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CONTENT

PROFESSOR D. M. BĂTINEŢU-GIURGIU TURNS 90 YEARS OLD

By Mihaly Bencze, Daniel Sitaru and Neculai Stanciu-Romania

Dumitru M. Bătineţu-Giurgiu was born on January 27, 1936, in Pietroşani commune, Vlaşca county, where he attended the General School; then he studied of the "Ion Maiorescu" National College from Giurgiu. In the autumn of 1960, he became a student of the University of Bucharest, Faculty of Mathematics (specialization in mathematical analysis).

He was an assistant professor at the Department of Mathematics of the Polytechnic Institute '' Gh. Asachi ʼʼ from Iaşi (1965-1968); scientific researcher at the Forest Research Institute in Bucharest (1968-1970); assistant professor at the mathematics-physics department of the "Nicolae Bălcescu" Agronomic Institute in Bucharest (1970-1972); professor at the "Ion Creangă" National College in Bucharest (1985-1989) and at the "Matei Basarab" National College in Bucharest (1976-1985; 1989-2003).

He dealt with: Lalescu's sequences, introducing various extensions and concepts of Lalescu functions and Euler-Lalescu functions; the Ioachimescu sequences, introducing the notion of Ioachimescu type constant, the Euler-Ioachimescu type sequences and generalizations of the Ioachimescu sequences; Ghermănescu's sequences and Euler-Ghermănescu constants.

The scientific activity of Professor D.M. Bătineţu-Giurgiu is extremely rich and varied through reference works, articles and studies, published books, notes and teaching materials. He has been and he is part of the editorial boards of several national and international math magazines.

He participated and presented his results of scientific research at international and national conferences, symposia and sessions of scientific communications.
Throughout his career, he has been honored with numerous medals and diplomas.

I cannot conclude the modest presentation of the activity of Professor D.M. Bătineţu-Giurgiu without making a brief reference to his wife, to Mrs. Maria Bătineţu-Giurgiu, who supported him without conditions and limits, creating full freedom in his professional activity.

In the end, as one who stayed around the teacher a little longer and from whom I had a lot to learn, I wish him health, joy, I bow respectfully thanking him and wishing him from the bottom of my heart Happy birthday and continued to success !

On January 27th, 2026, Dumitru M. Bătineţu-Giurgiu – DMBG, as we calls him - turned 90. His long career was, and continue to be, extremely fruitful and influential for students and teachers - lovers of mathematics from all over the world. DMBG always regarded and practical mathematics as a whole – this is a lesson he served us constantly. The activity of DMBG (for more details see the references below) proves that he didn't preach in the desert. This volume is a tribute from members of younger generations who directly or indirectly benefitted from DMBG's passion for mathematics.

DMBG has been the bedrock of the Problems and Solutions Section of the Octogon Journal for 40 years. DMBG, a mathematician at heart and an educator by vocation, proposals problems and solutions with unrelenting care and energy for all these years.

For DMBG, the core of mathematics enterprise, both in his teaching and research, has always been mathematics itself, with all its depth, complexity, and elegance. As a teacher, DMBG continually strives to engender an aesthetic sense of the subject while showing respect for the mathematical thinking of others, especially when such thinking is novel.

DMBG is generous toward people, whether close or distant. Those who personally know DMBG know him to be a warm and caring human being. Those who have come to know DMBG in correspondence with him find him most collegial, helpful, and kind. DMBG's mathematical career has touched many as attested by the dedication to his educational work in the below references. DMBG's salutary influence will be felt for many years to come.

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BASIC INTEGRALS – I

By Daniel Sitaru-Romania

Abstract: In this paper are presented a few basic integrals. The article is asked by young students who want to learn the way of calculus from the beginning.

