

$$\sum_{cyc}\sqrt{\frac{b+c}{m_a}}+\sum_{cyc}\sqrt{\frac{m_a}{b+c}}+\frac{R^5}{32r^5}\geq 1+\sum_{cyc}\sqrt{\frac{b+c}{n_a}}+\sum_{cyc}\sqrt{\frac{n_a}{b+c}}$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$n_a^2\overset{\text{Bogdan Fustei}}{=}s^2-2r_ah_a \therefore a^2n_a^2\overset{?}{\leq}4(R-r)^2s^2$$

$$\Leftrightarrow a^2(s^2-2h_ar_a)\overset{?}{\leq}4(R-r)^2s^2$$

$$\Leftrightarrow(4R^2\sin^2 A)s^2-4rs\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(s\tan\frac{A}{2}\right)\overset{?}{\leq}4(R^2-2Rr+r^2)s^2$$

$$\Leftrightarrow R^2(1-\sin^2 A)-2Rr\left(1-2\sin^2\frac{A}{2}\right)+r^2\overset{?}{\geq}0\Leftrightarrow R^2\cos^2 A-2Rr\cos A+r^2\overset{?}{\geq}0$$

$$\Leftrightarrow(R\cos A-r)^2\overset{?}{\geq}0\to\text{true}\therefore n_a\leq\frac{2(R-r)s}{a}\text{ and analogs}\Rightarrow$$

$$\sum_{cyc}\sqrt{\frac{n_a}{b+c}}\leq\sum_{cyc}\sqrt{\frac{2(R-r)s}{a(b+c)}}=\sum_{cyc}\sqrt{\frac{2(R-r)bcs}{4Rrs(b+c)}}=\frac{R-r}{\sqrt{4Rr(R-r)}}\cdot\sum_{cyc}\sqrt{\frac{2bc}{b+c}}$$

$$\overset{\substack{\text{Euler}\\\text{and}\\\text{AM-HM}}}{\leq}\frac{R-r}{\sqrt{8Rr^2}}\cdot\sum_{cyc}\sqrt{b+c}\overset{\text{CBS}}{\leq}\frac{R-r}{\sqrt{8Rr^2}}.\sqrt{3.4s}\overset{\text{Mitrinovic}}{\leq}\frac{R-r}{\sqrt{2Rr^2}}.\sqrt{3R.\frac{3\sqrt{3}}{2}}$$

$$=\frac{R-r}{2r}.3.\sqrt[4]{3}\therefore\sum_{cyc}\sqrt{\frac{n_a}{b+c}}\overset{(1)}{\leq}\left(t^2-\frac{1}{2}\right).3.\sqrt[4]{3}\left(t=\sqrt{\frac{R}{2r}}\right)\text{ and again,}$$

$$\sum_{cyc}\sqrt{\frac{m_a}{b+c}}\overset{Lascu}{\geq}\frac{1}{\sqrt{2}}.\sum_{cyc}\sqrt{\cos\frac{A}{2}}\overset{AM-GM}{\geq}\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{s}{4R}}\overset{Mitrinovic}{\geq}\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{3\sqrt{3}.r}{4R}}$$

$$=\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{3\sqrt{3}.rR^2}{4R^3}}\overset{Euler}{\geq}\frac{3}{\sqrt{2}}.\sqrt[6]{\frac{3\sqrt{3}.4r^3}{4R^3}}=\frac{3.\sqrt[4]{3}}{2}.\sqrt{\frac{2r}{R}}\therefore\sum_{cyc}\sqrt{\frac{m_a}{b+c}}\overset{②}{\geq}\frac{3.\sqrt[4]{3}}{2}.\frac{1}{t}\text{ and}$$

$$\because n_a\overset{\text{Bogdan Fustei}}{\geq}m_a\text{ and analogs}\therefore\sum_{cyc}\sqrt{\frac{b+c}{m_a}}\overset{③}{\geq}\sum_{cyc}\sqrt{\frac{b+c}{n_a}}\text{ and so,}$$

$$①,②\text{ and }③\Rightarrow\text{it suffices to prove}:\frac{3.\sqrt[4]{3}}{2}.\frac{1}{t}+t^{10}\overset{?}{\geq}1+\left(t^2-\frac{1}{2}\right).3.\sqrt[4]{3}$$

$$\Leftrightarrow t^{10}-1\overset{?}{\geq}\frac{3.\sqrt[4]{3}}{2}.\left(2t^2-1-\frac{1}{t}\right)\Leftrightarrow$$

$$(t-1)\left(t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t\right)\overset{?}{\geq}\frac{3.\sqrt[4]{3}}{2}.(t-1)\binom{2t^2+}{2t+1}$$

$$\text{and}\because t-1\overset{Euler}{\geq}0\text{ and }\frac{3.\sqrt[4]{3}}{2}<2\therefore\text{it suffices to prove}:$$

$$t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t\overset{?}{\geq}4t^2+4t+2\rightarrow\text{true}$$

$$\because t^{10}+t^9+t^8+t^7\overset{t\geq1}{\geq}t^2+t^2+t^2+t^2=4t^2\text{ and}$$

$$t^6+t^5+t^4+t^3\overset{t\geq1}{\geq}t+t+t+t=4t\text{ and finally},t^2+t\overset{t\geq1}{\geq}1+1=2$$

$$\therefore\sum_{cyc}\sqrt{\frac{b+c}{m_a}}+\sum_{cyc}\sqrt{\frac{m_a}{b+c}}+\frac{R^5}{32r^5}\geq1+\sum_{cyc}\sqrt{\frac{b+c}{n_a}}+\sum_{cyc}\sqrt{\frac{n_a}{b+c}}\ \forall\Delta ABC,$$

$$''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

3822. *In $\Delta ABC, I_a, I_b, I_c$ − excenters. Prove that:*

$$[I_aBC]+[I_bAC]+[I_cAB]\geq3F$$

Proposed by Sarkhan Adgozalov-Georgia

Solution 1 by Qurban Muellim-Azerbaijan

Let $a=x+y, b=y+z, c=x+z, s=x+y+z$

Lemma :

$$(x+y)(y+z)(x+z)\geq8xyz$$

$$[I_aBC]=\frac{a.r_a}{2}$$

$$LHS=\sum\left(\frac{r_a.a}{2}\right)=\frac{1}{2}\sum\left(\frac{aF}{s-a}\right)=\frac{F}{2}\sum\frac{1}{s-a}=\frac{F}{2}\sum\frac{x+y}{z}\geq$$

$$\ge \frac{3F}{2}\cdot\sqrt[3]{\frac{(x+y)(y+z)(x+z)}{xyz}} \ge \frac{3F}{2}\sqrt[3]{\frac{8xyz}{xyz}} = \frac{3F}{2}.2 = 3F$$

$$Equality\ holds\ for\ a=b=c.$$

Solution 2 by Ertan Yildirim-Turkiye

Lemma:

$$r_a r_b r_c = p^2 r$$

Proof:

$$r_a = \frac{F}{p-a}, r_b = \frac{F}{p-b}, r_c = \frac{F}{p-c}$$

$$r_a r_b r_c = \frac{F^3}{(p-a)(p-b)(p-c)} = \frac{F^3 p}{p(p-a)(p-b)(p-c)} = \frac{F^3 p}{F^2} = Fp = prp = p^2 r.$$

$$[I_aBC] = \frac{ar_a}{2}$$

$$[I_bAC] = \frac{br_b}{2}$$

$$[I_cAB] = \frac{cr_c}{2}$$

$$LHS = \frac{1}{2}\sum ar_a \ge \frac{1}{2}.3.\sqrt[3]{abcr_ar_br_c} = \frac{3}{2}.\sqrt[3]{4prRp^2r} = \frac{3}{2}.\sqrt[3]{4p^3Rr^2} \ge \frac{3}{2}.2pr = 3F$$

$$Equality\ holds\ for\ a=b=c.$$

3823. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2 + \left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{\frac{\sin^2 C}{\sin^2 A}(\sin B+\sin C)^2 + \frac{\sin^2 B}{\sin^2 A}(\sin C+\sin A)^2} \ge 1$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Soumava Chakraborty-Kolkata-India

$$\forall A', B', C', x', y', z' > 0,$$

$$\frac{x'}{y'+z'}(B'+C') + \frac{y'}{z'+x'}(C'+A') + \frac{z'}{x'+y'}(A'+B') \underset{①}{\overset{\text{Walter Janous}}{\ge}} \sqrt{3\sum_{cyc} A'B'}$$

$$\text{Now, } \sum_{cyc}\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2 + \left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{\frac{\sin^2 C}{\sin^2 A}(\sin B+\sin C)^2 + \frac{\sin^2 B}{\sin^2 A}(\sin C+\sin A)^2}$$

$$=\sum_{cyc}\frac{\sin^2 A\left(\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2+\left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{(\sin B+\sin C)^2(\sin C+\sin A)^2}\right)}{\frac{\sin^2 C}{(\sin C+\sin A)^2}+\frac{\sin^2 B}{(\sin B+\sin C)^2}}$$

$$=\sum_{cyc}\frac{\frac{\sin^2 A}{(\sin A+\sin B)^2}\cdot\left(\frac{1}{(\sin B+\sin C)^2}+\frac{1}{(\sin C+\sin A)^2}\right)}{\frac{\sin^2 B}{(\sin B+\sin C)^2}+\frac{\sin^2 C}{(\sin C+\sin A)^2}}$$

$$=\frac{x'}{y'+z'}(B'+C')+\frac{y'}{z'+x'}(C'+A')+\frac{z'}{x'+y'}(A'+B')$$

$$\left(\begin{array}{c}x'=\frac{\sin^2 A}{(\sin A+\sin B)^2},y'=\frac{\sin^2 B}{(\sin B+\sin C)^2},z'=\frac{\sin^2 C}{(\sin C+\sin A)^2},\\ A'=\frac{1}{(\sin A+\sin B)^2},B'=\frac{1}{(\sin B+\sin C)^2},C'=\frac{1}{(\sin C+\sin A)^2}\end{array}\right)$$

$$\overset{\text{via } ①}{\geq}\sqrt{3\sum_{cyc}\frac{1}{(\sin A+\sin B)^2(\sin B+\sin C)^2}}\overset{AM-GM}{\geq}$$

$$\frac{3}{\left(\sqrt[3]{(\sin A+\sin B)(\sin B+\sin C)(\sin C+\sin A)}\right)^2}\overset{AM-GM}{\geq}\frac{3}{\left(\frac{2(\sin A+\sin B+\sin C)}{3}\right)^2}\overset{Jensen}{\geq}$$

$$\frac{3}{(\sqrt{3})^2}=1 \text{ and so}, \sum_{cyc}\frac{\left(\frac{\sin B+\sin C}{\sin A+\sin B}\right)^2+\left(\frac{\sin C+\sin A}{\sin A+\sin B}\right)^2}{\frac{\sin^2 C}{\sin^2 A}(\sin B+\sin C)^2+\frac{\sin^2 B}{\sin^2 A}(\sin C+\sin A)^2}\geq 1\ \forall\ \Delta\ ABC,$$

$$''=''\ \text{iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3824. In acute ΔABC the following relationship holds:

$$\cos(A).\cos(B)+\cos(C).\cos(B)+\cos(A).\cos(C)\leq\frac{a^3+b^3+c^3}{16RS}$$

Proposed by Ertan Yildirim-Turkiye

Soltion by Mirsadix Muzefferov-Azerbaijan

$$RHS=\frac{a^3+b^3+c^3}{16RS}=\frac{a^3+b^3+c^3}{4abc}\overset{AM-GM}{\geq}\frac{3abc}{4abc}=\frac{3}{4}\quad(*)$$

$$LHS=\sum_{cyc}\cos(A).\cos(B)=\frac{p^2+r^2-4R^2}{4R^2}\overset{Gerretsen}{\leq}$$

$$\frac{4R^2+4Rr+3r^2+r^2-4R^2}{4R^2}=\frac{4Rr+4r^2}{4R^2}=\frac{r}{R}+\left(\frac{r}{R}\right)^2\leq\frac{1}{2}+\frac{1}{4}=\frac{3}{4}\quad(**)$$

$$LHS \overset{(**)}{\le} \frac{3}{4},\quad RHS \overset{(*)}{\ge} \frac{3}{4} \to LHS \le RHS\ (PROVED)$$

Equality holds for an equilateral triangle.

3825. In ΔABC the following relationship holds:

$$\cos\left(\frac{A-B}{2}\right).\cos\left(\frac{B-C}{2}\right).\cos\left(\frac{C-A}{2}\right) \le \frac{1}{4}\left(\frac{R}{r}+\frac{r}{R}\right)+\frac{3}{8}$$

Proposed by Ertan Yildirim-Turkiye

Solution by Mirsadix Muzefferov-Azerbaijan

By Mollweide's formula:

$$\cos\left(\frac{A-B}{2}\right) = \frac{a+b}{c}.\sin\left(\frac{C}{2}\right)$$

$$\prod_{cyc}\cos\left(\frac{A-B}{2}\right) = \prod_{cyc}\frac{a+b}{c}.\prod_{cyc}\sin\left(\frac{C}{2}\right) = \frac{r}{4R}\left(2+\left(\frac{a}{b}+\frac{b}{a}\right)+\left(\frac{b}{c}+\frac{c}{b}\right)+\left(\frac{c}{a}+\frac{a}{c}\right)\right) \le$$

$$\le \left(2+3.\frac{R}{r}\right).\frac{r}{4R} = \frac{r}{2R}+\frac{3}{4}$$

Let's prove that :

$$\frac{r}{2R}+\frac{3}{4} \le \frac{1}{4}\left(\frac{R}{r}+\frac{r}{R}\right)+\frac{3}{8}$$

$$\frac{2r}{R}+3 \le \frac{R}{r}+\frac{r}{R}+\frac{3}{2} \to 2R^2-3Rr-2r^2 \ge 0 \to$$

$$(R-2r)(2R+r) \ge 0 \to\quad R \ge 2r\ (Euler)\ True$$

Equality holds for an equilateral triangle.

3826. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{2r}{h_a^2}+\frac{2}{h_a}+\frac{r}{h_bh_c}} \ge \frac{4p^2r}{p^2+r^2+4Rr}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\frac{2r}{h_a^2}+\frac{2}{h_a}+\frac{r}{h_bh_c}} = \sum_{cyc}\frac{h_a^2}{2r+2h_a+\frac{Rrh_a^3}{2p^2r^2}} \overset{\text{Bergstrom}}{\ge}$$

$$\frac{\frac{(p^2+r^2+4Rr)^2}{4R^2}}{6r+\frac{p^2+r^2+4Rr}{R}+\frac{R}{2p^2r}\cdot\sum_{cyc}\frac{a^3b^3}{8R^3}}$$

$$=\frac{4p^2r(p^2+r^2+4Rr)^2}{96R^2r^2p^2+16Rrp^2(p^2+r^2+4Rr)+(p^2+r^2+4Rr)^3-24Rrp^2(p^2+r^2+2Rr)}\overset{?}{\geq}$$

$$\frac{4p^2r}{p^2+r^2+4Rr}\Leftrightarrow(p^2+r^2+4Rr)^3\overset{?}{\geq}96R^2r^2p^2+16Rrp^2(p^2+r^2+4Rr)+$$

$$(p^2+r^2+4Rr)^3-24Rrp^2(p^2+r^2+2Rr)$$

$$\Leftrightarrow 3(p^2+r^2+2Rr)\overset{?}{\geq}12Rr+2(p^2+r^2+4Rr)\Leftrightarrow p^2\overset{?}{\geq}14Rr-r^2$$

$$\Leftrightarrow(p^2-16Rr+5r^2)+2r(R-2r)\overset{?}{\geq}0\to\text{true}\because p^2-16Rr+5r^2\overset{\text{Gerretsen}}{\geq}0\text{ and}$$

$$2r(R-2r)\overset{\text{Euler}}{\geq}0\therefore\sum_{cyc}\frac{1}{\frac{2r}{h_a^2}+\frac{2}{h_a}+\frac{r}{h_bh_c}}\geq\frac{4p^2r}{p^2+r^2+4Rr}\ \forall\Delta ABC,$$

$''=''$ iff ΔABC is equilateral (QED)

3827. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}\geq\frac{p^2}{4R+r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}=\sum_{cyc}\frac{r_a^2}{2r+2r_a+\frac{rr_a^3}{p^2r}}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{p^2(4R+r)^2}{6rp^2+2p^2(4R+r)+(4R+r)^3-12Rp^2}\left(\because\sum_{cyc}r_a^3=(4R+r)^3-12Rp^2\right)$$

$$=\frac{p^2(4R+r)^2}{(8R+8r)p^2+(4R+r)^3-12Rp^2}\overset{?}{\geq}\frac{p^2}{4R+r}$$

$$\Leftrightarrow(4R+r)^3\overset{?}{\geq}(8R+8r)p^2+(4R+r)^3-12Rp^2\Leftrightarrow 4R\overset{?}{\geq}8r\to\text{true via Euler}$$

$$\therefore\sum_{cyc}\frac{1}{\frac{2r}{r_a^2}+\frac{2}{r_a}+\frac{r}{r_br_c}}\geq\frac{p^2}{4R+r}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

3828. In ΔABC the following relationship holds:

$$\sum \frac{\left(c+\sqrt{ab}\right)^2}{2(b+c-a)(a+c)} \le \frac{R}{r}+1$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$Known\ result: \sum \frac{1}{s-a} = \frac{4R+r}{sr}$$

$$\left(c+\sqrt{ab}\right)^2 \overset{C-S}{\le} (c+a)(c+b)$$

$$\sum \frac{\left(c+\sqrt{ab}\right)^2}{2(b+c-a)(a+c)} \le \sum \frac{(c+a)(c+b)}{4(s-a)(a+c)} = \frac{1}{4}\sum \frac{c+b}{s-a} = \frac{1}{4}\sum \frac{2s-a}{s-a} =$$

$$= \frac{1}{4}\sum \frac{(s-a)+s}{s-a} = \frac{1}{4}\left(3+s\sum \frac{1}{s-a}\right) = \frac{1}{4}\left(3+s.\frac{4R+r}{sr}\right) = \frac{1}{4}\left(4+\frac{4R}{r}\right) = \frac{R}{r}+1$$

Equality holds for an equilateral triangle.

3829. In acute ΔABC the following relationship holds:

$$\cot\frac{A}{2}\sqrt{cosA}+\cot\frac{B}{2}\sqrt{cosB}+\cot\frac{C}{2}\sqrt{cosC} > \sqrt{3}$$

Proposed by Vasile Mircea Popa-Romania

Solution by Tapas Das-India

$$as\ 0 < cosA < 1\ then\ \sqrt{cosA} > cosA\ (1)$$

$$WLOG\ A \ge B \ge C$$

$$\cot\frac{A}{2} \le \cot\frac{B}{2} \le \cot\frac{C}{2}\ \&\ cosA \le cosB \le cosC$$

$$\cot\frac{A}{2}\sqrt{cosA}+\cot\frac{B}{2}\sqrt{cosB}+\cot\frac{C}{2}\sqrt{cosC} \overset{(1)}{\ge}$$

$$\geq \cot\frac{A}{2}cosA+\cot\frac{B}{2}cosB+\cot\frac{C}{2}cosC\overset{Chebyshev}{\geq}\frac{1}{3}\left(\sum\cot\frac{A}{2}\right)\left(\sum cosA\right)=$$

$$=\frac{1}{3}\cdot\frac{s}{r}\cdot\frac{R+r}{R}>\frac{1}{3}\cdot\frac{s}{r}\cdot\frac{R}{R}\overset{Mitrinovic}{>}\frac{1}{3}\cdot\frac{3\sqrt{3}r}{r}=\sqrt{3}$$

3830. $I-$ *incenter in* ΔABC, $A(2,2),B(6,4),C(4,8),M(8,6)$.

Find MI.