Introduction

Gamma function. Definition:

$$\boldsymbol{\Gamma}: (\mathbf{0}, \infty) \to \mathbb{R}; \boldsymbol{\Gamma}(\boldsymbol{t}) = \int_{\mathbf{0}}^{\infty} \boldsymbol{e^{-x} \cdot x^{t-1} dx}$$

Properties:

1. $\boldsymbol{\Gamma(1) = 1}$

Proof:

$$\Gamma(1) = \int_0^{\infty} e^{-x} \cdot x^{1-1} dx = \int_0^{\infty} e^{-x}\, dx = -\frac{1}{e^x}\Big|_0^{\infty} = -\frac{1}{e^{\infty}} + \frac{1}{e^0} = -\frac{1}{\infty} + \frac{1}{1} = 1$$

2. $\boldsymbol{\Gamma(2) = 1}$

Proof:

$$\Gamma(2) = \int_0^{\infty} e^{-x}\, x^{2-1} dx = \int_0^{\infty} x\, e^{-x} dx = -\int_0^{\infty} x\, (e^x)' dx = -xe^{-x}|_0^{\infty} + \int_0^{\infty} e^{-x}\, dx =$$

$$= -\frac{x}{e^x}\Big|_0^{\infty} + 1 = -\lim_{x\to\infty} \frac{x}{e^x} + 0 + 1 = -\lim_{x\to\infty} \frac{1}{e^x} + 1 = -\frac{1}{e^{\infty}} + 1 = -\frac{1}{\infty} + 1 = 0 + 1 = 1$$

3. $\boldsymbol{\Gamma(t+1) = t\Gamma(t); t} > 0$

Proof:

$$\Gamma(t+1) = \int_0^{\infty} e^{-x} \cdot x^t dx = -\int_0^{\infty} (e^{-x})'\, x^t dt = -e^{-x} x^t|_0^{\infty} + \int_0^{\infty} e^{-x} \cdot t \cdot x^{t-1} dx =$$

$$= -\frac{x^t}{e^x}\Big|_0^{\infty} + t\int_0^{\infty} e^{-x} \cdot x^{t-1} dx = -\lim_{x\to\infty} \frac{x^t}{e^x} - \frac{0^t}{e^0} + t\Gamma(t) = -0 - 0 + t\Gamma(t) = t\Gamma(t)$$

4. $\boldsymbol{\Gamma(n+1) = n!; n} \in \mathbb{N}$

Proof:

$$\Gamma(n+1)=\Gamma(1+n)=n\Gamma(n)=n\Gamma(1+n-1)=$$

$$=n(n-1)\Gamma(n-1)=n(n-1)\Gamma(1+n-2)=n(n-1)(n-2)\Gamma(n-2)=\cdots=$$

$$=n(n-1)(n-2)\cdot\ldots\cdot 3\cdot 2\cdot\Gamma(1)=n(n-1)(n-2)\cdot\ldots\cdot 3\cdot 2\cdot 1=n!$$

5. $\boldsymbol{\Gamma(t)=2\int_0^\infty e^{-y^2}\cdot y^{2t-1}dy}$

Proof:

$$\Gamma(t)=\int_0^\infty e^{-x}\cdot x^{t-1}dx\overset{x=y^2}{=}\int_0^\infty e^{-y^2}\cdot(y^2)^{t-1}\cdot 2ydy=$$

$$=2\int_0^\infty e^{-y^2}\cdot y^{2t-2}\cdot ydy=2\int_0^\infty e^{-y^2}\cdot y^{2t-1}dy$$

6. $\boldsymbol{\Gamma(t)\cdot\Gamma(1-t)=\frac{\pi}{\sin(t\pi)};t\in(0,1)}$

7. $\boldsymbol{\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}}$

Proof:

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(1-\frac{1}{2}\right)=\frac{\pi}{\sin\frac{\pi}{2}}$$

$$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)=\pi\Rightarrow\left(\Gamma\left(\frac{1}{2}\right)\right)^2=\pi\Rightarrow\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

Beta function. Definition:

$B:(0,\infty)\times(0,\infty)\to\mathbb{R},\ B(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}dx$

Properties:

1.

$$\boldsymbol{B(p,q)=\int_0^\infty\frac{t^{p-1}}{(1+t)^{p+q}}dt}$$

Proof:

$$B(p,q)=\int_0^1 x^{p-1}\cdot(1-x)^{q-1}dx\overset{\frac{x}{1-x}=t}{=}\int_0^\infty\left(\frac{t}{1+t}\right)^{p-1}\cdot\left(1-\frac{t}{1+t}\right)^{q-1}\cdot\frac{1}{(1+t)^2}dt=$$

$$=\int_0^\infty\frac{t^{p-1}\cdot 1\cdot 1}{(1+t)^{p-1+q+2}}dt=\int_0^\infty\frac{t^{p-1}}{(1+t)^{p+q}}dt$$