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

We know that in ΔABC *with vertices* $A(x_1,y_1),B(x_2,y_2),C(x_3,y_3)$:

$$Incenter-I=\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\frac{ay_1+by_2+cy_3}{a+b+c}\right)$$

$$now\ A(x_1,y_1)=A(2,2),B(x_2,y_2)=B(6,4),C(x_3,y_3)=C(4,8)$$

$$a=BC=\sqrt{(6-4)^2+(4-2)^2}=2\sqrt{5},$$
$$b=CA=\sqrt{(4-2)^2+(8-2)^2}=2\sqrt{10},$$
$$c=AB=\sqrt{(6-2)^2+(4-2)^2}=2\sqrt{5}$$

$$I-Incenter=$$
$$=\left(\frac{2\sqrt{5}\times 2+2\sqrt{10}\times 6+2\sqrt{5}\times 4}{2\sqrt{5}+2\sqrt{5}+2\sqrt{10}},\frac{2\sqrt{5}\times 2+2\sqrt{10}\times 4+2\sqrt{5}\times 8}{2\sqrt{5}+2\sqrt{5}+2\sqrt{10}}\right)=\left(3\sqrt{2},6-\sqrt{2}\right)$$

$$MI=\sqrt{\left(8-3\sqrt{2}\right)^2+\left(6-6+\sqrt{2}\right)^2}=\sqrt{64+18-48\sqrt{2}+2}=2\sqrt{21-12\sqrt{2}}$$

3831. In ΔABC the following relationship holds:

$$\sum\sqrt{2R\left(n_a-\sqrt{4r^2+(n_a-g_a)^2}\right)}\geq\sum AI\sqrt{\frac{n_a}{h_a}}$$

Proposed by Bogdan Fuștei-Romania

Solution by Tapas Das-India

$$r=\frac{F}{s}=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=(s-a)tan\left(\frac{A}{2}\right)\Rightarrow(s-a)=\frac{r}{\tan(\frac{A}{2})}\ (1)$$

$$lemma: |b-c|\ge n_a-g_a, \frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$$

(Reference: About Nagel and Gergonne's cevian by Bogdan Fustei , www.ssmrmh.ro)

$$L.H.S=\sum\sqrt{2R\left(n_a-\sqrt{4r^2+(n_a-g_a)^2}\right)}\overset{lemma}{\ge}$$

$$\ge\sum\sqrt{2R\left(n_a-\sqrt{4r^2+(b-c)^2}\right)}\overset{lemma}{\ge}$$

$$\ge\sum\sqrt{2R\left(n_a-\frac{n_a}{h_a}.2r\right)}=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R(h_a-2r)}=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R.\frac{2r(s-a)}{a}}=$$

$$=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R.\frac{2r(s-a)}{4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}}.\overset{(1)}{=}$$

$$=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{2R.\frac{2r\frac{r}{\tan(\frac{A}{2})}}{4R\sin\left(\frac{A}{2}\right)\cos\left(\frac{A}{2}\right)}}.=\sum\sqrt{\frac{n_a}{h_a}}\sqrt{\frac{r^2}{\sin^2\left(\frac{A}{2}\right)}}.=\sum AI\sqrt{\frac{n_a}{h_a}}$$

Equality holds for an equilateral triangle.

3832. In any Δ ABC the following relationship holds :

$$2\sum_{cyc}\frac{n_a}{\sqrt{4r^2+(n_a-g_a)^2}}\ge3+\sum_{cyc}\frac{b+c}{a}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(n_a-g_a)^2=n_a^2+g_a^2-2n_ag_a\overset{\text{Bogdan Fustei}}{\le}$$

$$(b-c)^2+2s(s-a)-2s(s-a)=(b-c)^2$$

$$\therefore\sqrt{4r^2+(n_a-g_a)^2}\le\sqrt{4r^2+(b-c)^2}\overset{\text{Bogdan Fustei}}{=}\frac{an_a}{s}\Rightarrow\frac{2n_a}{\sqrt{4r^2+(n_a-g_a)^2}}\ge$$

$$\frac{2n_a s}{a n_a}=\frac{a+b+c}{a}\therefore\frac{2n_a}{\sqrt{4r^2+(n_a-g_a)^2}}\geq 1+\frac{b+c}{a}\ \textit{and analogs}$$

$$\therefore 2\sum_{cyc}\frac{n_a}{\sqrt{4r^2+(n_a-g_a)^2}}\geq 3+\sum_{cyc}\frac{b+c}{a}\ \forall\,\Delta\,ABC,$$

$$''='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3833.

If in ΔABC, $A'\in(BC), B'\in(CA), C'\in(AB)$ and

$$\frac{BA'}{CA'}=\frac{CB'}{AB'}=\frac{AC'}{BC'}=5\ \textit{ then prove that:}$$

$$AA'^2+BB'^2+CC'^2>\frac{31\sqrt{3}}{9}F$$

Proposed by Daniel Sitaru-Romania

Solution by Tapas Das-India

$$BC=a, CA=b, AB=c,\frac{BA'}{CA'}=\frac{CB'}{AB'}=\frac{AC'}{BC'}=5$$

$$\textit{then } BA':CA'=5:1\Rightarrow BA'=\frac{5a}{6}, CA'=\frac{a}{6}$$

$$\textit{from } \Delta ABA' \textit{ we have } AA'^2=AB^2+BA'^2-2AB.BA'\cos\angle ABA'$$

$$AA'^2=c^2+\left(\frac{5a}{6}\right)^2-2c.\frac{5a}{6}\cos B\overset{\cos B<1}{>}c^2+\frac{25a^2}{36}-\frac{10ac}{6}$$

$$\textit{Similary:}$$

$$BB'^2=a^2+\frac{25b^2}{36}-\frac{10ab}{6}, CC'=b^2+\frac{25c^2}{36}-\frac{10bc}{6}$$

$$AA'^2+BB'^2+CC'^2>c^2+\frac{25a^2}{36}-\frac{10ac}{6}+a^2+\frac{25b^2}{36}-\frac{10ab}{6}+b^2+\frac{25c^2}{36}-\frac{10bc}{6}=$$

$$=\frac{61}{36}(a^2+b^2+c^2)-\frac{60}{36}(ab+bc+ca)>\frac{61}{36}(a^2+b^2+c^2)-\frac{60}{36}(a^2+b^2+c^2)$$

$$=\frac{31}{36}(a^2+b^2+c^2)>\frac{31}{36}(ab+bc+ca)\overset{Gordon}{\geq}\frac{31}{36}.4\sqrt{3}F=\frac{31\sqrt{3}}{9}F$$

3834. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)}\right) \le 9\frac{R^2}{r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$First, let\ us\ prove\ that:$$
$$\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)} \le \frac{2R}{r}$$

$$For\ this:\ \sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)} = 1+\frac{(4R+r)^2}{s^2} \overset{Gerretsen}{\le}$$

$$1+\frac{(4R+r)^2}{16Rr-r^2} = \frac{16R^2+24Rr-4r^2}{16Rr-5r^2} \overset{?}{\le} \frac{2R}{r}$$
$$8R^2+12Rr-2r^2 \le 16Rr-5r^2$$
$$8R^2-17Rr+2r^2 \ge 0$$
$$(R-2r)(8R-r) \ge 0 \to R \ge 2r\ (Euler)\ True$$

$$Now:\ \left(\sum_{cyc} r_a\right)\left(\sum_{cyc}\frac{1}{\cos^2\left(\frac{A}{2}\right)}\right) \le (4R+r).\frac{2R}{r} \overset{Euler}{\le} \frac{9R}{2}.\frac{2R}{r} = \frac{9R^2}{r}$$

$$Equality\ holds\ for\ an\ equilateral\ triangle.$$

3835. In ΔABC the following relationship holds:

$$\sum (h_b-h_c)(b-c)+4s(R-2r) \ge 0$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum (h_b-h_c)(b-c) = \frac{1}{2R}\sum (ac-ab)(b-c) =$$

$$= -\frac{1}{2R}\sum a(b-c)^2 = -\frac{1}{2R}\left(\sum a(b^2+c^2)-6abc\right) =$$
$$= -\frac{1}{2R}\left(\sum a\sum a^2-\sum a^3-6abc\right) =$$
$$= \frac{1}{2R}\left(24Rrs+2s(s^2-6Rr-3r^2)-2s.2(s^2-r^2-4Rr)\right)$$

$$=\frac{s}{R}(14Rr-r^2-s^2)\overset{Gerretsen}{\geq}$$

$$\geq\frac{s}{R}(14Rr-r^2-4R^2-4Rr-3r^2)=\frac{s}{R}(10Rr-4r^2-4R^2)$$

We need to show:

$$\sum(h_b-h_c)(b-c)+4s(R-2r)\geq 0$$

$$\frac{s}{R}(10Rr-4r^2-4R^2)+4s(R-2r)\geq 0 \text{ or } \frac{s}{R}(2Rr-4r^2)\geq 0$$

$$\frac{2sr}{R}(R-2r)\geq 0 \text{ true by Euler.}$$

Equality holds for an equilateral triangle.

3836. In ΔABC the following relationship holds:

$$\sum\frac{(b+c)^2}{w_bw_c}\geq 16$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$w_bw_c\leq\sqrt{s(s-b)}\cdot\sqrt{s(s-c)}=s\sqrt{(s-b)(s-c)}\overset{AM-GM}{\leq}$$

$$\leq s.\frac{s-b+s-c}{2}=\frac{s\big(2s-(b+c)\big)}{2}=\frac{as}{2}$$

$$\sum\frac{(b+c)^2}{w_bw_c}\geq\sum\frac{(b+c)^2}{\frac{as}{2}}=\frac{2}{s}\sum\frac{(b+c)^2}{a}\overset{Bergstrom}{\geq}$$

$$\geq\frac{2}{s}\cdot\frac{\big(2(a+b+c)\big)^2}{a+b+c}=\frac{8(a+b+c)}{s}=8\times\frac{2s}{s}=16$$

Equality holds for an equilateral triangle.

3837. In ΔABC the following relationship holds:

$$\frac{ctg\left(\frac{A}{2}\right)}{ctg^2\left(\frac{B}{2}\right)}+\frac{ctg\left(\frac{B}{2}\right)}{ctg^2\left(\frac{C}{2}\right)}+\frac{ctg\left(\frac{C}{2}\right)}{ctg^2\left(\frac{A}{2}\right)}\geq\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{ctg\left(\frac{A}{2}\right)}{ctg^2\left(\frac{B}{2}\right)}+\frac{ctg\left(\frac{B}{2}\right)}{ctg^2\left(\frac{C}{2}\right)}+\frac{ctg\left(\frac{C}{2}\right)}{ctg^2\left(\frac{A}{2}\right)}\overset{AM-GM}{\geq}3\left(\frac{ctg\left(\frac{A}{2}\right)}{ctg^2\left(\frac{B}{2}\right)}\cdot\frac{ctg\left(\frac{B}{2}\right)}{ctg^2\left(\frac{C}{2}\right)}\cdot\frac{ctg\left(\frac{C}{2}\right)}{ctg^2\left(\frac{A}{2}\right)}\right)^{\frac{1}{3}}=$$

$$=3\left(\prod_{cyc}\tan\left(\frac{A}{2}\right)\right)^{\frac{1}{3}}=3\left(\frac{r}{s}\right)^{\frac{1}{3}}\overset{Mitrinovic}{\geq}3\left(\frac{1}{3\sqrt{3}}\right)^{\frac{1}{3}}=\sqrt{3}$$

$$Equality\ holds\ for\ A=B=C.$$

3838. In ΔABC the following relationship holds:

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}\geq\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\frac{h_a}{b}+\frac{h_b}{c}+\frac{h_c}{a}=\frac{2F}{ab}+\frac{2F}{cb}+\frac{2F}{ac}=$$

$$\frac{2F(a+b+c)}{abc}=\frac{4F.s}{4F.R}=s$$

$$\frac{r_a}{b}+\frac{r_b}{c}+\frac{r_c}{a}\overset{AM-GM}{\geq}3\left(\frac{r_ar_br_c}{abc}\right)^{\frac{1}{3}}=3\left(\frac{rs^2}{4RF}\right)^{\frac{1}{3}}=$$

$$3\left(\frac{rs^2}{4R.rs}\right)^{\frac{1}{3}}=3\left(\frac{s}{4R}\right)^{\frac{1}{3}}$$

$$Let's\ prove\ that:$$

$$3\left(\frac{s}{4R}\right)^{\frac{1}{3}}\geq\frac{S}{R}\rightarrow\frac{27}{4}.\frac{s}{R}\geq\left(\frac{s}{R}\right)^3\rightarrow\frac{27}{4}\geq\left(\frac{s}{R}\right)^2\rightarrow s\leq\frac{3\sqrt{3}}{2}R\ (Mitrinovic)\ True$$

$$Equality\ holds\ for\ a=b=c.$$

3839. In ΔABC the following relationship holds:

$$\frac{3r}{2R}\leq\sum\frac{a}{2a+b+c}\leq\frac{3}{4}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Tapas Das-India

$$\sum\frac{a}{2a+b+c}=\sum\frac{a}{(a+b)+(a+c)}\overset{AM-HM}{\leq}\frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{a}{a+c}\right)=$$

$$=\frac{1}{4}\sum\left(\frac{a}{a+b}+\frac{b}{a+b}\right)=\frac{1}{4}\sum\frac{a+b}{a+b}=\frac{3}{4}$$

$$abc=4Rrs\overset{Euler\ \&\ Mitrinovic}{\geq}4.2r.r.3\sqrt{3}r=\left(2\sqrt{3}r\right)^3(1)$$

$$\sum\frac{a}{2a+b+c}\overset{AM-GM}{\geq}3\sqrt[3]{\frac{abc}{(2a+b+c)(2b+c+a)(2c+a+b)}}\overset{AM-GM}{\geq}$$

$$\geq\frac{\sqrt[3]{abc}}{\frac{2a+b+c+2b+c+a+2c+a+b}{3}}=\frac{3\sqrt[3]{abc}}{4(a+b+c)}\overset{(1)}{\geq}3\times\frac{2\times3\sqrt{3}r}{8s}\overset{Mitrinovic}{\geq}$$

$$\geq\ 3\times\frac{2\times3\sqrt{3}r}{8\times\frac{3\sqrt{3}R}{2}}=\frac{3r}{2R}$$

Equality holds for an equilateral triangle.

3840. In any Δ ABC the following relationship holds :

$$\frac{a-b}{a^2+bc}+\frac{b-c}{b^2+ca}+\frac{c-a}{c^2+ab}\leq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a-b}{a^2+bc}+\frac{b-c}{b^2+ca}+\frac{c-a}{c^2+ab}$$

$$=\frac{1}{(a^2+bc)(b^2+ca)(c^2+ab)}\cdot\sum_{cyc}\left((b-c)(c^2a^2+bc^3+a^3b+abc.b)\right)$$

$$=\frac{1}{(a^2+bc)(b^2+ca)(c^2+ab)}\cdot\begin{pmatrix}abc\sum_{cyc}ab+\sum_{cyc}b^2c^3+\sum_{cyc}a^3b^2+abc\sum_{cyc}a^2-\\\sum_{cyc}a^3b^2-\sum_{cyc}bc^4-abc\sum_{cyc}a^2-abc\sum_{cyc}ab\end{pmatrix}$$

$$=\frac{1}{(a^2+bc)(b^2+ca)(c^2+ab)}\cdot\left(\sum_{cyc}b^2c^3-\sum_{cyc}bc^4\right)\text{ and so, it remains}$$

$$\text{to prove}:\sum_{cyc}bc^4\underset{(*)}{\overset{?}{\geq}}\sum_{cyc}b^2c^3\ \&\ \text{now, we assign}: s-a\equiv x, s-b\equiv y, s-c\equiv z$$

$$\text{and then}: (*)\Leftrightarrow\sum_{cyc}((z+x)(x+y)^4)\overset{?}{\geq}\sum_{cyc}((z+x)^2(x+y)^3)$$

$$\Leftrightarrow\sum_{cyc}x^4y+\sum_{cyc}x^3y^2+\sum_{cyc}x^2y^3\underset{(**)}{\overset{?}{\geq}}3xyz\sum_{cyc}xy$$

$$\text{Now, }\sum_{cyc}x^4y+\sum_{cyc}x^2y^3+\sum_{cyc}x^3y^2\overset{AM-GM}{\geq}2\sum_{cyc}x^3y^2+\sum_{cyc}x^3y^2=3\sum_{cyc}\frac{x^3y^3}{y}$$

$$\overset{Holder}{\geq}3.\frac{\left(\sum_{cyc}xy\right)^3}{3\sum_{cyc}x}\geq\frac{\left(\sum_{cyc}xy\right).3xyz\sum_{cyc}x}{\sum_{cyc}x}=3xyz\sum_{cyc}xy\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\frac{a-b}{a^2+bc}+\frac{b-c}{b^2+ca}+\frac{c-a}{c^2+ab}0\ \forall\ \Delta\ ABC,$$

$$''='' \text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

3841. In acute ΔABC the following relationship holds:

$$\frac{a^2}{\cos(A)}+\frac{b^2}{\cos(B)}+\frac{c^2}{\cos(C)}\geq 72r^2$$

Proposed by Gheorghe Crăciun-Romania

Solution 1 by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a^2}{\cos(A)}\overset{AM-GM}{\geq}3\left(\frac{(abc)^2}{\prod_{cyc}\cos(A)}\right)^{\frac{1}{3}}=$$

$$=3\left(\frac{16R^2F^2}{\prod_{cyc}\cos(A)}\right)^{\frac{1}{3}}\overset{Euler}{\geq}3(8.16.4r^2.p^2r^2)^{\frac{1}{3}}\overset{Mitrinovic}{\geq}3(8.16.27r^2.r^2.r^2)^{\frac{1}{3}}=72r^2$$

Equality holds for an equilateral triangle

Solution 2 by Chew Cheong-Malaysia

$$(\cos(A)+\cos(B)+\cos(C))\sum_{cyc}\frac{a^2}{\cos(A)}\geq(a+b+c)^2$$

$$\sum_{cyc}\frac{a^2}{\cos(A)}\ge\frac{(a+b+c)^2}{(\cos(A)+\cos(B)+\cos(C))}\ge\frac{2(a+b+c)^2}{3}=\frac{2(a+b+c)^3}{3(a+b+c)}\ge$$

$$\ge\frac{2(27abc)}{3(a+b+c)}=\frac{36Rabc}{2R(a+b+c)}=36Rr\ge 72r^2$$

$$Equality\ \ holds\ \ when\ A=B=C=60^\circ$$

$$Note:\cos(A)+\cos(B)+\cos(C)\le\frac{3}{2}$$

$$Euler's\ \ inequality\ :\ \ R\ge 2r$$

3842. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}rr_a\right)\left(\sum_{cyc}\frac{a^2}{h_a}\right)\le 9\frac{R^3(2R-r)}{2r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a^2}{h_a}=\sum_{cyc}\frac{a^2}{\frac{2F}{a}}=\sum_{cyc}\frac{a^3}{2F}=\frac{2(S^3-3Sr^2-6SRr)}{2Sr}=$$

$$\frac{S^2-3r^2-6Rr}{2r}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2-3r^2-6Rr}{r}=$$

$$=\frac{4R^2-2Rr}{r}=\frac{2R(2R-r)}{r}\quad(*)$$

$$\sum_{cyc}rr_a=r(4R+r)\overset{Euler}{\le}\frac{9Rr}{2}\quad(**)$$

$$From\ \ (*)\ \ and\ \ (**)\ \ we\ \ have$$

$$\left(\sum_{cyc}rr_a\right)\left(\sum_{cyc}\frac{a^2}{h_a}\right)\overset{(*),(**)}{\le}\frac{9Rr}{2}\cdot\frac{2R(2R-r)}{r}\overset{Euler}{\le}9\frac{R^3(2R-r)}{2r}$$

$$Equality\ \ holds\ \ for\ \ a=b=c$$

3843.

In any Δ ABC the following relationship holds :

$$\frac{9}{16}\le\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\cos^3 A\right)\le 3-39\left(\frac{r}{R}\right)^4$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\cos^3 A = \sum_{cyc}\cos A - \sum_{cyc}\cos A\sin^2 A$$

$$= \sum_{cyc}\cos A - \sum_{cyc}\sin^2 A + 2\sum_{cyc}\sin^2\frac{A}{2}\sin^2 A$$

$$= \frac{R+r}{R} - \frac{s^2-4Rr-r^2}{2R^2} + \frac{2}{4Rrs.4R^2}\cdot\sum_{cyc}\left(a^3(-s^2+sa+bc)\right)$$

$$= \frac{R+r}{R} - \frac{s^2-4Rr-r^2}{2R^2} +$$

$$\frac{2}{4Rrs.4R^2}\cdot\left(\begin{array}{c}-2s^3(s^2-6Rr-3r^2)+2s((s^2+4Rr+r^2)^2-16Rrs^2-8r^2s^2)+\\ 8Rrs(s^2-4Rr-r^2)\end{array}\right)$$

$$= \frac{4R^3+12R^2r+6Rr^2+r^3-3rs^2}{4R^3} \therefore \left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\cos^3 A\right)$$

$$= \frac{R+r}{R}\cdot\frac{4R^3+12R^2r+6Rr^2+r^3-3rs^2}{4R^3} \to (1)$$

$$\text{Now, } \frac{R+r}{R}\cdot\frac{4R^3+12R^2r+6Rr^2+r^3-3rs^2}{4R^3} \overset{\text{Gerretsen}}{\leq}$$

$$\frac{R+r}{R}\cdot\frac{4R^3+12R^2r+6Rr^2+r^3-3r(16Rr-5r^2)}{4R^3} \overset{?}{\leq} 3-39\left(\frac{r}{R}\right)^4$$

$$\Leftrightarrow 4t^4-8t^3+15t^2+13t-86 \overset{?}{\geq} 0 \quad \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow 4t^3(t-2)+15(t^2-4)+13(t-2) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2$$

$$\therefore \left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\cos^3 A\right) \leq 3-39\left(\frac{r}{R}\right)^4 \text{ and again, via (1) and Gerretsen,}$$

$$\left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\cos^3 A\right) \geq$$

$$\frac{R+r}{R}\cdot\frac{4R^3+12R^2r+6Rr^2+r^3-3r(4R^2+4Rr+3r^2)}{4R^3} \overset{?}{\geq} \frac{9}{16}$$

$$\Leftrightarrow 7t^4+16t^3-24t^2-56t-32 \overset{?}{\geq} 0 \Leftrightarrow (t-2)(7t^3+30t^2+36t+16) \overset{?}{\geq} 0$$

$$\to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \therefore \left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\cos^3 A\right) \geq \frac{9}{16} \text{ and so,}$$

$$\frac{9}{16} \leq \left(\sum_{cyc}\cos A\right)\left(\sum_{cyc}\cos^3 A\right) \leq 3-39\left(\frac{r}{R}\right)^4 \quad \forall \Delta ABC,$$

$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$

3844. **In any Δ ABC the following relationship holds :**

$$36r \le \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \le \frac{4(R^4 - 7r^4)}{r^3}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \overset{Panaitopol}{\le} \sum_{cyc} \left(\frac{Rs}{a}\left(1 + \cot^2 \frac{A}{2}\right)\right)$$

$$= \frac{Rs(s^2 + 4Rr + r^2)}{4Rrs} + \sum_{cyc} \frac{Rs^3 . bc(s-a)^2}{4Rrs . r^2 s^2}$$

$$= \frac{s^2 + 4Rr + r^2}{4r} + \frac{1}{4r^3}(s^2(s^2 + 4Rr + r^2) - 24Rrs^2 + 8Rrs^2)$$

$$= \frac{s^4 - (12Rr - 2r^2)s^2 + r^3(4R + r)}{4r^3} \overset{Gerretsen}{\le} \frac{(4R^2 - 8Rr + 5r^2)s^2 + r^3(4R + r)}{4r^3}$$

$$\overset{Gerretsen}{\le} \frac{(4R^2 - 8Rr + 5r^2)(4R^2 + 4Rr + 3r^2) + r^3(4R + r)}{4r^3} \overset{?}{\le} \frac{4(R^4 - 7r^4)}{r^3}$$

$$\Leftrightarrow 16r(R^3 - 8r^3) \overset{?}{\ge} 0 \to \text{true} \because R \overset{Euler}{\ge} 2r \therefore \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \le \frac{4(R^4 - 7r^4)}{r^3} \text{ and again,}$$

$$\sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \overset{AM-GM}{\ge} 3.\sqrt[3]{\frac{m_a m_b m_c}{\frac{r^2}{16R^2}}} \overset{Lascu + AM-GM}{\ge} 3.\sqrt[3]{16R^2 . \frac{\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}}{r^2}}$$

$$= 3.\sqrt[3]{\frac{16R^2 s^2}{r}} \overset{Gerretsen + Euler}{\ge} 3.\sqrt[3]{\frac{16R^2 . \frac{27Rr}{2}}{r}} = 18R \overset{Euler}{\ge} 36r \therefore \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \ge 36r$$

$$\text{and so, } 36r \le \sum_{cyc} \frac{m_a}{\sin^2 \frac{A}{2}} \le \frac{4(R^4 - 7r^4)}{r^3} \forall \Delta \text{ ABC,}$$

$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$

3845. **In ΔABC the following relationship holds:**

$$\frac{2r}{R} \le \frac{w_a w_b w_c}{r_a r_b r_c} \le \frac{R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$w_aw_bw_c=\prod\frac{2bc}{b+c}\cos\frac{A}{2}=\frac{8abc}{(a+b)(b+c)(c+a)}\prod\cos\frac{A}{2}=\frac{16Rr^2s^2}{s^2+r^2+2Rr}$$

$$\frac{w_aw_bw_c}{r_ar_br_c}=\left(\frac{16Rr^2s^2}{s^2+r^2+2Rr}\right)\cdot\frac{1}{s^2r}=\frac{16Rr}{s^2+r^2+2Rr}\overset{\substack{Mitrinovic\\Euler}}{\geq}$$

$$\geq\frac{16Rr}{\frac{27}{4}R^2+\frac{R^2}{4}+R^2}=\frac{16Rr}{8R^2}=\frac{2r}{R}$$

$$\bullet\ \frac{w_aw_bw_c}{r_ar_br_c}=\frac{16Rr}{s^2+r^2+2Rr}\overset{\substack{Mitrinovic\\Euler}}{\leq}\frac{16Rr}{27r^2+r^2+4r^2}=\frac{R}{2r}$$

Equality holds for an equilateral triangle.

3846. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\big((\sin A-\sin B)(\cot B+\cot C)\big)\geq 0$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\big((\sin A-\sin B)(\cot B+\cot C)\big)=$$

$$=\sum_{cyc}\left((\sin A-\sin B)\left(\sum_{cyc}\cot A-\cot A\right)\right)$$

$$=\left(\sum_{cyc}\cot A\right)\left(\sum_{cyc}(\sin A-\sin B)\right)+\sum_{cyc}\frac{a(b^2+c^2-a^2)(b-a)}{2abc.\,2R}$$

$$=\frac{1}{4Rabc}\cdot\sum_{cyc}\big(a(b^2+c^2-a^2)(b-a)\big)\text{ and so, it remains to prove :}$$

$$\sum_{cyc}\big(a(b^2+c^2-a^2)(b-a)\big)\underset{(*)}{\overset{?}{\geq}}0$$

Now, we assign : $s-a\equiv x, s-b\equiv y, s-c\equiv z$ and then : $(*)\Leftrightarrow$

$$\sum_{cyc}\left((y+z)(x-y)((z+x)^2+(x+y)^2-(y+z)^2)\right)\overset{?}{\geq}0$$

$$\Leftrightarrow\sum_{cyc}x^3y+\sum_{cyc}x^2y^2\overset{?}{\geq}2xyz\sum_{cyc}x\to\text{true}\because\sum_{cyc}x^3y=xyz.\sum_{cyc}\frac{x^2}{z}\overset{\text{Bergstrom}}{\geq}$$

$$\frac{xyz\left(\sum_{cyc}x\right)^2}{\sum_{cyc}x}=xyz\sum_{cyc}x\ \text{ and }\ \sum_{cyc}x^2y^2\geq xyz\sum_{cyc}x\Rightarrow(*)\text{ is true}$$

$$\therefore\sum_{cyc}\left((\sin A-\sin B)(\cot B+\cot C)\right)\geq 0\ \forall\ \Delta\ ABC,$$

$$''=''\ \text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

3847. In ΔABC the following relationship holds:

$$\frac{a^2}{w_ar_a}+\frac{b^2}{w_br_b}+\frac{c^2}{w_cr_c}\geq 4$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$w_a=\frac{2bc}{b+c}\cos\left(\frac{A}{2}\right),r_a=s\cdot\tan\left(\frac{A}{2}\right)\text{ and }\sin\left(\frac{A}{2}\right)=\sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$w_a\cdot r_a=\frac{2bcs}{b+c}\cdot\sin\left(\frac{A}{2}\right)=\frac{2s\sqrt{bc(s-b)(s-c)}}{b+c}$$

$$\frac{b+c}{2}\overset{AM-GM}{\geq}\sqrt{bc}\to\frac{2\sqrt{bc}}{b+c}\leq 1$$

$$w_ar_a\leq s\sqrt{(s-b)(s-c)}\to\sqrt{(s-b)(s-c)}\overset{AM-GM}{\leq}\frac{2s-(b+c)}{2}=\frac{a}{2}$$

$$w_ar_a\leq\frac{as}{2}\to\frac{1}{w_ar_a}\geq\frac{2}{as}\to\frac{a^2}{w_ar_a}\geq\frac{2a}{s}$$

$$\sum_{cyc}\frac{a^2}{w_ar_a}\geq\frac{2}{s}(a+b+c)=\frac{4s}{s}=4$$

Equality holds for $a=b=c$.

3848. If I $-$incenter in ΔABC then:

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\geq\frac{3}{4r^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}=\sum_{cyc}\frac{1}{AI^2}=\sum_{cyc}\frac{1}{\frac{r^2}{sin^2\frac{A}{2}}}=\frac{1}{r^2}\sum_{cyc}sin^2\frac{A}{2}=$$

$$=\frac{1}{r^2}\left(1-\frac{r}{2R}\right)\overset{EULER}{\geq}\frac{1}{r^2}\left(1-\frac{\frac{R}{2}}{2R}\right)=\frac{1}{r^2}\left(1-\frac{1}{4}\right)=\frac{3}{4r^2}$$

Equality holds for $a=b=c$.

3849. **In ΔABC the following relationship holds:**

$$\frac{1+\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}+\frac{1+\sin\left(\frac{B}{2}\right)}{\cos\left(\frac{B}{2}\right)}+\frac{1+\sin\left(\frac{C}{2}\right)}{\cos\left(\frac{C}{2}\right)}\geq 3\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$Let\ :\ f(x)=\frac{1+\sin(x)}{\cos(x)}\ and\ x\in\left(0;\frac{\pi}{2}\right)\ Then\ we\ get$$

$$f'(x)=\frac{1+\sin(x)}{cos^2(x)},\qquad f''(x)=\left(\frac{1+\sin(x)}{cos^2(x)}\right)'=$$

$$=\frac{(1+\sin(x))'.cos^2(x)-(1+\sin(x))\left(cos^2(x)\right)'}{cos^4(x)}=\frac{(1+\sin(x))^2}{cos^3(x)}>0$$

$$The\ function\ f(x)=\frac{1+\sin(x)}{\cos(x)}\ satisfies\ the\ conditions\ of$$

$$Jensen's\ inequality,\quad f(x)\ is\ a\ convex\ function.$$

$$\sum_{cyc}\frac{1+\sin\left(\frac{A}{2}\right)}{\cos\left(\frac{A}{2}\right)}\geq 3.\frac{1+\sin\left(\frac{A+B+C}{6}\right)}{\cos\left(\frac{A+B+C}{6}\right)}=3.\frac{1+\sin\left(\frac{\pi}{6}\right)}{\cos\left(\frac{\pi}{6}\right)}=3.\frac{1+\frac{1}{2}}{\frac{\sqrt{3}}{2}}=3\sqrt{3}$$

Equality holds for $A=B=C$.

3850. **In any Δ ABC the following relationship holds :**

$$\frac{20}{9}(m_a m_b + m_b m_c + m_c m_a) > ab + bc + ca$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$5\sum_{cyc} ab \overset{?}{>} 4\sum_{cyc} m_a m_b \Leftrightarrow 5(s^2 + 4Rr + r^2) \underset{(*)}{\overset{?}{>}} 2\left(\sum_{cyc} m_a\right)^2 - 2\sum_{cyc} m_a^2 \text{ and}$$

$$\because \left(\sum_{cyc} m_a\right)^2 \overset{\text{Chu-Yang}}{\leq} 4s^2 - 16Rr + 5r^2 \therefore \text{in order to prove } (*), \text{it suffices}$$

$$\text{to prove : } 5(s^2 + 4Rr + r^2) + 3(s^2 - 4Rr - r^2) \overset{?}{>} 2(4s^2 - 16Rr + 5r^2)$$

$$\Leftrightarrow 8(5Rr - r^2) \overset{?}{>} 0 \to \text{true} \because 5Rr \overset{\text{Euler}}{\geq} 10r^2 > r^2 \Rightarrow (*) \text{ is true}$$

$$\therefore 5\sum_{cyc} ab > 4\sum_{cyc} m_a m_b \text{ and implementing it on a triangle with sides :}$$

$$\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}, \text{whose medians as a consequence of trivial calculations} =$$

$$\frac{a}{2}, \frac{b}{2}, \frac{c}{2} \text{ respectively, we get : } 5.\frac{4}{9}\sum_{cyc} m_a m_b > \frac{4}{4}\sum_{cyc} ab$$

$$\Rightarrow \frac{20}{9}(m_a m_b + m_b m_c + m_c m_a) > ab + bc + ca \;\forall\, \Delta\, ABC \text{ (QED)}$$

3851. **In any Δ ABC the following relationship holds :**

$$\cos 2A + \cos 2B - \cos 2C \leq \frac{3}{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\cos 2A + \cos 2B - \cos 2C = -2\cos C\cos(A - B) - 2\cos^2 C + 1 \overset{?}{\leq} \frac{3}{2}$$

$$\Leftrightarrow \cos^2 C + \cos C\cos(A - B) + \frac{1}{4} \overset{?}{\geq} 0$$

$$\Leftrightarrow \cos^2 C + \cos C\cos(A - B) + \frac{\cos^2(A - B)}{4} + \frac{1 - \cos^2(A - B)}{4} \overset{?}{\geq} 0$$

$$\Leftrightarrow\left(\cos C+\frac{\cos(A-B)}{2}\right)^2+\frac{1-\cos^2(A-B)}{4}\overset{?}{\geq}0\to \text{true} \because 1\geq\cos^2(A-B),$$

$$''=''\ \text{iff } A=B\wedge\cos C=-\frac{\cos(A-B)}{2}=-\frac{1}{2}\text{ and so,}$$

$$\cos 2A+\cos 2B-\cos 2C\leq\frac{3}{2},''=''\ \text{iff }\left(A=B=\frac{\pi}{6};C=\frac{2\pi}{3}\right)$$

3852. If in $\Delta ABC, I_a, I_b, I_c -$ *excenters* then:

$$[I_aBC]+[I_bAC]+[I_cAB]\geq 3F$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$$Let\ a=x+y, b=y+z, c=x+z, s=x+y+z$$

$$Lemma\ 1(Cesaro):(x+y)(y+z)(x+z)\geq 8xyz$$

$$[I_aBC]=\frac{a.r_a}{2}$$

$$LHS=\sum\left(\frac{r_a.a}{2}\right)=\frac{1}{2}\sum\left(\frac{aF}{s-a}\right)=\frac{F}{2}\sum\frac{1}{s-a}=\frac{F}{2}\sum\frac{x+y}{z}\geq$$

$$\geq\frac{3F}{2}\cdot\sqrt[3]{\frac{(x+y)(y+z)(x+z)}{xyz}}\geq\frac{3F}{2}\sqrt[3]{\frac{8xyz}{xyz}}=\frac{3F}{2}.2=3F$$

$$Equality\ holds\ for\ a=b=c.$$

3853. In ΔABC the following relationship holds:

$$2\sqrt{3}F\leq\sum AH.h_a\leq\frac{9R^2}{2}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$$\sum AH.h_a=\sum_{cyc}2R\cos A\cdot\frac{2F}{a}=\sum_{cyc}2R\cos A\cdot\frac{bc\sin A}{2R\sin A}=$$

$$=\sum bc.\cos(A)=\sum\left(\frac{b^2+c^2-a^2}{2}\right)=\frac{\sum a^2}{2}\geq\frac{4\sqrt{3}F}{2}=2\sqrt{3}F$$

$$\sum AH.h_a=\frac{\sum a^2}{2}\leq\frac{9R^2}{2}\ (Leibniz).Equality\ for\ holds\ a=b=c.$$

3854. *Let k_a,k_b,k_c be the symmedian cevians of ΔABC. Prove that:*

$$\sum[k_a(b^2+c^2)]\geq 216r^3$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

Lemma 1:

$$k_a=\frac{2bc}{b^2+c^2}m_a$$

Proof:

Let the symmedian of vertex A divide the side BC into two parts, m and n, at point D.

$$\frac{b^2}{m}=\frac{c^2}{n}.\quad BD=b^2t,CD=c^2t.$$

$$b^2t+c^2t=a\Rightarrow t=\frac{a}{b^2+c^2}$$

According to Stewart's theorem:

$$k_a^2=\frac{b^2c^2t+c^2b^2t}{a}-b^2c^2t=\frac{2b^2c^2t}{b^2t+c^2t}-b^2c^2t=\frac{2b^2c^2}{b^2+c^2}-b^2c^2.\left(\frac{a}{b^2+c^2}\right)=$$

$$=\frac{b^2c^2(2b^2+2c^2-a^2)}{(b^2+c^2)^2}=\frac{4b^2c^2}{(b^2+c^2)^2}.m_a^2\ \Rightarrow k_a=\frac{2bc}{b^2+c^2}.m_a$$

$$LHS=2\sum bcm_a\geq 6\sqrt[3]{(abc)^2m_am_bm_c}\geq 6\sqrt[3]{16r^2s^2R^2s^2r}=6r\sqrt[3]{16s^4R^2}\geq$$

$$\geq 6r\sqrt[3]{16.729r^4 4r^2}=6r.4.9r^2=216r^3$$

$$Equality\ holds\ for\ a=b=c.$$

3855. In ΔABC the following relationship holds:

$$\left(\sum_{cyc}\frac{ab}{a+b}\right)\left(3+\sum_{cyc}\tan^2\left(\frac{A}{2}\right)\right)\leq 3\sqrt{3}\frac{R^2}{r}$$

Proposed by Kostantinos Geronikolas-Greece

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{ab}{a+b}\le\frac{1}{4}\sum_{cyc}\frac{(a+b)^2}{a+b}=\frac{1}{4}\sum_{cyc}(a+b)=p$$

$$3+\sum_{cyc}tan^2\left(\frac{A}{2}\right)=1+\frac{(4R+r)^2}{p^2}=\frac{p^2+(4R+r)^2}{p^2}$$

$$LHS\le p.\frac{p^2+(4R+r)^2}{p^2}\overset{Gerretsen}{\le}\frac{4R^2+4Rr+3r^2+16R^2+8Rr+r^2}{p}=$$

$$\frac{20R^2+12Rr+4r^2}{p}\overset{Euler}{\le}\frac{27R^2}{p}\overset{Mitrinovic}{\le}\frac{27R^2}{3\sqrt{3}r}=3\sqrt{3}\frac{R^2}{r}$$

Equality holds for $a=b=c$.

3856. *In ΔABC holds :*

$$\sum_{cyc}\left(\frac{m_b+m_c}{a}\right)\ge 3^{\frac{n+2}{2}}\left(\frac{2r}{R}\right)^n,\quad n\in N$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\left(\frac{m_b+m_c}{a}\right)\overset{AM-GM}{\ge}\sum_{cyc}\left(\frac{2\sqrt{m_bm_c}}{a}\right)^n\overset{AM-GM}{\ge}3\left(\frac{8m_am_bm_c}{abc}\right)^{\frac{n}{3}}\ge$$

$$\overset{m_a\ge\sqrt{p(p-a)}}{\ge}3\left(\frac{8p\sqrt{p(p-a)(p-b)(p-c)}}{abc}\right)^{\frac{n}{3}}=3\left(\frac{8pF}{abc}\right)^{\frac{n}{3}}=$$

$$=3\left(\frac{8pF}{4RF}\right)^{\frac{n}{3}}\overset{Mitrinovic}{\ge}3\left(\frac{2.3\sqrt{3}r}{R}\right)^{\frac{n}{3}}=3^{\frac{n+2}{2}}\left(\frac{2r}{R}\right)^{\frac{n}{3}}\ge 3^{\frac{n+2}{2}}\left(\frac{2r}{R}\right)^n$$

Equality holds for $a=b=c$.