2.

$$\boldsymbol{B(p,q)=\frac{\Gamma(p)\cdot\Gamma(q)}{\Gamma(p+q)}}$$

3.

$$\boldsymbol{B(p,q)=B(q,p)}$$

4.

$$\boldsymbol{B(p,q)=\frac{p-1}{p+q-1}\cdot B(p-1,q);p}>1;q>0$$

5.

$$\boldsymbol{B(p,q)=\frac{q-1}{p+q-1}\cdot B(p,q-1);p}>0;q>1$$

6.

$$\boldsymbol{B\left(\frac{1}{2},\frac{1}{2}\right)=\pi}$$

Application 1. Find:

$$\boldsymbol{\Omega(n)=\int_0^\infty\frac{1}{1+x^{2n}}dx;n}\in\mathbb{N}^*$$

Proof:

$$\Omega(n)=\int_0^\infty\frac{1}{1+x^{2n}}dx\overset{y=x^{2n}}{=}\int_0^\infty\frac{1}{1+y}\cdot\frac{1}{2n\cdot\sqrt[2n]{y^{2n-1}}}dy=$$

$$=\frac{1}{2n}\int_0^\infty\frac{y^{-\frac{2n-1}{2n}}}{1+y}dy=\frac{1}{2n}\int_0^\infty\frac{y^{\frac{1}{2n}-1}}{(1+y)^{\frac{1}{2n}+\frac{2n-1}{2n}}}dy=$$

$$=\frac{1}{2n}B\left(\frac{1}{2n},1-\frac{1}{2n}\right)=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{1}{2n}\right)\cdot\Gamma\left(1-\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2n}+1-\frac{1}{2n}\right)}=\frac{1}{2n}\cdot\frac{\frac{\pi}{\sin\frac{\pi}{2n}}}{\Gamma(1)}=\frac{\pi}{2n\sin\frac{\pi}{2n}}$$

Corolaries:

1.a.

$$\boldsymbol{\Omega(1)=\int_0^\infty\frac{1}{1+x^2}dx=\frac{\pi}{2\sin\frac{\pi}{2}}=\frac{\pi}{2}}$$

1.b.

$\boldsymbol{\Omega(2)=\int_0^\infty \frac{1}{1+x^4}dx=\frac{\pi}{4\sin\frac{\pi}{4}}=\frac{\pi}{4\cdot\frac{\sqrt{2}}{2}}=\frac{\pi}{2\sqrt{2}}}$

1.c.

$$\boldsymbol{\Omega(3)=\int_0^\infty \frac{1}{1+x^6}dx=\frac{\pi}{6\sin\frac{\pi}{6}}=\frac{\pi}{6\cdot\frac{1}{2}}=\frac{\pi}{1}}$$

1.d.

$$\boldsymbol{\Omega(4)=\int_0^\infty \frac{1}{1+x^8}dx=\frac{\pi}{8\sin\frac{\pi}{8}}=\frac{\pi}{8\cdot\frac{\sqrt{2-\sqrt{2}}}{2}}=\frac{\pi}{4\sqrt{2-\sqrt{2}}}}$$

Application 2. Find:

$$\boldsymbol{\Omega(n)=\int_0^\infty \frac{dx}{\sqrt{1+x^{2n}}};\ n\in\mathbb{N}^*;\ n\geq 2}$$

Proof:

$$\Omega(n)=\int_0^\infty \frac{dx}{\sqrt{1+x^{2n}}}\overset{y=x^{2n}}{=}\int_0^\infty \frac{1}{\sqrt{1+y}}\cdot\frac{1}{2n\cdot\sqrt[2n]{y^{2n-1}}}dy=$$

$$=\frac{1}{2n}\int_0^\infty \frac{y^{\frac{-2n+1}{2n}}}{(1+y)^{\frac{1}{2}}}dy=\frac{1}{2n}\int_0^\infty \frac{y^{\frac{1}{2n}-1}}{(1+y)^{\frac{1}{2n}+\frac{1}{2}-\frac{1}{2n}}}=$$