3857. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{1}{h_a\cdot\sqrt{\frac{2}{h_b}+\frac{1}{h_c}}}\ge\frac{1}{\sqrt{r}}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } x=\frac{3r}{h_a}, y=\frac{3r}{h_b}, z=\frac{3r}{h_c} \text{ and then : } \sum_{cyc}\frac{1}{h_a.\sqrt{\frac{2}{h_b}+\frac{1}{h_c}}}=\sum_{cyc}\frac{1}{\frac{3r}{x}.\sqrt{\frac{2y}{3r}+\frac{z}{3r}}}$$

$$=\frac{1}{\sqrt{3r}}.\sum_{cyc}\frac{x}{\sqrt{2y+z}}=\frac{1}{\sqrt{3r}}.\sum_{cyc}\frac{x^2}{x.\sqrt{2y+z}}\overset{\text{Bergstrom}}{\geq}\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(x.\sqrt{2y+z}\right)}$$

$$=\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sum_{cyc}\left(\sqrt{x}.\sqrt{2xy+zx}\right)}\overset{\text{CBS}}{\geq}\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{\sum_{cyc}x}.\sqrt{3\sum_{cyc}xy}}$$

$$\geq\frac{1}{\sqrt{3r}}.\frac{\left(\sum_{cyc}x\right)^2}{\sqrt{\sum_{cyc}x}.\sqrt{\left(\sum_{cyc}x\right)^2}}=\frac{1}{\sqrt{3r}}.\sqrt{\sum_{cyc}x}=\frac{\sqrt{3}}{\sqrt{3r}}\left(\because\sum_{cyc}x=\sum_{cyc}\frac{3r}{h_a}=3\right)$$

$$=\frac{1}{\sqrt{r}} \text{ and so, } \sum_{cyc}\frac{1}{h_a.\sqrt{\frac{2}{h_b}+\frac{1}{h_c}}}\geq\frac{1}{\sqrt{r}}\ \forall\,\Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3858. In any Δ ABC the following relationship holds :

$$\sum_{cyc}\frac{a^2}{h_b+h_c}\leq\sum_{cyc}\frac{a^2}{r_b+r_c}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a^2}{h_b+h_c}\overset{?}{\leq}\sum_{cyc}\frac{a^2}{r_b+r_c}\Leftrightarrow 2R\sum_{cyc}\frac{a^2}{ca+ab}\overset{?}{\leq}\sum_{cyc}\frac{16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}}{4R\cos^2\frac{A}{2}}$$

$$\Leftrightarrow 2R\sum_{cyc}\frac{2s-(b+c)}{b+c}\overset{?}{\leq}4R.\frac{2R-r}{2R}$$

$$\Leftrightarrow\frac{2s}{2s(s^2+2Rr+r^2)}.\left(\left(\sum_{cyc}a\right)^2+\sum_{cyc}ab\right)\overset{?}{\leq}\frac{2R-r}{R}+3$$

$$\Leftrightarrow\frac{5s^2+4Rr+r^2}{s^2+2Rr+r^2}\overset{?}{\leq}\frac{5R-r}{R}\Leftrightarrow s^2\overset{?}{\leq}6R^2+2Rr-r^2$$

$$\Leftrightarrow s^2-(4R^2+4Rr+3r^2)-2(R+r)(R-2r)\overset{?}{\leq}0$$

$$\rightarrow \text{true} \because s^2\overset{\text{Gerretsen}}{\leq}4R^2+4Rr+3r^2 \text{ and } R\overset{\text{Euler}}{\geq}2r$$

$$\therefore \sum_{cyc} \frac{a^2}{h_b + h_c} \le \sum_{cyc} \frac{a^2}{r_b + r_c}\ \forall\ \Delta\ ABC, ''='' \text{ iff } \Delta\ ABC \text{ is equilateral (QED)}$$

3859. In ΔABC the following relationship holds:

$$12r \le \sum \frac{m_a}{\cos^2 \frac{A}{2}} \le \frac{3R^2}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$WLOG\ a \ge b \ge c \text{ then } m_a \le m_b \le m_c \text{ and } \cos\frac{A}{2} \le \cos\frac{B}{2} \le \cos\frac{C}{2}$$

$$\sum \frac{m_a}{\cos^2 \frac{A}{2}} \overset{Chebyshev}{\le} \frac{1}{3}\left(\sum m_a\right)\left(\sum \frac{1}{\cos^2\left(\frac{A}{2}\right)}\right) = \frac{1}{3}\left(\sum m_a\right)\left(\sum \sec^2\left(\frac{A}{2}\right)\right) \le$$

$$\overset{Gotman\ II}{\le} \frac{1}{3}\cdot\frac{9R}{2}\left(\frac{s^2+(4R+r)^2}{s^2}\right) \overset{Doucet}{\le} \frac{3R}{2}\left(\frac{\frac{(4R+r)^2}{3}+(4R+r)^2}{3r(4R+r)}\right) =$$

$$= \frac{3R}{2}\cdot\frac{4(4R+r)}{9r} \overset{Euler}{\le} \frac{2R}{3r}\left(\frac{9R}{2}\right) = \frac{3R^2}{r}$$

$$\sum \frac{m_a}{\cos^2 \frac{A}{2}} = \sum \frac{\sec^2 \frac{A}{2}}{\frac{1}{m_a}} \overset{Bergstrom}{\ge} \frac{\left(\sec\frac{A}{2}+\sec\frac{B}{2}+\sec\frac{C}{2}\right)^2}{\frac{1}{m_a}+\frac{1}{m_b}+\frac{1}{m_c}} \overset{\substack{Jensen \\ m_a \ge h_a}}{\ge}$$

$$\ge \frac{\left(3\sec\frac{\pi}{6}\right)^2}{\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}} = \frac{\left(3\times\frac{2}{\sqrt{3}}\right)^2}{\frac{1}{r}} = 12r$$

Equality holds for an equilateral triangle.

3860. In ΔABC the following relationship holds:

$$\frac{2}{r} \le \sum \frac{h_b + h_c}{h_a^2} \le \frac{R}{r^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$\sum\frac{h_b+h_c}{h_a^2}=\sum\frac{h_a+h_b+h_c-h_a}{h_a^2}=\sum h_a\cdot\sum\frac{1}{h_a^2}-\sum\frac{1}{h_a}=$$

$$=\frac{bc+ca+ab}{2R}\cdot\frac{a^2+b^2+c^2}{4r^2s^2}-\frac{1}{r}=\frac{(s^2+r^2+4Rr)}{2R}\frac{2(s^2-r^2-4Rr)}{4r^2s^2}-\frac{1}{r}$$

$$=\frac{s^4-r^2(4R+r)^2}{4Rr^2s^2}-\frac{1}{r}=\frac{s^2}{4Rr^2}-\frac{1}{4R}\left(\frac{4R+r}{s}\right)^2-\frac{1}{r}\overset{Doucet\ Gerretsen}{\leq}$$

$$\leq\frac{4R^2+4Rr+3r^2}{4Rr^2}-\frac{3}{4R}-\frac{1}{r}=\frac{R}{r^2}+\frac{1}{r}+\frac{3}{4R}-\frac{3}{4R}-\frac{1}{r}=\frac{R}{r^2}$$

$$\sum\frac{1}{h_b+h_c}\overset{AM-GM}{\leq}\frac{1}{4}\sum\left(\frac{1}{h_b}+\frac{1}{h_c}\right)=\frac{1}{2}\sum\frac{1}{h_a}=\frac{1}{2r}\ (1)$$

$$\sum\frac{h_b+h_c}{h_a^2}=\sum\frac{\frac{1}{h_a^2}}{\frac{1}{h_b+h_c}}\overset{Bergstrom}{\geq}\frac{\left(\sum\frac{1}{h_a}\right)^2}{\sum\frac{1}{h_b+h_c}}\overset{(1)}{\geq}\frac{\left(\frac{1}{r}\right)^2}{\frac{1}{2r}}=\frac{2}{r}$$

Equality holds for an equilateral triangle.

3861. In any Δ ABC the following relationship holds :

$$\frac{r}{2R^3}\leq\sum_{cyc}\frac{\cos A}{a(b+c)}\leq\frac{R^3}{128r^5}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cos A}{a(b+c)}\ \boxed{=}\ \frac{1}{2abc.2s(s^2+2Rr+r^2)}\cdot\sum_{cyc}\left((b^2+c^2-a^2)\left(a^2+\sum_{cyc}ab\right)\right)$$

$$=\frac{1}{16Rrs^2(s^2+2Rr+r^2)}\cdot\left(\left(2\sum_{cyc}a^2b^2-\sum_{cyc}a^4\right)+\left(\sum_{cyc}ab\right)\left(\sum_{cyc}a^2\right)\right)$$

$$\boxed{=}\ \frac{8r^2s^2+s^4-(4Rr+r^2)^2}{8Rrs^2(s^2+2Rr+r^2)} \overset{?}{\geq} \frac{r}{2R^3}$$

$$\Leftrightarrow (R^2-4r^2)s^4+r^2(8R^2-8Rr-4r^2)s^2-R^2r^2(4R+r)^2 \boxed{\overset{?}{\underset{(*)}{\geq}}} 0$$

Now, LHS of $(*) \overset{\text{Gerretsen}}{\geq} (R^2-4r^2)(16Rr-5r^2)s^2+r^2(8R^2-8Rr-4r^2)s^2-$
$R^2r^2(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow (16R^3+3R^2r-72Rr^2+16r^3)s^2 \boxed{\overset{?}{\underset{(**)}{\geq}}} R^2r(4R+r)^2$

and again, RHS of $(**) \overset{\text{Gerretsen}}{\geq} (16R^3+3R^2r-72Rr^2+16r^3)(16Rr-5r^2)$

$$\left(\begin{matrix}\because 16R^3+3R^2r-72Rr^2+16r^3 \\ =(R-2r)(16R^2+35Rr-2r^2)+12r^3 \overset{\text{Euler}}{\geq} 12r^3>0\end{matrix}\right) \overset{?}{\geq} R^2r(4R+r)^2$$

$$\Leftrightarrow 30R^4-5R^3r-146R^2r^2+77Rr^3-10r^4 \overset{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(30R^3+55R^2r-36Rr^2+5r^3) \overset{?}{\geq} 0 \to \text{true} \because R \overset{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{\text{cyc}} \frac{\cos A}{a(b+c)} \geq \frac{r}{2R^3} \text{ and again}, \sum_{\text{cyc}} \frac{\cos A}{a(b+c)} \overset{?}{\leq} \frac{R^3}{128r^5}$$

$$\Leftrightarrow \frac{8r^2s^2+s^4-(4Rr+r^2)^2}{8Rrs^2(s^2+2Rr+r^2)} \overset{?}{\leq} \frac{R^3}{128r^5}$$

$$\Leftrightarrow (R^4-16r^4)s^4+r\left(2R^5+R^4r-128r^5\right)s^2+16r^6(4R+r)^2 \boxed{\overset{?}{\underset{(\bullet)}{\geq}}} 0 \text{ and now},$$

LHS of $(\bullet) \overset{\text{Gerretsen}}{\geq} (R^4-16r^4)(16Rr-5r^2)s^2+r\left(2R^5+R^4r-128r^5\right)s^2+$
$16r^6(4R+r)^2 \overset{?}{\geq} 0 \Leftrightarrow \left(9R^5-2R^4r-128Rr^4-24r^5\right)s^2+8r^5(4R+r)^2 \boxed{\overset{?}{\underset{(\bullet\bullet)}{\geq}}} 0$

and it's trivi*ally* true when : $9R^5-2R^4r-128Rr^4-24r^5 \geq 0$ *and* when :

$$9R^5-2R^4r-128Rr^4-24r^5<0, \text{then : LHS of } (\bullet\bullet) \overset{\text{Gerretsen}}{\geq}$$
$$\left(9R^5-2R^4r-128Rr^4-24r^5\right)(4R^2+4Rr+3r^2)+8r^5(4R+r)^2$$

$$\Leftrightarrow 36t^7+28t^6+19t^5-6t^4-512t^3-480t^2-416t-64 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\left(36t^6+100t^5+219t^4+432t^3+352t^2+224t+32\right) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (\bullet\bullet) \Rightarrow (\bullet) \text{ is true} \therefore \sum_{\text{cyc}} \frac{\cos A}{a(b+c)} \leq \frac{R^3}{128r^5} \text{ and so, combining,}$$

$$\frac{r}{2R^3} \leq \sum_{\text{cyc}} \frac{\cos A}{a(b+c)} \leq \frac{R^3}{128r^5}\ \forall \Delta\,ABC, ''='' \text{ iff } \Delta\,ABC \text{ is equilateral (QED)}$$

3862. Prove that in any non isosceles triangle ABC holds:

$$\sqrt{\sum_{cyc}\frac{4(a^2-ab+b^2)}{(a-b)^2}}>3$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$4(a^2-ab+b^2)=2\times 2(a^2+b^2)-4ab=$$
$$=2((a+b)^2+(a-b)^2)-((a+b)^2-(a-b)^2)=(a+b)^2+3(a-b)^2$$

$$\frac{4(a^2-ab+b^2)}{(a-b)^2}=\frac{(a+b)^2+3(a-b)^2}{(a-b)^2}=\frac{(a+b)^2}{(a-b)^2}+3>3\ (1)$$

$$\sqrt{\sum_{cyc}\frac{4(a^2-ab+b^2)}{(a-b)^2}}\overset{(1)}{>}\sqrt{3+3+3}=3$$

3863. In ΔABC the following relationship holds:

$$\frac{ctg^2(A)}{\sin(A)}+\frac{ctg^2(B)}{\sin(B)}+\frac{ctg^2(C)}{\sin(C)}\geq\frac{2\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{ctg^2(A)}{\sin(A)}\overset{Bergstrom}{\geq}\frac{\left(\sum_{cyc}ctg\,(A)\right)^2}{\sum_{cyc}\sin\,(A)}=\frac{\left(\frac{s^2-4Rr-r^2}{2sr}\right)^2}{\frac{s}{R}}\overset{Gerretsen}{\geq}$$
$$\geq\frac{(16Rr-5r^2-4Rr-r^2)^2}{4s^2r^2}.\frac{R}{s}=\frac{(12Rr-6r^2)^2}{4s^2r^2}.\frac{R}{s}=$$
$$=\frac{R.(6R-3r)^2}{s^3}\overset{Euler}{\geq}\frac{R.(6R-1.5R)^2}{s^3}\overset{Mitrinovic}{\geq}\frac{8R.20.25R^2}{81\sqrt{3}R^3}=\frac{2\sqrt{3}}{3}$$

Equality holds for $A=B=C$.

3864. In ΔABC the following relationship holds:

$$\frac{\sin\left(\frac{A}{2}\right)}{\sin^2(B)}+\frac{\sin\left(\frac{B}{2}\right)}{\sin^2(C)}+\frac{\sin\left(\frac{C}{2}\right)}{\sin^2(A)}\geq 2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{\sin\left(\frac{A}{2}\right)}{\sin^2(B)} \overset{AM-GM}{\geq} 3\sqrt[3]{\frac{\prod_{cyc}\sin\left(\frac{A}{2}\right)}{\left(\prod_{cyc}\sin(A)\right)^2}} = 3\sqrt[3]{\frac{\prod_{cyc}\sin\left(\frac{A}{2}\right)}{64\prod_{cyc}\left(\sin\left(\frac{A}{2}\right).\cos\left(\frac{A}{2}\right)\right)^2}} =$$

$$= 3\sqrt[3]{\frac{1}{8\left(\prod_{cyc}\sin(A)\right)\left(\prod_{cyc}\cos\left(\frac{A}{2}\right)\right)}} = \frac{3}{2}\sqrt[3]{\frac{1}{\frac{F}{2R^2}\cdot\frac{p}{4R}}} =$$

$$= \frac{3}{2}\sqrt[3]{\frac{8R^3}{F.p}} = 3\sqrt[3]{\frac{R.R^2}{r.p^2}} \overset{Euler\ (R\geq 2r)}{\geq} 3\sqrt[3]{2.\frac{4p^2}{27p^2}} = 2$$

$$Equality\ \ holds\ \ for\ \ A = B = C$$

3865. **In any Δ ABC the following relationship holds :**

$$\frac{a}{\sqrt{b^2+c^2}} + \frac{b}{\sqrt{c^2+a^2}} + \frac{c}{\sqrt{a^2+b^2}} < 2\sqrt{2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{\sqrt{b^2+c^2}} \overset{\text{Reverse CBS}}{\leq} \sqrt{2}.\sum_{cyc}\frac{a}{b+c} = \sqrt{2}.\sum_{cyc}\frac{a}{s+s-a} < \sqrt{2}.\sum_{cyc}\frac{a}{s}$$

$$= \sqrt{2}.\frac{2s}{s} = 2\sqrt{2}\ \forall\ \Delta\ ABC$$

Solution 2 by Amin Hajiyev-Azerbaijan

$$Any\ \Delta ABC \to \begin{cases} a+b > c \\ b+c > a \\ a+c > b \end{cases} \to \begin{cases} \dfrac{c}{a+b} < 1 \\ \dfrac{a}{b+c} < 1 \\ \dfrac{b}{a+c} < 1 \end{cases}$$

$$a < b+c \to\ a+b+c < 2(b+c)$$

$$\to \frac{1}{b+c} < \frac{2}{a+b+c} \to \frac{a}{b+c} < \frac{2a}{a+b+c}$$

$$\sum_{cyc}\frac{a}{b+c}<\sum_{cyc}\frac{2a}{a+b+c}\to\sum_{cyc}\frac{a}{b+c}<2$$

$$(c-b)^2\geq 0\to c^2+b^2\geq 2cb\to c^2+b^2\geq\frac{(c+b)^2}{2}$$

$$\frac{c+b}{\sqrt{2}}\leq\sqrt{c^2+b^2}\to\frac{1}{\sqrt{c^2+b^2}}\leq\frac{\sqrt{2}}{c+b}\to\frac{a}{\sqrt{c^2+b^2}}\leq\frac{a\sqrt{2}}{b+c}$$

$$\sum_{cyc}\frac{a}{\sqrt{b^2+c^2}}\leq\sqrt{2}\sum_{cyc}\frac{a}{b+c}$$

$$\sum_{cyc}\frac{a}{b+c}<2\to\sum_{cyc}\frac{a}{\sqrt{b^2+c^2}}<2\sqrt{2}$$

3866. In ΔABC the following relationship holds:

$$\sqrt{\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}}}+\sqrt{\frac{\tan\frac{C}{2}\tan\frac{A}{2}}{\tan\frac{B}{2}}}\geq\sqrt[4]{27}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$We\ know\ that\ in\ any\ triangle\ \sum\tan\frac{A}{2}\tan\frac{B}{2}=1(1)$$

$$let\ \tan\frac{A}{2}=x,\tan\frac{B}{2}=y,\tan\frac{C}{2}=z\ then\ xy+yz+zx\overset{(1)}{=}1\ (2)$$

$$\sqrt{\frac{\tan\frac{A}{2}\tan\frac{B}{2}}{\tan\frac{C}{2}}}+\sqrt{\frac{\tan\frac{B}{2}\tan\frac{C}{2}}{\tan\frac{A}{2}}}+\sqrt{\frac{\tan\frac{C}{2}\tan\frac{A}{2}}{\tan\frac{B}{2}}}=$$

$$=\sqrt{\frac{xy}{z}}+\sqrt{\frac{yz}{x}}+\sqrt{\frac{zx}{y}}=\frac{xy+yz+zx}{\sqrt{xyz}}\overset{(2)}{=}\frac{1}{\sqrt{xyz}}=\frac{1}{\sqrt[4]{x^2y^2z^2}}=$$

$$=\sqrt[4]{\frac{1}{xy.yz.zx}}\overset{AM-GM}{\geq}\sqrt[4]{\frac{27}{(xy+yz+zx)^3}}\overset{(2)}{=}\sqrt[4]{27}$$

$$Equality\ holds\ for\ x=y=z=\frac{1}{\sqrt{3}}\Rightarrow A=B=C=\frac{\pi}{3}$$

3867. In any acute Δ ABC the following relationship holds :

$$\frac{\cos A}{1+\tan^2 A}+\frac{\cos B}{1+\tan^2 B}+\frac{\cos C}{1+\tan^2 C}\geq\frac{3}{8}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{\cos A}{1+\tan^2 A}=\sum_{cyc}\cos^3 A=$$

$$=3\prod_{cyc}\cos A+\frac{1}{2}\left(\sum_{cyc}\cos A\right)\left(3\sum_{cyc}\cos^2 A-\left(\sum_{cyc}\cos A\right)^2\right)$$

$$=\frac{3(s^2-(2R+r)^2)}{4R^2}+\frac{R+r}{2R}.\left(3\left(3-\frac{s^2-4Rr-r^2}{2R^2}\right)-\frac{(R+r)^2}{R^2}\right)$$

$$=\frac{3(s^2-(2R+r)^2)-2(R+r)^3+18R^2(R+r)-3(R+r)(s^2-4Rr-r^2)}{4R^3}\overset{?}{\geq}\frac{3}{8}$$

$$\Leftrightarrow 5R^3+24R^2r+12Rr^2+2r^3\underset{(*)}{\overset{?}{\geq}}6rs^2$$

$$\text{Now, }6rs^2\overset{\text{Gerretsen}}{\leq}6r(4R^2+4Rr+3r^2)\overset{?}{\leq}R^3+24R^2r+12Rr^2+2r^3$$

$$\Leftrightarrow 5R^3-12Rr^2-16r^3\overset{?}{\geq}0\Leftrightarrow(R-2r)(5R^2+10Rr+8r^2)\overset{?}{\geq}0\to\text{true}$$

$$\because R\overset{\text{Euler}}{\geq}2r\Rightarrow(*)\text{ is true}\therefore\frac{\cos A}{1+\tan^2 A}+\frac{\cos B}{1+\tan^2 B}+\frac{\cos C}{1+\tan^2 C}\geq\frac{3}{8}$$

$\forall\,\Delta$ ABC (QED)," =" iff Δ ABC is equilateral (QED)

3868.

Let ΔDEF be the Gergonne's triangle of ΔABC.
I_a, I_b, I_c – excenters of ΔABC. Prove that:

$$9\sqrt{3}Rr\geq\sum EF.AI_a\geq 18\sqrt{3}r^2$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$$AI_a = \frac{r_a}{\sin\left(\frac{A}{2}\right)}$$

$$\frac{FE}{\sin\left(\measuredangle FDE = 90 - \frac{A}{2}\right)} = 2r \Rightarrow FE = 2r\cos\left(\frac{A}{2}\right)$$

$$EF.AI_a = 2r\cos\left(\frac{A}{2}\right)\cdot\frac{r_a}{\sin\left(\frac{A}{2}\right)} = 2rr_a\cdot\cot\left(\frac{A}{2}\right) = 2rr_a\cdot\frac{s}{r_a} = 2rs$$

$$\sum EF\cdot AI_a = \sum 2rs = 6rs$$

$$6rs \ge 6r\cdot 3\sqrt{3}r = 18\sqrt{3}r^2 \ (1)$$

$$6rs \le 6r\cdot\frac{3\sqrt{3}R}{2} = 9\sqrt{3}Rr \ \ (2)$$

$$(1)\wedge(2) \Rightarrow 9\sqrt{3}Rr \ge \sum EF\cdot AI_a \ge 18\sqrt{3}r^2$$

$$Equality\ holds\ for\ a = b = c.$$

3869. If H −orthocenter in acute ΔABC then:

$$HA + HB + HC \ge 6r$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$HA + HB + HC = \sum_{cyc} HA = 2R\sum_{cyc} cosA = 2R\left(1 + \frac{r}{R}\right) =$$

$$= 2R + 2r \overset{EULER}{\ge} 2\cdot 2r + 2r = 6r$$

Equality holds for an equilateral triangle.

3870. In ΔABC the following relationship holds:

$$w_a^2 + w_b^2 + w_c^2 \le \frac{27R^2}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$w_a^2+w_b^2+w_c^2=\sum_{cyc} w_a^2 \le \sum_{cyc} s(s-a)=s\left(\sum_{cyc} s-\sum_{cyc} a\right)=$$

$$=s(3s-2s)=s^2 \overset{MITRINOVIC}{\le} \left(\frac{3\sqrt{3}R}{2}\right)^2=\frac{27R^2}{4}$$

Equality holds for an equilateral triangle.

3871. In ΔABC the following relationship holds:

$$a^2\cos\frac{A}{2}+b^2\cos\frac{B}{2}+c^2\cos\frac{C}{2}\ge 6F$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$a^2\cos\frac{A}{2}+b^2\cos\frac{B}{2}+c^2\cos\frac{C}{2} \overset{CEBYSHEV}{\ge} \frac{1}{3}\sum_{cyc} a^2\cdot\sum_{cyc}\cos\frac{A}{2}\ge$$

$$\overset{IONESCU-WEITZENBOCK}{\ge} \frac{1}{3}\cdot 4\sqrt{3}F\cdot\sum_{cyc}\cos\frac{A}{2} \overset{JENSEN}{\ge} \frac{1}{3}\cdot 4\sqrt{3}F\cdot 3\cos\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=$$

$$=4\sqrt{3}F\cos\left(\frac{\pi}{6}\right)=4\sqrt{3}F\cdot\frac{\sqrt{3}}{2}=6F$$

Equality holds for an equilateral triangle.

3872. If I –incenter in ΔABC then holds:

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\ge\frac{3}{4r^2}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Daniel Sitaru-Romania

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}=\sum_{cyc}\frac{1}{AI^2}=\sum_{cyc}\frac{1}{\frac{r^2}{\sin^2\frac{A}{2}}}=\frac{1}{r^2}\sum_{cyc}\sin^2\frac{A}{2}=$$

$$=\frac{1}{r^2}\left(1-\frac{r}{2R}\right)\overset{EULER}{\geq}\frac{1}{r^2}\left(1-\frac{\frac{R}{2}}{2R}\right)=\frac{1}{r^2}\left(1-\frac{1}{4}\right)=\frac{3}{4r^2}$$

Equality holds for an equilateral triangle.

3873. In any Δ ABC the following relationship holds :

$$\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\leq\frac{b^2c^2}{16r^4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Let } s_0=\text{semiperimeter and then : }\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\overset{?}{\leq}\frac{b^2c^2}{16r^4}$$

$$\Leftrightarrow\frac{s_0}{r}.\frac{s_0(s_0-a)}{r.s_0}\overset{?}{\leq}\frac{16R^2r^2s_0^2}{16r^4.a^2}\Leftrightarrow a^2(s_0-a)\overset{?}{\leq}R^2s_0$$

$$\Leftrightarrow 16R^2\cos^2\frac{A}{2}\sin^2\frac{A}{2}.4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\overset{?}{\leq}R^2.4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

$$\Leftrightarrow 16s^2(1-s^2)(c-s)\overset{?}{\leq}s+c\ \left(s=\sin\frac{A}{2},c=\cos\frac{B-C}{2}\right)$$

$$\Leftrightarrow(16s^2(1-s^2)-1)c\underset{(*)}{\overset{?}{\leq}}s+16s^3(1-s^2)$$

Case 1 $16s^2(1-s^2)>1$ and then, since $c\leq 1\therefore$ LHS of $(*)\leq 16s^2(1-s^2)-1$

$$\overset{?}{\leq}s+16s^3(1-s^2)\Leftrightarrow 16s^2(1-s)\overset{?}{\leq}1+16s^3(1-s)\Leftrightarrow 16s^2(1-s)^2\overset{?}{\leq}1$$

$$\Leftrightarrow 4s(1-s)\overset{?}{\leq}1\Leftrightarrow 4s^2-4s+1\overset{?}{\geq}0\Leftrightarrow(1-2s)^2\overset{?}{\geq}0\to\text{true}\Rightarrow(*)\text{ is true}$$

Case 2 $16s^2(1-s^2)\leq 1$ and then, since $c=\cos\frac{B-C}{2}=\frac{b+c}{a}.\sin\frac{A}{2}>\sin\frac{A}{2}=s$

$$\therefore\text{LHS of }(*)\leq(16s^2(1-s^2)-1)s\overset{?}{\leq}s+16s^3(1-s^2)\Leftrightarrow 2s\overset{?}{\geq}0\to\text{true}$$

(strict inequality) $\Rightarrow(*)$ is true $\therefore$ combining both cases, $(*)$ is true $\forall\,\Delta$ ABC

$$\therefore\cot^2\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\leq\frac{b^2c^2}{16r^4}\ \forall\,\Delta\text{ ABC},''=''\text{ iff }\cos\frac{B-C}{2}=1\text{ and }\sin\frac{A}{2}=\frac{1}{2}$$

$\Rightarrow ''=''$ iff Δ ABC is equilateral (QED)

3874. In any Δ ABC the following relationship holds :

$$\frac{a}{l_b+l_c}+\frac{b}{l_c+l_a}+\frac{c}{l_a+l_b}\geq\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{a}{l_b+l_c}=\sum_{cyc}\frac{a^2}{a(l_b+l_c)}\overset{\text{Bergstrom}}{\geq}\frac{4s^2}{\sum_{cyc}(a(l_b+l_c))}=\frac{4s^2}{\sum_{cyc}((b+c)l_a)}$$

$$=\frac{4s^2}{\sum_{cyc}\left((b+c)\frac{2bc}{b+c}\cos\frac{A}{2}\right)}=\frac{2s^2}{\sum_{cyc}\left(bc.\sqrt{\frac{s(s-a)}{bc}}\right)}=\frac{2s^2}{\sum_{cyc}\left(\sqrt{bc}.\sqrt{s(s-a)}\right)}$$

$$\overset{\text{CBS}}{\geq}\frac{2s^2}{\sqrt{\sum_{cyc}ab}.\sqrt{s\sum_{cyc}(s-a)}}\geq\frac{2s^2}{\sqrt{\frac{4s^2}{3}}.\sqrt{s^2}}=\sqrt{3}\text{ and so,}$$

$$\frac{a}{l_b+l_c}+\frac{b}{l_c+l_a}+\frac{c}{l_a+l_b}\geq\sqrt{3}\ \forall\ \Delta\ ABC,''=''\text{ iff }\Delta\ ABC\text{ is equilateral (QED)}$$

3875. In any acute Δ ABC the following relationship holds :

$$\frac{m_a^2+m_b^2}{\cos C}+\frac{m_b^2+m_c^2}{\cos A}+\frac{m_c^2+m_a^2}{\cos B}\geq 27R^2$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\text{We have }\forall\ m,n,p,u,v,w>0,(n+p)u+(p+m)v+(m+n)w\geq$$

$$2\sqrt{(mn+np+pm)(uv+vw+wu)}\to(1)$$

$$\text{So, }\sum_{cyc}\frac{m_b^2+m_c^2}{\cos A}=\sum_{cyc}\left(m_a^2\left(\frac{1}{\cos B}+\frac{1}{\cos C}\right)\right)=(n+p)u+(p+m)v+(m+n)w$$

$$\left(m=\frac{1}{\cos A},n=\frac{1}{\cos B},p=\frac{1}{\cos C}\ and\ u=m_a^2,v=m_b^2,w=m_c^2\right)\overset{\text{via }(1)}{\geq}$$

$$2.\sqrt{\sum_{cyc}m_a^2m_b^2}.\sqrt{\sum_{cyc}\frac{1}{\cos B\cos C}}$$

$$=2.\sqrt{\frac{9}{16}\left(\sum_{cyc}a^2b^2\right).\frac{R+r}{R\left(\frac{s^2-4R^2-4Rr-r^2}{4R^2}\right)}}\overset{?}{\geq}27R^2$$

$$\Leftrightarrow(R+r)((s^2+4Rr+r^2)^2-16Rrs^2)\overset{?}{\geq}81R^3(s^2-4R^2-4Rr-r^2)$$

$$\Leftrightarrow(R+r)s^4-(81R^3+8R^2r+6Rr^2-2r^3)s^2+$$

$$324R^5+324R^4r+97R^3r^2+24R^2r^3+9Rr^4+r^5\overset{?}{\underset{(*)}{\geq}}0$$

$$\text{Now, since }P=(s^2-16Rr+5r^2)^2\overset{\text{Gerretsen}}{\geq}0\therefore\text{ in order to prove }(*),$$

$$\text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow 324R^5 + 324R^4r - 159R^3r^2 - 72R^2r^3 + 144Rr^4 - 24r^5 \overset{?}{\underset{(**)}{\geq}}$$

$$(81R^3 - 24R^2r - 16Rr^2 + 8r^3)s^2$$

$$\text{Again, } (81R^3 - 24R^2r - 16Rr^2 + 8r^3)s^2 \overset{\text{Gerretsen}}{\leq}$$

$$(81R^3 - 24R^2r - 16Rr^2 + 8r^3)(4R^2 + 4Rr + 3r^2) \overset{?}{\leq} \text{LHS of } (**)$$

$$\Leftrightarrow 48t^4 - 121t^3 + 16t^2 + 80t - 24 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t-2)\Big((t-2)(48t^2 + 71t + 108) + 228\Big) \overset{?}{\geq} 0 \to \text{true} \because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (**) \Rightarrow (*)$$

$$\text{is true} \therefore \frac{m_a^2 + m_b^2}{\cos C} + \frac{m_b^2 + m_c^2}{\cos A} + \frac{m_c^2 + m_a^2}{\cos B} \geq 27R^2 \ \forall \ acute \ \Delta \ ABC,$$

$$'' ='' \text{ iff } \Delta \text{ ABC is equilateral (QED)}$$

3876. In ΔABC the following relationship holds:

$$\frac{w_a}{b+c} + \frac{w_b}{a+c} + \frac{w_c}{a+b} \leq \frac{3\sqrt{3}}{4}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Amin Hajiyev-Azerbaijan

$$w_a = \frac{2 \cdot b \cdot c \cdot \cos\left(\frac{A}{2}\right)}{b+c} \to \frac{w}{b+c} = \frac{2 \cdot b \cdot c \cdot \cos\left(\frac{A}{2}\right)}{(b+c)^2}$$

$$\frac{b+c}{2} \overset{AM-GM}{\geq} \sqrt{bc} \to \frac{4}{(b+c)^2} \leq \frac{1}{bc} \to \frac{bc}{(b+c)^2} \leq \frac{1}{4}$$

$$\frac{w_a}{b+c} \leq \frac{1}{2}\cos\left(\frac{A}{2}\right) \to \sum_{cyc} \frac{w_a}{b+c} \leq \frac{1}{2}\sum_{cyc} \cos\left(\frac{A}{2}\right)$$

$$RHS = \frac{1}{2}\sum_{cyc} \cos\left(\frac{A}{2}\right) \to f(x) = \cos\left(\frac{x}{2}\right) \to \frac{d^2}{dx^2} f(x) = -\frac{1}{4}\cos\left(\frac{x}{2}\right) < 0$$

$$f(x) \text{ is concave function}$$

$$\frac{\cos\left(\frac{A}{2}\right) + \cos\left(\frac{B}{2}\right) + \cos\left(\frac{C}{2}\right)}{3} \overset{JENSEN}{\leq} \cos\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) \to \angle A + \angle B + \angle C = \pi$$

$$\sum_{cyc} \cos\left(\frac{A}{2}\right) \leq 3\cos\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{2} \to RHS = \frac{1}{2}\sum_{cyc} \cos\left(\frac{A}{2}\right) \leq \frac{3\sqrt{3}}{4}$$

$$\sum_{cyc}\frac{I_a}{b+c}\le\frac{1}{2}\sum_{cyc}\cos\left(\frac{A}{2}\right)\to\sum_{cyc}\frac{I_a}{b+c}\le\frac{3\sqrt{3}}{4}$$

The equality holds for an equilateral triangle.

3877. In ΔABC the following relationship holds:

$$\frac{1+\cot A}{\sin^2 A}+\frac{1+\cot B}{\sin^2 B}+\frac{1+\cot C}{\sin^2 C}\ge\frac{12+4\sqrt{3}}{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$Let\ x=\cot A, y=\cot B, z=\cot C\ then\ \sum xy=\sum cotA.cotB=1\ (1)$$

$$\sum x\ge\sqrt{3(xy+yz+zx)}\overset{(1)}{\triangleq}\sqrt{3}\ (2)$$

$$\frac{1+\cot A}{\sin^2 A}+\frac{1+\cot B}{\sin^2 B}+\frac{1+\cot C}{\sin^2 C}=\sum\frac{1+\cot A}{\sin^2 A}=\sum(1+cotA)\csc^2 A=$$

$$=\sum(1+x)(1+x^2)=\sum(1+x+x^2+x^3)=$$

$$=3+\sum x+\sum x^2+\sum x^3\overset{CBS}{\ge}3+\sum x+\frac{1}{3}\left(\sum x\right)^2+\frac{1}{9}\left(\sum x\right)^3$$

$$\overset{(2)}{\ge}3+\sqrt{3}+\frac{3}{3}+\frac{1}{9}\left(\sqrt{3}\right)^3=4+\sqrt{3}+\frac{\sqrt{3}}{3}=\frac{12+4\sqrt{3}}{3}$$

Equality holds for an equilateral triangle.

3878.

In any Δ ABC the following relationship holds :

$$9(\mathrm{R}-\mathrm{r})\le\sum_{cyc}\frac{\mathrm{m}_a^2}{\mathrm{r}_a}\le\frac{9\mathrm{R}^3}{8\mathrm{r}^2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{m_a^2}{r_a}=\frac{1}{4rs}\cdot\sum_{cyc}\left((s-a)\left(2\sum_{cyc}a^2-3a^2\right)\right)$$

$$=\frac{s^2-4Rr-r^2}{rs}\cdot\sum_{cyc}(s-a)-\frac{3}{4rs}\cdot\left(s\sum_{cyc}a^2-\sum_{cyc}a^3\right)$$

$$=\frac{s^2-4Rr-r^2}{r}-\frac{3}{4rs}\cdot\left(2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)\right)$$

$$=\frac{s^2-7Rr-4r^2}{r}\overset{\text{Gerretsen}}{\leq}\frac{4R^2-3Rr-r^2}{r}\overset{?}{\leq}\frac{9R^3}{8r^2}$$

$$\Leftrightarrow 9R^3-32R^2r+24Rr^2+8r^3\overset{?}{\geq}0\Leftrightarrow(R-2r)\left((R-2r)(9R+4r)+4r^2\right)\overset{?}{\geq}0$$

$$\rightarrow \text{true} \because R\overset{\text{Euler}}{\geq}2r \therefore \sum_{cyc}\frac{m_a^2}{r_a}\leq\frac{9R^3}{8r^2}\ \textit{and again},\sum_{cyc}\frac{m_a^2}{r_a}=\frac{s^2-7Rr-4r^2}{r}$$

$$\overset{\text{Gerretsen}}{\geq}\frac{9Rr-9r^2}{r}\therefore\sum_{cyc}\frac{m_a^2}{r_a}\geq 9(R-r)\ \textit{and}\ \text{so},$$

$$9(R-r)\leq\sum_{cyc}\frac{m_a^2}{r_a}\leq\frac{9R^3}{8r^2}\ \forall\Delta\, ABC,''=''\ \text{iff}\ \Delta\, ABC\ \text{is equil}\textit{ater}\text{al}\ (\text{QED})$$

3879. In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a}{h_a^2}\geq\sum_{cyc}\frac{a}{r_a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sum_{cyc}\frac{a}{h_a^2}=\sum_{cyc}\frac{a^3}{4F^2}=\frac{1}{4F^2}\sum_{cyc}a^3\quad(LHS)$$

$$\sum_{cyc}\frac{a}{r_a^2}=\sum_{cyc}\frac{a}{\frac{F^2}{(p-a)^2}}=\sum_{cyc}\frac{a(p-a)^2}{F^2}=\frac{1}{F^2}\sum_{cyc}a(p-a)^2\quad(RHS)$$

$$\textit{Let's prove that } LHS\overset{?}{\geq}RHS$$

$$\frac{1}{4F^2}\sum_{cyc}a^3\overset{?}{\geq}\frac{1}{F^2}\sum_{cyc}a(p-a)^2$$

$$\sum_{cyc} a^3 \overset{?}{\geq} \frac{1}{F^2}\sum_{cyc} a(2p-2a)^2$$

$$\sum_{cyc} a^3 \overset{?}{\geq} \frac{1}{F^2}\sum_{cyc} a(b+c-a)^2$$

$$\sum_{cyc} a^3 \overset{?}{\geq} \frac{1}{F^2}\sum_{cyc} a^3 - a^2(b+c) - b^2(a+c) - c^2(a+b) + 6abc$$

$$a^2(b+c) + b^2(a+c) + c^2(a+b) \geq 6abc$$

$$a^2.2\sqrt{bc} + b^2.2\sqrt{ac} + c^2.2\sqrt{ab} \overset{A-G}{\geq} 3\sqrt[3]{8a^2.b^2.c^2.\sqrt{bc}.\sqrt{ac}.\sqrt{ab}} \geq$$

$$6\sqrt[3]{(abc)^3} = 6abc$$

$$\textit{Equality holds for } a=b=c.$$

3880. In ΔABC the following relationship holds:

$$6r \leq \sqrt[3]{(h_a+h_b)(h_b+h_c)(h_a+h_c)} \leq 3R$$

Proposed by Marin Chirciu-Romania

Solution by Mirsadix Muzefferov-Azerbaijan

$$\sqrt[3]{(h_a+h_b)(h_b+h_c)(h_a+h_c)} \overset{AM-GM}{\geq}$$

$$\geq \sqrt[3]{2\sqrt{h_ah_b}.2\sqrt{h_bh_c}.2\sqrt{h_ah_c}} = 2\sqrt[3]{h_ah_bh_c} =$$

$$= 2\sqrt[3]{\frac{2F^2}{R}} \geq 2\sqrt[3]{\frac{2S^2r^2}{R}} \geq 2\sqrt[3]{\frac{27Rr.r^2}{R}} = 6r \quad (LHS)$$

$$\sqrt[3]{(h_a+h_b)(h_b+h_c)(h_a+h_c)} \overset{AM-GM}{\leq} \frac{2}{3}(h_a+h_b+h_c) \leq$$

$$\leq \frac{2}{3}(m_a+m_b+m_c) \overset{Gotman}{\leq} \frac{2}{3}.\frac{9R}{2} = 3R \quad (RHS)$$

$$\textit{Equality holds for } a=b=c.$$

3881.