$$=\frac{1}{2n}\int_0^\infty \frac{y^{\frac{1}{2n}-1}}{(1+y)^{\frac{1}{2n}+\frac{n-1}{2n}}}dy=\frac{1}{2n}B\left(\frac{1}{2n},\frac{n-1}{2n}\right)=$$

$$=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{1}{2n}\right)\cdot\Gamma\left(\frac{n-1}{2n}\right)}{\Gamma\left(\frac{1}{2n}+\frac{n-1}{2n}\right)}=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{1}{2n}\right)\cdot\Gamma\left(\frac{n-1}{2n}\right)}{\Gamma\left(\frac{1}{2}\right)}=\frac{\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{n-1}{2n}\right)}{2n\sqrt{\pi}}$$

Corolaries:

2.a.

$$\boldsymbol{\Omega(2)=\int_0^\infty \frac{dx}{\sqrt{1+x^4}}=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\cdot 2\sqrt{\pi}}=\frac{\Gamma^2\left(\frac{1}{4}\right)}{4\sqrt{\pi}}}$$

2.b.

$$\boldsymbol{\Omega(4)=\int_0^\infty \frac{dx}{\sqrt{1+x^8}}=\frac{\Gamma\left(\frac{1}{8}\right)\Gamma\left(\frac{3}{8}\right)}{8\sqrt{\pi}}}$$

2.c.

$$\boldsymbol{\Omega(3)=\int_0^\infty \frac{dx}{\sqrt{1+x^6}}=\frac{\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{1}{3}\right)}{6\sqrt{\pi}}}$$

2.d.

$$\boldsymbol{\Gamma(5)=\int_0^\infty \frac{dx}{\sqrt{1+x^{10}}}=\frac{\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{2}{5}\right)}{10\sqrt{\pi}}}$$

Application 3. Find:

$$\boldsymbol{\Omega(n)=\int_0^1 x\sqrt[n]{1-x}dx;n\in\mathbb{N};n\geq 2}$$

Proof:

$$\Omega(n)=\int_0^\infty x\cdot\sqrt[n]{1-x}dx=\int_0^1 x\cdot(1-x)^{\frac{1}{n}}dx=$$

$$=\int_0^1 x^{2-1}\cdot(1-x)^{\frac{n+1}{n}}dx=B\left(2,\frac{n+1}{n}\right)=\frac{\Gamma(2)\Gamma\left(\frac{n+1}{n}\right)}{\Gamma\left(2+\frac{n+1}{n}\right)}=\frac{\Gamma(2)\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(3+\frac{1}{n}\right)}=$$

$$=\frac{1\cdot\Gamma\left(1+\frac{1}{n}\right)}{\left(3+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\Gamma\left(1+\frac{1}{n}\right)}=\frac{1}{\frac{3n+1}{n}\cdot\frac{2n+1}{n}}=\frac{n^2}{(3n+1)(2n+1)}$$

Corolaries:

3.a.

$$\boldsymbol{\Omega(2)=\int_0^1 x\sqrt{1-x}\,dx=\frac{2^2}{(6+1)(4+1)}=\frac{4}{35}}$$

3.b.

$$\boldsymbol{\Omega(3)=\int_0^1 x\sqrt[3]{1-x}\,dx=\frac{3^2}{(9+1)(6+1)}=\frac{9}{10\cdot 7}=\frac{9}{70}}$$

3.c.

$$\boldsymbol{\Omega(4)=\int_0^1 x\sqrt[4]{1-x}\,dx=\frac{4^2}{(12+1)(8+1)}=\frac{16}{117}}$$

Application 4. Find:

$$\boldsymbol{\Omega(n)=\int_0^\infty \frac{x}{(1+x^{2n})^2}dx;n\in\mathbb{N}^*}$$

Proof.

$$\Omega(n)=\frac{\int_0^\infty x}{(1+x^{2n})^2}dx\overset{y=x^{2n}}{=}\int_0^\infty\frac{\sqrt[n]{y}}{(1+y)^2}\cdot\frac{dy}{2n\cdot\sqrt[2n]{y^{2n-1}}}=$$

$$=\frac{1}{2n}\int_0^\infty\frac{y^{\frac{1}{2n}-\frac{2n-1}{2n}}}{(1+y)^2}dy=\frac{1}{2n}\int_