In any Δ ABC the following relationship holds :

$$\prod_{cyc}\frac{h_b+h_c}{2h_a} \leq \frac{R}{2r}.\frac{\sum_{cyc} h_a^2}{\sum_{cyc} h_ah_b}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\prod_{cyc}\frac{h_b+h_c}{2h_a}\overset{?}{\le}\frac{R}{2r}\cdot\frac{\sum_{cyc}h_a^2}{\sum_{cyc}h_ah_b}\Leftrightarrow\frac{1}{8}\cdot\prod_{cyc}\frac{ca+ab}{bc}\overset{?}{\le}\frac{R}{2r}\cdot\frac{\sum_{cyc}b^2c^2}{\sum_{cyc}(bc.ca)}$$

$$\Leftrightarrow\frac{1}{8}\cdot\frac{4Rrs.2s(s^2+2Rr+r^2)}{16R^2r^2s^2}\overset{?}{\le}\frac{R}{2r}\cdot\frac{(s^2+4Rr+r^2)^2-16Rrs^2}{8Rrs^2}$$

$$\Leftrightarrow(R-r)s^4-r(8R^2+r^2)s^2+Rr^2(4R+r)^2\underset{(*)}{\overset{?}{\ge}}0$$

$$\text{Now, LHS of }(*)\overset{\text{Gerretsen}}{\ge}\left((R-r)(16Rr-5r^2)-r(8R^2+r^2)\right)s^2+$$

$$Rr^2(4R+r)^2\overset{?}{\ge}0\Leftrightarrow(8R^2-21Rr+4r^2)s^2+Rr(4R+r)^2\underset{(**)}{\overset{?}{\ge}}0\text{ and it's}$$

trivially true when : $8R^2-21Rr+4r^2\ge 0$ and when : $8R^2-21Rr+4r^2<0$,

$$\text{then : LHS of }(**)\overset{\text{Gerretsen}}{\ge}(8R^2-21Rr+4r^2)(4R^2+4Rr+3r^2)+Rr(4R+r)^2$$

$$\overset{?}{\ge}0\Leftrightarrow 16t^4-18t^3-18t^2-23t+6\overset{?}{\ge}0\left(t=\frac{R}{r}\right)$$

$$\Leftrightarrow(t-2)(16t^3+14t^2+10t-3)\overset{?}{\ge}0\to\text{true}\because t\overset{\text{Euler}}{\ge}2\Rightarrow(**)\Rightarrow(*)\text{ is true}$$

$$\therefore\prod_{cyc}\frac{h_b+h_c}{2h_a}\le\frac{R}{2r}\cdot\frac{\sum_{cyc}h_a^2}{\sum_{cyc}h_ah_b}\ \forall\Delta ABC,''=''\text{ iff }\Delta ABC\text{ is equilateral (QED)}$$

3882. In any Δ ABC the following relationship holds :

$$\sqrt{\frac{R}{r}+3-\sqrt{2}}\le\frac{\sqrt{r_br_c}}{h_a}+\frac{\sqrt{r_cr_a}}{h_b}+\frac{\sqrt{r_ar_b}}{h_c}\le\sqrt{\frac{2R}{r}+1}$$

Proposed by Mohamed Amine Ben Ajiba-Tanger-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{s(s-a)}{h_a^2}=\frac{s}{4r^2s^2}\sum_{cyc}a^2(s-a)$$

$$=\frac{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{4r^2s}=\frac{4Rrs-8r^2s}{4r^2s}\Rightarrow\sum_{cyc}\frac{r_br_c}{h_a^2}\overset{①}{=}\frac{R}{r}+1\text{ and}$$

$$\text{now, }\sum_{cyc}\left(\frac{\sqrt{r_br_c}}{h_a}\cdot\frac{\sqrt{r_cr_a}}{h_b}\right)=R.\frac{\sqrt{s(s-a)}.\sqrt{s(s-b)}.\sqrt{s(s-c)}}{2r^2s^2}\cdot\sum_{cyc}\frac{h_a}{\sqrt{s(s-a)}}$$

$$=\frac{Rs.rs.2rs}{2r^2s^2.\sqrt{s}}\cdot\sum_{cyc}\frac{1}{a.\sqrt{s-a}}=\frac{Rs}{\sqrt{s}}\cdot\sum_{cyc}\frac{1}{a.\sqrt{s-a}}\overset{\text{Bergstrom}}{\ge}\frac{Rs}{\sqrt{s}}\cdot\frac{9}{\sum_{cyc}(a.\sqrt{s-a})}$$

$$= \frac{Rs}{\sqrt{s}} \cdot \frac{9}{\sum_{cyc} \sqrt{a} . \sqrt{a(s-a)}} \overset{CBS}{\geq} \frac{Rs}{\sqrt{s}} \cdot \frac{9}{\sqrt{2s} . \sqrt{\sum_{cyc} a(s-a)}}$$

$$= \frac{Rs}{\sqrt{s}} \cdot \frac{9}{\sqrt{2s} . \sqrt{s(2s) - 2(s^2 - 4Rr - r^2)}} = \frac{Rs}{2s} \cdot \frac{9}{\sqrt{4Rr + r^2}} \overset{Euler}{\geq} \frac{9R}{2 . \sqrt{\frac{9Rr}{2}}}$$

$$\therefore 2 \sum_{cyc} \left(\frac{\sqrt{r_b r_c}}{h_a} . \frac{\sqrt{r_c r_a}}{h_b} \right) \geq 3\sqrt{2} . \sqrt{\frac{R}{r}} \overset{via \text{ ①}}{\Rightarrow} \sum_{cyc} \frac{r_b r_c}{h_a^2} + 2 \sum_{cyc} \left(\frac{\sqrt{r_b r_c}}{h_a} . \frac{\sqrt{r_c r_a}}{h_b} \right) \geq$$

$$\frac{R}{r} + 1 + 3\sqrt{2} . \sqrt{\frac{R}{r}} \overset{?}{\geq} \left(\sqrt{\frac{R}{r}} + 3 - \sqrt{2} \right)^2 = \frac{R}{r} + 11 - 6\sqrt{2} + 2 . \sqrt{\frac{R}{r}} . (3 - \sqrt{2})$$

$$\Leftrightarrow \sqrt{\frac{R}{r}} . (5\sqrt{2} - 6) \overset{?}{\geq} 10 - 6\sqrt{2} \to \text{true} \because \sqrt{\frac{R}{r}} . (5\sqrt{2} - 6) \overset{Euler}{\geq} \sqrt{2} . (5\sqrt{2} - 6)$$

$$(\because 5\sqrt{2} - 6 > 0) = 10 - 6\sqrt{2} \Rightarrow \left(\sum_{cyc} \frac{\sqrt{r_b r_c}}{h_a} \right)^2 \geq \left(\sqrt{\frac{R}{r}} + 3 - \sqrt{2} \right)^2$$

$$\Rightarrow \boxed{\sum_{cyc} \frac{\sqrt{r_b r_c}}{h_a} \geq \sqrt{\frac{R}{r}} + 3 - \sqrt{2}} \text{ and again, } \sum_{cyc} \frac{\sqrt{r_b r_c}}{h_a} = \sum_{cyc} \frac{\sqrt{s(s-a)} . a}{2\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \sum_{cyc} \frac{y+z}{2\sqrt{yz}} \; (x = s-a, y = s-b, z = s-c) =$$

$$\sum_{cyc} \frac{\beta^2 + \gamma^2}{2\beta\gamma} \; (\alpha = \sqrt{x}, \beta = \sqrt{y}, \gamma = \sqrt{z}) = \frac{1}{2\alpha\beta\gamma} \left(\left(\sum_{cyc} \alpha \right) \left(\sum_{cyc} \alpha\beta \right) - 3\alpha\beta\gamma \right)$$

$$\Rightarrow \sum_{cyc} \frac{\sqrt{r_b r_c}}{h_a} - 1 = \frac{1}{2\alpha\beta\gamma} \left(\left(\sum_{cyc} \alpha \right) \left(\sum_{cyc} \alpha\beta \right) - 5\alpha\beta\gamma \right) \overset{?}{\leq} \sqrt{\frac{2R}{r}}$$

$$= \sqrt{\frac{(y+z)(z+x)(x+y)}{2xyz}} = \sqrt{\frac{(\alpha^2 + \beta^2)(\beta^2 + \gamma^2)(\gamma^2 + \alpha^2)}{2\alpha^2\beta^2\gamma^2}} \Leftrightarrow$$

$$2 \left(\left(\sum_{cyc} \alpha^2 \right) \left(\sum_{cyc} \alpha^2\beta^2 \right) - \alpha^2\beta^2\gamma^2 \right) \boxed{\overset{?}{\underset{(*)}{\geq}}} \left(\left(\sum_{cyc} \alpha \right) \left(\sum_{cyc} \alpha\beta \right) - 5\alpha\beta\gamma \right)^2$$

Since $\alpha, \beta, \gamma > 0 \therefore$ assigning $\beta + \gamma = X, \gamma + \alpha = Y, \alpha + \beta = Z \Rightarrow X + Y - Z = 2\gamma > 0, Y + Z - X = 2\alpha > 0, Z + X - Y = 2\beta > 0 \Rightarrow X + Y > Z, Y + Z > X, Z + X > Y$ $\Rightarrow X, Y, Z$ form sides of a triangle XYZ with semiperimeter, circumradius and inradius

$$= s_0, R_0, r_0\ (say); \text{then}: \sum_{cyc} \alpha = s_0, \alpha\beta\gamma = r_0^2 s_0, \sum_{cyc} \alpha\beta = 4R_0 r_0 + r_0^2,$$

$$\sum_{cyc} a^2 = s_0^2 - 8R_0 r_0 - 2r_0^2, \sum_{cyc} \alpha^2\beta^2 = r_0^2\left((4R_0 + r_0)^2 - 2s_0^2\right), \text{and then,}$$

$$(*) \text{ becomes}: 2\left(\left(s_0^2 - 8R_0 r_0 - 2r_0^2\right)\left(r_0^2\left((4R_0 + r_0)^2 - 2s_0^2\right)\right) - r_0^4 s_0^2\right) \overset{?}{\geq}$$

$$\left((s_0)(4R_0 r_0 + r_0^2) - 5r_0^2 s_0\right)^2 \Leftrightarrow$$

$$4r_0^2\left(-s_0^4 + (4R_0^2 + 20R_0 r_0 - 2r_0^2)s_0^2 - r_0(4R_0 + r_0)^3\right) \overset{?}{\geq} 0$$

$$\text{Indeed, Rouche} \Rightarrow s_0^2 - (m - n) \geq 0 \text{ and } s_0^2 - (m + n) \leq 0, \text{where } m =$$

$$2R_0^2 + 10R_0 r_0 - r_0^2 \text{ and } n = 2(R_0 - 2r_0).\sqrt{R_0^2 - 2R_0 r_0}$$

$$\therefore \left(s_0^2 - (m + n)\right)\left(s_0^2 - (m - n)\right) \leq 0 \Rightarrow s_0^4 - (4R_0^2 + 20R_0 r_0 - 2r_0^2)s_0^2 +$$

$$r_0(4R_0 + r_0)^3 \leq 0 \Rightarrow (*) \text{ is true} \therefore \sum_{cyc} \frac{\sqrt{r_b r_c}}{h_a} \leq \sqrt{\frac{2R}{r} + 1}$$

$$\therefore \sqrt{\frac{R}{r} + 3 - \sqrt{2}} \leq \sum_{cyc} \frac{\sqrt{r_b r_c}}{h_a} \leq \sqrt{\frac{2R}{r} + 1}\ \forall \Delta ABC, \text{"}=\text{" iff } a = b = c\ (QED)$$

3883. In ΔABC the following relationship holds:

$$\frac{1 + \cot A}{\sin A} + \frac{1 + \cot B}{\sin B} + \frac{1 + \cot C}{\sin C} \geq 2 + 2\sqrt{3}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\text{Let } x = \cot A, y = \cot B, z = \cot C \text{ then } \sum xy = \sum \cot A.\cot B = 1\ (1)$$

$$\frac{1 + \cot A}{\sin A} + \frac{1 + \cot B}{\sin B} + \frac{1 + \cot C}{\sin C} = \sum \frac{1 + \cot A}{\sin A} =$$

$$= \sum (1 + \cot A)\sqrt{1 + \cot^2 A} = \sum (1 + x)\sqrt{(1 + x^2)} =$$

$$= \sum \sqrt{(1 + x^2)} + \sum \sqrt{(x^2 + x^4)} \overset{Minkowski}{\geq}$$

$$\geq \sqrt{(1 + 1 + 1)^2 + (x + y + z)^2} + \sqrt{(x + y + z)^2 + (x^2 + y^2 + z^2)^2}$$

$$\geq \sqrt{9+3(xy+yz+zx)}+\sqrt{3(xy+yz+zx)+(xy+y+zx)^2} \overset{(1)}{\geq}$$

$$\geq \sqrt{9+3}+\sqrt{3+1} = 2+2\sqrt{3}$$

Equality holds for an equilateral triangle.

3884. In ΔABC the following relationship holds:

$$\sum_{cyc}(\cos A+\cos B)^2+2\sum_{cyc}\cos A \leq 6$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Tapas Das-India

$$\sum_{cyc}(\cos A+\cos B)^2+2\sum_{cyc}\cos A = \sum_{cyc}\left(2\cos\frac{A+B}{2}.\cos\frac{A-B}{2}\right)^2+2\sum_{cyc}\cos A \leq$$

$$\overset{\cos\frac{A-B}{2}\leq 1}{\leq} \sum_{cyc}\left(2\cos\frac{\pi-C}{2}\right)^2+2\sum_{cyc}\cos A = 4\sum \sin^2\left(\frac{C}{2}\right)+2\sum_{cyc}\cos A =$$

$$= 2\sum 2\sin^2\left(\frac{C}{2}\right)+2\sum_{cyc}\cos A = 2\sum(1-\cos C)+2\sum_{cyc}\cos A =$$

$$= 6-2\sum \cos A+2\sum_{cyc}\cos A = 6$$

Equality holds for an equilateral triangle.

3885. *Inspired by a problem of Professor Daniel Sitaru*

In any acute Δ ABC the following relationship holds :

$$8m_a m_b m_c \geq (w_a+w_b)(w_b+w_c)(w_c+w_a) \geq 8s_a s_b s_c$$

Proposed byMohamed Amine Ben Ajiba-Morocco

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{w_a} \overset{CBS}{\leq} \sqrt{\sum_{cyc}\frac{1}{h_a}}.\sqrt{\sum_{cyc}\frac{h_a}{w_a^2}} = \sqrt{\frac{1}{2Rr}.\sum_{cyc}\frac{bc}{bc-\frac{a^2bc}{(b+c)^2}}}$$

$$= \sqrt{\frac{1}{2Rr} \cdot \sum_{cyc} \frac{(s+s-a)^2}{4s(s-a)}} = \sqrt{\frac{1}{2Rr} \cdot \left(\frac{s(4Rr+r^2)}{4r^2s} + \frac{s}{4s} + \frac{3}{2}\right)} = \sqrt{\frac{R+2r}{2Rr^2}} \overset{?}{\leq} \frac{3R+2r}{4Rr}$$

$$\Leftrightarrow (R-2r)^2 \overset{?}{\geq} 0 \to \text{true} \therefore \sum_{cyc} \frac{1}{w_a} \leq \frac{3R+2r}{4Rr} \to (1) \text{ and now, } 8\prod_{cyc} m_a =$$

$$8\prod_{cyc} w_a \cdot \prod_{cyc} \frac{m_a}{w_a} \overset{Lascu}{\geq} 8\prod_{cyc} w_a \cdot \prod_{cyc} \frac{\frac{b+c}{2}\cos\frac{A}{2}}{\frac{2bc\cos\frac{A}{2}}{b+c}} = 8\prod_{cyc} w_a \cdot \prod_{cyc} \frac{(b+c)^2}{4bc}$$

$$= 8\prod_{cyc} w_a \cdot \frac{4s^2(s^2+2Rr+r^2)^2}{64.16R^2r^2s^2} \therefore 8m_am_bm_c \geq \prod_{cyc} w_a \cdot \frac{(s^2+2Rr+r^2)^2}{32R^2r^2} \to (2)$$

$$\text{and again, } \prod_{cyc}(w_b+w_c) = \prod_{cyc} w_a \cdot \left(\left(\sum_{cyc} w_a\right)\left(\sum_{cyc} \frac{1}{w_a}\right) - 1\right)$$

$$\overset{via\ (1)}{\leq} \prod_{cyc} w_a \cdot \left(\sqrt{\frac{2s^2+41Rr+26r^2}{2}} \cdot \left(\frac{3R+2r}{4Rr}\right) - 1\right)$$

(Reference : Inequality in Triangle by Dang Ngoc Minh – 113; published at www.ssmrmh.ro)

$$\overset{via\ (2)}{\Rightarrow} \text{ it suffices to prove : } \frac{(s^2+2Rr+r^2)^2}{32R^2r^2} + 1 \overset{?}{\geq} \sqrt{\frac{2s^2+41Rr+26r^2}{2}} \cdot \left(\frac{3R+2r}{4Rr}\right)$$

$$\Leftrightarrow \frac{((s^2+2Rr+r^2)^2+32R^2r^2)^2}{32R^2r^2} \overset{?}{\geq} (2s^2+41Rr+26r^2)(3R+2r)^2$$

$$\Leftrightarrow s^8 + (8Rr+4r^2)s^6 + r^2(88R^2+24Rr+6r^2)s^4 - r^2(576R^4+480R^3r+80R^2r^2-24Rr^3-4r^4)s^2 -$$

$$r^3(11808R^5+21936R^4r+14944R^3r^2+3240R^2r^3-8Rr^4-r^5) \underset{(*)}{\overset{?}{\geq}} 0 \text{ and } \because P =$$

$$(s^2-2R^2-8Rr-3r^2)^4 + 8(R^2+5Rr+2r^2)(s^2-2R^2-8Rr-3r^2)^3$$

$$\overset{Walker}{\geq} 0 \therefore \text{to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} P$$

$$\Leftrightarrow (3R^4+30R^3r+95R^2r^2+60Rr^3+12r^4)s^4 -$$

$$(8R^6+108R^5r+594R^4r^2+1136R^3r^3+898R^2r^4+312Rr^5+40r^6)s^2 +$$

$$6R^8+104R^7r+712R^6r^2+960R^5r^3+1613R^4r^4+2114R^3r^5+1503R^2r^6+$$

$$460Rr^7+44r^8 \underset{(**)}{\overset{?}{\geq}} 0 \text{ and } \because Q =$$

$$(3R^4+30R^3r+95R^2r^2+60Rr^3+12r^4)(s^2-2R^2-8Rr-3r^2)^2 \overset{Walker}{\geq} 0$$

$$\therefore \text{to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} Q \Leftrightarrow$$

$$\left(2R^6+30R^5r+142R^4r^2+402R^3r^3+340R^2r^4+120Rr^5+16r^6\right)s^2 \overset{?}{\underset{(***)}{\geq}}$$

$$3R^8+56R^7r+428R^6r^2+2372R^5r^3+4521R^4r^4+3830R^3r^5+1572R^2r^6+$$
$$328Rr^7+32r^8 \text{ and finally, LHS of } (***) \overset{Walker}{\geq}$$
$$\left(2R^6+30R^5r+142R^4r^2+402R^3r^3+340R^2r^4+120Rr^5+16r^6\right)\left(2R^2+8Rr+3r^2\right)$$
$$\overset{?}{\geq} \text{RHS of } (***) \Leftrightarrow$$
$$t^8+20t^7+102t^6-342t^5-199t^4+336t^3+440t^2+160t+16 \overset{?}{\geq} 0 \left(t=\frac{R}{r}\right) \Leftrightarrow$$
$$(t-2)\left((t-2)(t^6+24t^5+194t^4+338t^3+377t^2+492t+900)+1792\right) \overset{?}{\geq} 0$$
$$\to \text{true} \because t \overset{Euler}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true } \therefore 8m_am_bm_c \geq \prod_{cyc}(w_b+w_c)$$
$$\forall \text{ acute } \Delta\, ABC \text{ and } \forall \text{ acute } \Delta\, ABC, \frac{m_a}{w_a} \overset{Tsintsifas}{\leq} \frac{b^2+c^2}{2bc} \text{ and analogs} \Rightarrow w_a \geq s_a$$
$$\text{and analogs} \therefore (w_a+w_b)(w_b+w_c)(w_c+w_a) \overset{Cesaro}{\geq} 8w_aw_bw_c \geq 8s_as_bs_c$$
$$\therefore 8m_am_bm_c \geq (w_a+w_b)(w_b+w_c)(w_c+w_a) \geq 8s_as_bs_c\ \forall \text{ acute } \Delta\, ABC,$$
$$'' ='' \text{ iff } \Delta\, ABC \text{ is equilateral (QED)}$$

3886. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{1}{3\cot^2 A+1} \geq \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} = \sum_{cyc} \frac{s^2}{3r_a^2+s^2} =$$
$$= \frac{s^2}{(3r_a^2+s^2)(3r_b^2+s^2)(3r_c^2+s^2)} \cdot \sum_{cyc}\left((3r_b^2+s^2)(3r_c^2+s^2)\right)$$
$$= \frac{s^2\left(9\left(\left(\sum_{cyc} r_ar_b\right)^2-2r_ar_br_c\left(\sum_{cyc} r_a\right)\right)+6\left(\left(\sum_{cyc} r_a\right)^2-2\sum_{cyc} r_ar_b\right)+3s^4\right)}{27r^2s^4+9s^2\left(\left(\sum_{cyc} r_ar_b\right)^2-2r_ar_br_c\left(\sum_{cyc} r_a\right)\right)+3s^4\left(\left(\sum_{cyc} r_a\right)^2-2\sum_{cyc} r_ar_b\right)+s^6}$$
$$= \frac{s^2\left(9\left(s^4-2s^2r(4R+r)\right)+6((4R+r)^2-2s^2)+3s^4\right)}{27r^2s^4+9s^2\left(s^4-2s^2r(4R+r)\right)+3s^4((4R+r)^2-2s^2)+s^6} \overset{?}{\geq} \frac{3}{2}$$

$$\Leftrightarrow 4R^2+8Rr-5r^2-s^2 \overset{?}{\geq} 0 \Leftrightarrow 4R^2+4Rr+3r^2-s^2+4r(R-2r) \overset{?}{\geq} 0 \to \text{true}$$

$$\because 4R^2+4Rr+3r^2 \overset{\text{Gerretsen}}{\geq} s^2 \text{ and } R \overset{\text{Euler}}{\geq} 2r \therefore \sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} \geq \frac{3}{2}\ \forall\, \Delta\, ABC$$

and implementing it on a triangle with angles : $(\pi-2A), (\pi-2B), (\pi-2C)$,

we get : $\sum_{cyc} \frac{1}{3\tan^2\frac{\pi-2A}{2}+1} \geq \frac{3}{2} \Rightarrow \sum_{cyc} \frac{1}{3\cot^2 A+1} \geq \frac{3}{2}$ *and since the latter*

triangle is an acute one $\therefore \sum_{cyc} \frac{1}{3\cot^2 A+1} \geq \frac{3}{2}\ \forall$ *acute* $\Delta\, ABC$,

$'' ='' $ iff $\Delta\, ABC$ is equilateral (QED)

3887. In any $\Delta\, ABC$ the following relationship holds :

$$\sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} \geq \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} = \sum_{cyc} \frac{s^2}{3r_a^2+s^2} =$$

$$= \frac{s^2}{(3r_a^2+s^2)(3r_b^2+s^2)(3r_c^2+s^2)} \cdot \sum_{cyc}\left((3r_b^2+s^2)(3r_c^2+s^2)\right)$$

$$= \frac{s^2\left(9\left(\left(\sum_{cyc} r_a r_b\right)^2 - 2r_a r_b r_c\left(\sum_{cyc} r_a\right)\right) + 6\left(\left(\sum_{cyc} r_a\right)^2 - 2\sum_{cyc} r_a r_b\right) + 3s^4\right)}{27r^2s^4 + 9s^2\left(\left(\sum_{cyc} r_a r_b\right)^2 - 2r_a r_b r_c\left(\sum_{cyc} r_a\right)\right) + 3s^4\left(\left(\sum_{cyc} r_a\right)^2 - 2\sum_{cyc} r_a r_b\right) + s^6}$$

$$= \frac{s^2\left(9\left(s^4-2s^2r(4R+r)\right)+6((4R+r)^2-2s^2)+3s^4\right)}{27r^2s^4+9s^2(s^4-2s^2r(4R+r))+3s^4((4R+r)^2-2s^2)+s^6} \overset{?}{\geq} \frac{3}{2}$$

$$\Leftrightarrow 4R^2+8Rr-5r^2-s^2 \overset{?}{\geq} 0 \Leftrightarrow 4R^2+4Rr+3r^2-s^2+4r(R-2r) \overset{?}{\geq} 0 \to \text{true}$$

$$\because 4R^2+4Rr+3r^2 \overset{\text{Gerretsen}}{\geq} s^2 \text{ and } R \overset{\text{Euler}}{\geq} 2r \therefore \sum_{cyc} \frac{1}{3\tan^2\frac{A}{2}+1} \geq \frac{3}{2}\ \forall\, \Delta\, ABC,$$

$'' ='' $ iff $\Delta\, ABC$ is equilateral (QED)

3888. In ΔABC the following relationship holds:

$$3\sum a^2 \ge 16F^2\sum \frac{1}{a^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tapas Das-India

$$R.H.S = 16F^2\sum \frac{1}{a^2} \overset{Steining}{\le} 16r^2s^2 \times \frac{1}{4r^2} = 4s^2$$

$$L.H.S = 3\sum a^2 = 3\times 2(s^2 - r^2 - 4Rr) = 6\left(s^2 - r(4R+r)\right) \overset{s^2\ge 3r(4R+r)}{\ge}$$

$$\ge 6\left(s^2 - \frac{s^2}{3}\right) = 4s^2,\ \ so\ L.H.S \ge R.H.S$$

Equality holds for an equilateral triangle.

3889. In ΔABC the following relationship holds:

$$7\sum a^4 + 64s^2Rr \ge 5(s^2 + r^2 + 4Rr)^2$$

Proposed by Neculai Stanciu-Romania

Solution by Tapas Das-India

$$ab + bc + ca = s^2 + r^2 + 4Rr,\qquad abc = 4Rrs$$

$$7\sum a^4 + 64s^2Rr \ge 5(s^2 + r^2 + 4Rr)^2$$

$$7\sum a^4 + 8.(2s).(4Rrs) \ge 5(s^2 + r^2 + 4Rr)^2$$

$$7\sum a^4 + 8abc(a+b+c) \ge 5(ab+bc+ca)^2\ \ (1)$$

$$7\sum a^4 + 8abc(a+b+c) =$$

$$= 7\left((a^2+b^2+c^2)^2 - 2(a^2b^2+b^2c^2+c^2a^2)\right) + 8abc(a+b+c) =$$

$$= 7(a^2+b^2+c^2)^2 - 14(a^2b^2+b^2c^2+c^2a^2) + 8abc(a+b+c) \ge$$

$$\overset{\substack{\forall x,y,z>0 \\ x^2+y^2+z^2\ge xy+yz+zx \\ (x+y+z)^2\ge 3(xy+yz+zx)}}{\ge} 7\times 3(a^2b^2+b^2c^2+c^2a^2) - 14(a^2b^2+b^2c^2+c^2a^2)$$

$$+ 8abc(a+b+c) = 7(a^2b^2+b^2c^2+c^2a^2) + 8abc(a+b+c)$$

$$5(ab+bc+ca)^2 = 5(a^2b^2+b^2c^2+c^2a^2) + 10abc(a+b+c)$$

$$\textit{Now from}(1)\textit{we need to show}:$$
$$7(a^2b^2+b^2c^2+c^2a^2)+8abc(a+b+c)\ge$$
$$\ge 5(a^2b^2+b^2c^2+c^2a^2)+10abc(a+b+c)$$

$$2(a^2b^2+b^2c^2+c^2a^2)\ge 2abc(a+b+c)$$
$$\textit{this is true as } 2(a^2b^2+b^2c^2+c^2a^2)\ge 2(ab.bc+bc.ca+ca.ab)=$$
$$=2abc(a+b+c)$$

Equality holds for an equilateral triangle.

3890. **In any acute Δ ABC the following relationship holds:**

$$\frac{1}{IA}+\frac{1}{IB}+\frac{1}{IC}\le\frac{1}{HA}+\frac{1}{HB}+\frac{1}{HC}$$

Proposed by Nguyen Hung Cuong-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{1}{HA}=\frac{1}{4R\cos A\cos B\cos C}.2\sum_{cyc}(\cos B\cos C)=$$
$$=\frac{4R^2}{4R(s^2-(2R+r)^2)}.\left(\left(\frac{R+r}{R}\right)^2-\left(3-\frac{2(s^2-4Rr-r^2)}{4R^2}\right)\right)$$
$$=\frac{4R^2}{4R(s^2-(2R+r)^2)}.\frac{s^2-4R^2+r^2}{2R^2}=\frac{s^2-4R^2+r^2}{2R(s^2-(2R+r)^2)}\overset{?}{\ge}\frac{3}{2r}$$
$$\Leftrightarrow 12R^3+8R^2r+3Rr^2+r^3\underbrace{\overset{?}{\ge}}_{(*)}(3R-r)s^2$$
$$\text{Now, }(3R-r)s^2\overset{Blundon-Gerretsen}{\le}(3R-r).\frac{R(4R+r)^2}{4R-2r}\overset{?}{\le}12R^3+8R^2r+3Rr^2+r^3$$
$$\Leftrightarrow r^2(R^2-Rr-2r^2)\overset{?}{\ge}0\Leftrightarrow r^2(R-2r)(R+r)\overset{?}{\ge}0\to \text{true} \because R\overset{Euler}{\ge}2r$$
$$\Rightarrow(*)\text{ is true }\therefore\sum_{cyc}\frac{1}{HA}\ge\frac{3}{2r}\overset{Jensen}{\ge}\frac{1}{r}\sum_{cyc}\sin\frac{A}{2}=\sum_{cyc}\frac{1}{IA}\ \textit{and so},$$
$$\frac{1}{IA}+\frac{1}{IB}+\frac{1}{IC}\le\frac{1}{HA}+\frac{1}{HB}+\frac{1}{HC}\ \forall\ \textit{acute}\ \Delta\ ABC,''=''\text{ iff }\Delta\text{ ABC is equilateral (QED)}$$

3891. **In acute ΔABC the following relationship holds:**

$$a^2\sqrt{\tan A}+b^2\sqrt{\tan B}+c^2\sqrt{\tan C}\ge 36\sqrt[4]{3}r^2$$

Proposed by Vasile Mircea Popa-Romania

Solution by Tapas Das-India

As triangle is acute then :

$$\sum \tan A \overset{Jensen}{\geq} 3\tan\frac{A+B+C}{3} = 3\tan\frac{\pi}{3} = 3\sqrt{3}$$

$$\textit{We know that in any } \Delta ABC: \sum \tan A = \prod \tan A \geq 3\sqrt{3} = \left(\sqrt{3}\right)^3 \ (1)$$

$$a^2\sqrt{\tan A} + b^2\sqrt{\tan B} + c^2\sqrt{\tan C} \overset{Chebyshev}{\geq} \frac{1}{3}\left(\sum a^2\right)\left(\sum \sqrt{\tan A}\right) \overset{AM-GM}{\geq}$$

$$\geq \frac{1}{3}\left(\sum a^2\right).3.\sqrt[6]{\prod \tan A} \underset{Neuberg}{\overset{(1)}{\geq}} \frac{1}{3}36r^2.\sqrt[4]{3} = 36\sqrt[4]{3}r^2$$

Equality holds for an equilateral triangle.

3892. In any acute Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{h_a^2}{\sqrt{9r^2+3h_a^2}} \leq \frac{9R}{4}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{h_a^2}{\sqrt{9r^2+3h_a^2}} = \sum_{cyc}\left(h_a.\frac{h_a}{\sqrt{9r^2+3h_a^2}}\right) \overset{CBS}{\leq}$$

$$\sqrt{\sum_{cyc} h_a^2}.\sqrt{\frac{1}{3}\sum_{cyc}\frac{3h_a^2+9r^2-9r^2}{9r^2+3h_a^2}} \leq \sqrt{\sum_{cyc} s(s-a)}.\sqrt{1-\sum_{cyc}\frac{r^2}{3r^2+h_a^2}}$$

$$= s.\sqrt{1-\sum_{cyc}\frac{r^2\left(3r^2+h_b^2\right)\left(3r^2+h_c^2\right)}{\left(3r^2+h_a^2\right)\left(3r^2+h_b^2\right)\left(3r^2+h_c^2\right)}}$$

$$= s.\sqrt{1-\frac{27r^6+\frac{6r^4}{4R^2}.\sum_{cyc}a^2b^2+\frac{r^4s^2}{R^2}.\sum_{cyc}a^2}{27r^6+\frac{4r^4s^4}{R^2}+\frac{9r^4}{4R^2}.\sum_{cyc}a^2b^2+\frac{3r^4s^2}{R^2}.\sum_{cyc}a^2}}$$

$$= s.\sqrt{\frac{3\sum_{cyc}a^2b^2+8s^2\sum_{cyc}a^2+16s^4}{108R^2r^2+16s^4+9\sum_{cyc}a^2b^2+12s^2\sum_{cyc}a^2}} \overset{?}{\leq} \frac{9R}{4}$$

$$\Leftrightarrow s^2.\frac{3\sum_{cyc}a^2b^2+8s^2\sum_{cyc}a^2+16s^4}{108R^2r^2+16s^4+9\sum_{cyc}a^2b^2+12s^2\sum_{cyc}a^2}\overset{?}{\leq}\frac{81R^2}{16}$$

$$\Leftrightarrow -560s^6+(3969R^2+1408Rr+160r^2)s^4- r(13608R^3+1254R^2r+384Rr^2+48r^3)s^2+ R^2r^2(20412R^2+5832Rr+729r^2)\underset{(*)}{\overset{?}{\geq}}0$$

; now, Rouche $\Rightarrow s^2-(m-n)\geq 0$ *and* $s^2-(m+n)\leq 0$, where $m=2R^2+10Rr-r^2$ *and* $n=2(R-2r).\sqrt{R^2-2Rr}$

$$\therefore \left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\leq 0\Rightarrow s^4-s^2(2m)+m^2-n^2\leq 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{①}{\leq}0\Rightarrow P= -560s^2(s^4-(4R^2+20Rr-2r^2)s^2+r(4R+r)^3)\geq 0$$

$\therefore$ in order to prove $(*)$, it suffices to prove : LHS of $(*)\overset{?}{\geq}P\Leftrightarrow (1729R^2-9792Rr+1280r^2)s^4+ r(22232R^3+25626R^2r+6336Rr^2+512r^3)s^2+ R^2r^2(20412R^2+5832Rr+729r^2)\underset{(**)}{\overset{?}{\geq}}0$ *and* it's trivi*ally* true if :

$1729R^2-9792Rr+1280r^2\geq 0$ *and* when : $1729R^2-9792Rr+1280r^2<0$,

then : $Q=(1729R^2-9792Rr+1280r^2)\begin{pmatrix}s^4-(4R^2+20Rr-2r^2)s^2+\\ r(4R+r)^3\end{pmatrix}\overset{via\ ①}{\geq}0$

$\therefore$ in order to prove $(**)$, it suffices to prove : LHS of $(**)\overset{?}{\geq}Q$

$$\Leftrightarrow (1729R^4+4411R^3r-42138R^2r^2+12880Rr^3-512r^4)s^2\underset{(***)}{\overset{?}{\geq}} 16r\begin{pmatrix}1729R^5-8815R^4r-5831R^3r^2-861R^2r^3+\\ 87Rr^4+20r^5\end{pmatrix}+13R^4r+R^3r^2+10R^2r^3$$

Case 1 $1729R^4+4411R^3r-42138R^2r^2+12880Rr^3-512r^4\geq 0$ *and* then :

$LHS_{(***)}\overset{Gerretsen}{\geq}(1729R^4+4411R^3r-42138R^2r^2+12880Rr^3-512r^4)\begin{pmatrix}16Rr\\ -5r^2\end{pmatrix}$

$\overset{?}{\geq}RHS_{(***)}\Leftrightarrow 16(6342t^4-18843t^3+13454t^2-2312t+70)+7t^4+4t^3+4t^2$

$\overset{?}{\geq}0\ \left(t=\frac{R}{r}\right)\Leftrightarrow (t-2)(16(6342t^3-6158t^2+1138t-35)+7t^3+2t^2+8t)\overset{?}{\geq}0$

$\rightarrow$ true $\because t\overset{Euler}{\geq}2\Rightarrow (***)$ is true

Case 2 $1729R^4+4411R^3r-42138R^2r^2+12880Rr^3-512r^4<0$ *and* then :

$LHS_{(***)}\overset{Gerretsen}{\geq}\begin{pmatrix}1729R^4+4411R^3r-42138R^2r^2+\\ 12880Rr^3-512r^4\end{pmatrix}(4R^2+4Rr+3r^2)\overset{?}{\geq}$

$RHS_{(***)}\Leftrightarrow 3458t^6-1552t^5-2347t^4-5252t^3-31588t^2+17600t-928\overset{?}{\geq}0$

$\Leftrightarrow (t-2)(3458t^5+5364t^4+8381t^3+11510t^2-8568t+464)\overset{?}{\geq}0\rightarrow$ true

$\because t\overset{Euler}{\geq}2\Rightarrow (***)$ is true $\therefore$ combining both *cases*, $(***)\Rightarrow(**)\Rightarrow(*)$ is true

$$\therefore \sum_{cyc} \frac{h_a^2}{\sqrt{9r^2 + 3h_a^2}} \le \frac{9R}{4} \ \forall \Delta ABC, '' ='' \text{ iff } \Delta ABC \text{ is equilateral}$$

3893. In any acute ΔABC the following relationship holds :

$$\sum_{cyc} \frac{r_a^2}{\sqrt{9r^2 + 3r_a^2}} \le \frac{9R^2}{8r}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc} \frac{r_a^2}{\sqrt{9r^2 + 3r_a^2}} = \sum_{cyc} \left(r_a . \frac{r_a}{\sqrt{9r^2 + 3r_a^2}} \right) \overset{CBS}{\le} \sqrt{\sum_{cyc} r_a^2} . \sqrt{\frac{1}{3} \sum_{cyc} \frac{3r_a^2 + 9r^2 - 9r^2}{9r^2 + 3r_a^2}}$$

$$= \sqrt{(4R+r)^2 - 2s^2} . \sqrt{1 - \sum_{cyc} \frac{r^2}{3r^2 + r_a^2}}$$

$$= \sqrt{(4R+r)^2 - 2s^2} . \sqrt{1 - \sum_{cyc} \frac{r^2 (3r^2 + r_b^2)(3r^2 + r_c^2)}{(3r^2 + r_a^2)(3r^2 + r_b^2)(3r^2 + r_c^2)}}$$

$$= \sqrt{(4R+r)^2 - 2s^2} . \sqrt{1 - \frac{27r^6 + 6r^4 . \sum_{cyc} r_a^2 + r^2 . \sum_{cyc} r_b^2 r_c^2}{27r^6 + r^2 s^4 + 9r^4 . \sum_{cyc} r_a^2 + 3r^2 . \sum_{cyc} r_b^2 r_c^2}} =$$

$$\sqrt{(4R+r)^2 - 2s^2} . \sqrt{\frac{r^2 s^4 + 3r^4((4R+r)^2 - 2s^2) + 2r^2(s^4 - 2r(4R+r)s^2)}{27r^6 + r^2 s^4 + 9r^4((4R+r)^2 - 2s^2) + 3r^2(s^4 - 2r(4R+r)s^2)}} \overset{?}{\le} \frac{9R^2}{8r}$$

$$\overset{\text{squaring and re-arranging}}{\Leftrightarrow} 96r^2 s^6 + (81R^4 - 768R^2r^2 - 896Rr^3 - 368r^4)s^4 -$$
$$r(486R^5 + 486R^4 r - 4096R^3r^2 - 6144R^2r^3 - 2304Rr^4 - 256r^5)s^2 +$$
$$r^2(2916R^6 + 1458R^5 r - 11559R^4r^2 - 12288R^3r^3 - 4608R^2r^4 - 768Rr^5 - 48r^6)$$
$$\underset{(*)}{\overset{?}{\ge}} 0 \text{ and } \because P = 96r^2(s^2 - 16Rr + 5r^2)^3 +$$

$$(81R^4 - 768R^2r^2 + 3712Rr^3 - 1808r^4)(s^2 - 16Rr + 5r^2)^2 \overset{Gerretsen}{\ge} 0$$
$$\left(\begin{array}{c} \because 81t^4 - 768t^2 + 3712t - 1808 \left(t = \frac{R}{r} \right) \\ = (t-2)(81t^3 + 162t^2 - 444t + 2824) + 3840 \overset{Euler}{\ge} 3840 > 0 \end{array} \right)$$

$$\therefore \text{ in order to prove } (*), \text{ it suffices to prove : LHS of } (*) \overset{?}{\ge} P$$

$$\Leftrightarrow (1053R^5 - 648R^4 r - 10240R^3r^2 + 29440R^2r^3 - 23296Rr^4 + 5568r^5)s^2 \underset{(**)}{\overset{?}{\ge}}$$

$$32r\begin{pmatrix}278R^6-225R^5r-2860R^4r^2+10816R^3r^3-10980R^2r^4+4182Rr^5-\\518r^6\end{pmatrix}+$$
$$14R^6-9R^5r+8R^4r^2; \because 1053t^5-648t^4-10240t^3+29440t^2-23296t+5568$$
$$=(t-2)\left((t-2)(1053t^3+3564t^2-196t+14400)+35088\right)+18144$$
$$\overset{Euler}{\geq} 18144>0 \therefore \text{LHS of } (**) \overset{Gerretsen}{\geq}$$
$$(1053R^5-648R^4r-10240R^3r^2+29440R^2r^3-23296Rr^4+5568r^5)\begin{pmatrix}16Rr-\\5r^2\end{pmatrix}$$
$$\overset{?}{\geq} RHS_{(**)} \Leftrightarrow 3969t^6-4212t^5-34544t^4+88064t^3-84288t^2+35872t-$$
$$5632\overset{?}{\geq}0 \Leftrightarrow (t-2)\begin{pmatrix}(t-2)\begin{pmatrix}3969t^4+11664t^3-3764t^2+\\26352t+36176\end{pmatrix}+\\75168\end{pmatrix}\overset{?}{\geq}0 \rightarrow \text{true}$$
$$\because t \overset{Euler}{\geq} 2 \Rightarrow (**) \Rightarrow (*) \text{ is true} \therefore \sum_{cyc}\frac{r_a^2}{\sqrt{9r^2+3r_a^2}} \leq \frac{9R^2}{8r}\ \forall\, \Delta ABC,$$
$$"=" \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3894. ***H, O orhocenter and circumcenter of acute ΔABC.***
O_a, O_b, O_c circumcenters of $\Delta BHC, \Delta AHC, \Delta AHB$.
I_a, I_b, I_c – excenters of ΔABC. Prove that:

$$\sum\frac{OO_a}{AI\cdot AI_a}\leq\frac{1}{3}\left(\frac{1}{r}+\frac{1}{R}\right)$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$$Lemma:\ AI=\frac{r}{\sin\left(\frac{A}{2}\right)} \text{ and } AI_a=\frac{r_a}{\sin\left(\frac{A}{2}\right)}$$
$$Let\ OO_a=2x,\ \cot A=\frac{2x}{a}\Rightarrow 2x=a.\cot A\ \Rightarrow OO_a=2R.\cos A$$
$$LHS=\sum\frac{OO_a}{AI\cdot AI_a}=\sum\frac{2R\cos A}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}}=\sum\frac{2R\cos A\sin^2\left(\frac{A}{2}\right)}{rs.\tan\left(\frac{A}{2}\right)}=\sum\frac{R\cos A\sin A}{rs}=$$
$$=\sum\frac{R\cdot\sin(2A)}{2rs}=\frac{R}{2rs}\sum\sin(2A)=\frac{R}{2rs}\prod\sin(A)=\frac{4R}{2rs}\cdot\frac{abc}{8R^3}=\frac{1}{R}$$

$$LHS=\frac{1}{R}=\frac{1}{3}\left(\frac{2}{R}+\frac{1}{R}\right)\le\frac{1}{3}\left(\frac{2}{2r}+\frac{1}{R}\right)=\frac{1}{3}\left(\frac{1}{r}+\frac{1}{R}\right)$$

Equality holds for $a=b=c$.

3895.

Let Ω *be the Brocard's point of* ΔABC. O_a, O_b, O_c *circumcenters of* $\Delta B\Omega C, \Delta A\Omega C, \Delta A\Omega B$. I, I_a, I_b, I_c *– incenter and excenters of* ΔABC. *Prove that:*

$$\sum\frac{OO_a\cdot c}{AI\cdot AI_a}\ge\frac{9r}{s}$$

Proposed by Sarkhan Adgozalov-Georgia

Solution by Qurban Muellim-Azerbaijan

$$Lemma:\ AI=\frac{r}{\sin\left(\frac{A}{2}\right)},\ AI_a=\frac{r_a}{\sin\left(\frac{A}{2}\right)}\ and\ ab+bc+ca\ge 36r^2$$

$$Let\ OO_{a\cap BC}=K,\ OK=x\ and\ O_aK=y.\ \cot A=\frac{2x}{a}\ and\ \cot C=\frac{2y}{a}\Rightarrow$$

$$\Rightarrow OO_a=\frac{a}{2}\cdot\frac{\sin B}{\sin A\cdot\sin C}=\frac{a}{2}\cdot\frac{\frac{b}{2R}}{\frac{a}{2R}\cdot\frac{c}{2R}}=\frac{bR}{c}$$

$$LHS=\sum\frac{\frac{bR}{c}\cdot c}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}}=\sum\frac{bR}{\frac{rr_a}{\sin^2\left(\frac{A}{2}\right)}}=\frac{R}{r}\sum\frac{b\cdot\sin^2\left(\frac{A}{2}\right)}{r_a}=\frac{R}{r}\sum\frac{b\cdot\sin^2\left(\frac{A}{2}\right)}{s\cdot\tan\left(\frac{A}{2}\right)}=$$

$$=\frac{R}{r}\sum\frac{b\sin A}{2s}=\sum\frac{ab}{4rs}\ge\frac{36r^2}{4rs}=\frac{9r}{s}$$

Equality holds for $a=b=c$.

3896.

In any Δ ABC the following relationship holds :

$$n_a\le AI+\sqrt{(b-c)^2+r^2}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\mathbf{n_a \overset{?}{\le} AI + \sqrt{(b-c)^2 + r^2}}$$

$$\Leftrightarrow \mathbf{n_a^2} \boxed{\underset{(*)}{\overset{?}{\le}}} \mathbf{AI^2 + (b-c)^2 + r^2 + 2AI.\sqrt{(b-c)^2 + r^2}}$$

Let $\mathbf{s_0 = semiperimeter}$ *and* **then :** $\mathbf{n_a^2 - AI^2 - (b-c)^2 - r^2 \overset{Bogdan\ Fustei}{=}}$

$$\mathbf{s_0^2\left(1 - \frac{r}{R}\sec^2\frac{A}{2}\right) - 16R^2\sin^2\frac{B}{2}\sin^2\frac{C}{2} - 16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2} -}$$

$$\mathbf{16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} = 16R^2\cos^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}\left(1 - \frac{2s(c-s)}{1-s^2}\right) -}$$

$$\mathbf{4R^2(c-s)^2 - 16R^2s^2(1-c^2) - 4R^2s^2(c-s)^2 \ \left(s = \sin\frac{A}{2}, c = \cos\frac{B-C}{2}\right)}$$

$$\mathbf{= 4R^2\left((s+c)^2(1+2s^2-2sc) - (c-s)^2 - 4s^2(1-c^2) - s^2(c-s)^2\right)}$$

$$\mathbf{= 8R^2s\big(2(c-s) - c(c-s)(c+s)\big)}$$

$$\mathbf{\therefore n_a^2 - AI^2 - (b-c)^2 - r^2 \boxed{\overset{①}{=}} 8R^2s(c-s)(2-c^2-cs)}$$

$$\mathbf{Also, 2AI.\sqrt{(b-c)^2 + r^2}}$$

$$\mathbf{= 8R\sin\frac{B}{2}\sin\frac{C}{2}.\sqrt{16R^2\sin^2\frac{A}{2}\sin^2\frac{B-C}{2} + 16R^2\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}}}$$

$$\mathbf{= 4R(c-s).\sqrt{16R^2s^2(1-c^2) + 4R^2s^2(c-s)^2}}$$

$$\mathbf{\therefore 2AI.\sqrt{(b-c)^2 + r^2} \boxed{\overset{②}{=}} 8R^2s(c-s).\sqrt{(c-s)^2 + 4(1-c^2)}}$$

$$\mathbf{\therefore ① \text{ and } ② \Rightarrow (*) \Leftrightarrow 2 - c^2 - cs \overset{?}{\le} \sqrt{(c-s)^2 + 4(1-c^2)}}$$

$$\mathbf{\Leftrightarrow c^2 - c^4 + 2cs - 2sc^3 + s^2 - c^2s^2 \overset{?}{\ge} 0}$$

$$\mathbf{\Leftrightarrow c^2(1-c^2) + 2cs(1-c^2) + s^2(1-c^2) \overset{?}{\ge} 0}$$

$$\mathbf{\Leftrightarrow \left(1 - \cos^2\frac{B-C}{2}\right)\left(\sin\frac{A}{2} + \cos\frac{B-C}{2}\right)^2 \overset{?}{\ge} 0 \to true \because \sin\frac{A}{2}, \cos\frac{B-C}{2} > 0}$$

$$\mathbf{and\ \cos^2\frac{B-C}{2} \le 1 \Rightarrow (*)\ is\ true \therefore n_a \le AI + \sqrt{(b-c)^2 + r^2}\ \forall \Delta ABC,}$$

$$\mathbf{'' ='' \text{ iff } b = c \text{ (QED)}}$$

3897.

In *any acute* Δ ABC the *following* relationship *holds* :

$$\mathbf{m_a m_b m_c \ge \frac{(R + 16r)s^2}{18}}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$(m_a m_b m_c)^2 = \frac{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3}{16}$$

$$\left(\begin{array}{c}\text{Reference : Solution to Inequality in Triangle by}\\ \text{Dang Ngoc Minh} - 124;\text{ published at www.ssmrmh.ro}\end{array}\right) \overset{?}{\geq} \frac{(R + 16r)^2 s^4}{324}$$

$$\Leftrightarrow 81s^6 - (4R^2 + 1100Rr - 1649r^2)s^4 - r^2(4860R^2 + 9720Rr + 2673r^2)s^2 -$$

$$81r^3(4R + r)^3 \underset{(*)}{\overset{?}{\geq}} 0 \text{ and } \because 81(s^2 - 2R^2 - 8Rr - 3r^2)^3 \overset{Walker}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove } (*), \text{it suffices to prove : LHS of } (*) \overset{?}{\geq} 81(s^2 - 2R^2 - 8Rr - 3r^2)^3$$

$$\Leftrightarrow (241R^2 + 422Rr + 1189r^2)s^4 -$$

$$(486R^4 + 3888R^3r + 11664R^2r^2 + 10692Rr^3 + 2430r^4)s^2 + 324R^6 + 3888R^5r +$$

$$17010R^4r^2 + 29808R^3r^3 + 23571R^2r^4 + 8262Rr^5 + 1053r^6 \underset{(**)}{\overset{?}{\geq}} 0 \text{ and } \because P =$$

$$(241R^2 + 422Rr + 1189r^2)(s^2 - 2R^2 - 8Rr - 3r^2)^2 \overset{Walker}{\geq} 0 \therefore \text{in order}$$

$$\text{to prove } (**), \text{it suffices to prove : LHS of } (**) \overset{?}{\geq} P$$

$$\Leftrightarrow (239R^4 + 828R^3r + 645R^2r^2 + 5432Rr^3 + 2352r^4)s^2 \underset{(***)}{\overset{?}{\geq}}$$

$$320R^6 + 2756R^5r + 9783R^4r^2 + 25940R^3r^3 + 44609R^2r^4 + 26304Rr^5 + 4824r^6$$

$$\text{and finally, LHS of } (***) \overset{Walker}{\geq}$$

$$(239R^4 + 828R^3r + 645R^2r^2 + 5432Rr^3 + 2352r^4)(2R^2 + 8Rr + 3r^2) \overset{?}{\geq}$$

$$RHS_{(***)} \Leftrightarrow 79t^6 + 406t^5 - 576t^4 - 3716t^3 + 2743t^2 + 4404t + 1116 \overset{?}{\geq} 0$$

$$\left(t = \frac{R}{r}\right) \Leftrightarrow (t - 2)^2(79t^4 + 722t^3 + 1996t^2 + 1380t + 279) \overset{?}{\geq} 0 \to \text{true}$$

$$\because t \overset{Euler}{\geq} 2 \Rightarrow (***) \Rightarrow (**) \Rightarrow (*) \text{ is true and so, } m_a m_b m_c \geq \frac{(R + 16r)s^2}{18}$$

$$\forall \text{ acute ABC}, '' = '' \text{ iff } \Delta\text{ ABC is equilateral (QED)}$$

3898.

In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{s^2 + r^2 - 4rr_a}{p_a^2} + \sum_{cyc} \frac{r^2}{s^2 - n_a^2} = \frac{11}{2}$$

Proposed by Dang Ngoc Minh-Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$p_a=\frac{2s}{2s+a}\cdot\sqrt{s^2-3r^2-16Rr\sin^2\frac{A}{2}}\to (1)$$

(Reference : Solution to Inequality in Triangle by Mohamed Amine Ben Ajiba − 68; relation (•••); published at www.ssmrmh.ro)

$$\text{Now, } s^2-3r^2-16Rr\sin^2\frac{A}{2}=s^2+r^2-4r^2-\frac{16Rra(s-b)(s-c)}{4Rrs}$$
$$=s^2+r^2-\frac{4s(s-a)(s-b)(s-c)}{s^2}-\frac{4a(s-b)(s-c)}{s}$$
$$=s^2+r^2-\frac{4(s-b)(s-c)}{s}.(s-a+a)=s^2+r^2-4(s-b)(s-c)$$
$$=s^2+r^2-\frac{4(s-a)(s-b)(s-c)}{s-a}=s^2+r^2-\frac{4r^2s}{s-a}=s^2+r^2-4rr_a\overset{\text{via }(1)}{\Rightarrow}$$
$$p_a^2=\frac{4s^2}{(2s+a)^2}.(s^2+r^2-4rr_a)\Rightarrow\frac{s^2+r^2-4rr_a}{p_a^2}=\frac{(2s+a)^2}{4s^2}\text{ and analogs}$$
$$\Rightarrow\sum_{cyc}\frac{s^2+r^2-4rr_a}{p_a^2}+\sum_{cyc}\frac{r^2}{s^2-n_a^2}=\sum_{cyc}\frac{(2s+a)^2}{4s^2}+\sum_{cyc}\frac{r^2}{s^2-n_a^2}\overset{\text{Bogdan Fustei}}{=}$$
$$\frac{1}{4s^2}\left(12s^2+4s(2s)+2(s^2-4Rr-r^2)\right)+\sum_{cyc}\frac{r^2}{2r_ah_a}$$
$$=\frac{11s^2-4Rr-r^2}{2s^2}+\sum_{cyc}\frac{r^2}{\frac{r}{R}.s^2\sec^2\frac{A}{2}}=\frac{11s^2-4Rr-r^2}{2s^2}+\frac{Rr}{s^2}.\frac{4R+r}{2R}$$
$$=\frac{11s^2-4Rr-r^2+4Rr+r^2}{2s^2}\therefore\sum_{cyc}\frac{s^2+r^2-4rr_a}{p_a^2}+\sum_{cyc}\frac{r^2}{s^2-n_a^2}=\frac{11}{2}$$
$$\forall\text{ ABC (QED)}$$

3899. In any acute Δ ABC the following relationship holds :

$$\frac{1}{p}\left(\frac{R^2}{r^2}-1\right)\le\sum_{cyc}\frac{AH}{ar_a}\le\frac{1}{p}\left(\frac{2R^2}{r^2}-5\right)$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\sum_{cyc}\frac{AH}{ar_a}=\frac{2R\cos A}{(2R\sin A)p\tan\frac{A}{2}}=\sum_{cyc}\frac{1-\tan^2\frac{A}{2}}{2p\tan^2\frac{A}{2}}=\frac{p}{2}\left(\sum_{cyc}\frac{1}{r_a^2}\right)-\frac{3}{2p}$$

$$= \frac{p}{2}\left(\frac{1}{r^2} - \frac{2(4R+r)}{rp^2}\right) - \frac{3}{2p} \therefore \sum_{cyc} \frac{AH}{ar_a} = \frac{p^2 - 8Rr - 5r^2}{2r^2p} \overset{Gerretsen}{\leq}$$

$$\frac{4R^2 - 4Rr - 2r^2}{2r^2p} \overset{Euler}{\leq} \frac{2R^2 - 4r^2 - r^2}{r^2p} \therefore \sum_{cyc} \frac{AH}{ar_a} \leq \frac{1}{p}\left(\frac{2R^2}{r^2} - 5\right) \textit{ and again},$$

$$\sum_{cyc} \frac{AH}{ar_a} = \frac{p^2 - 8Rr - 5r^2}{2r^2p} \overset{Walker}{\geq} \frac{2R^2 - 2r^2}{2r^2p} \therefore \sum_{cyc} \frac{AH}{ar_a} \geq \frac{1}{p}\left(\frac{R^2}{r^2} - 1\right) \text{ and so},$$

$$\frac{1}{p}\left(\frac{R^2}{r^2} - 1\right) \leq \sum_{cyc} \frac{AH}{ar_a} \leq \frac{1}{p}\left(\frac{2R^2}{r^2} - 5\right) \forall \textit{ acute } \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

3900. In any Δ ABC the following relationship holds :

$$\sum_{cyc} \frac{h_a}{h_b} + \frac{\sum_{cyc} r_a r_b}{\sum_{cyc} r_a^2} \geq \frac{8r}{R}$$

Proposed by Marin Chirciu-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Via AM} - \text{GM}, \sum_{cyc} \frac{h_a}{h_b} + \frac{\sum_{cyc} r_a r_b}{\sum_{cyc} r_a^2} \geq 3 + \frac{s^2}{(4R+r)^2 - 2s^2}$$

$$= \frac{3(4R+r)^2 - 5s^2}{(4R+r)^2 - 2s^2} \overset{Gerretsen}{\geq} \frac{3(4R+r)^2 - 5(4R^2 + 4Rr + 3r^2)}{(4R+r)^2 - 2(16Rr - 5r^2)} \overset{?}{\geq} \frac{8r}{R}$$

$$\Leftrightarrow 7t^3 - 31t^2 + 45t - 222 \overset{?}{\geq} 0 \left(t = \frac{R}{r}\right) \Leftrightarrow (t-2)\left((t-2)(7t-3) + 5\right) \overset{?}{\geq} 0$$

$$\rightarrow \text{true} \because t \overset{Euler}{\geq} 2 \therefore \sum_{cyc} \frac{h_a}{h_b} + \frac{\sum_{cyc} r_a r_b}{\sum_{cyc} r_a^2} \geq \frac{8r}{R} \forall \Delta ABC,$$

$$'' ='' \text{ iff } \Delta ABC \text{ is equilateral (QED)}$$

It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru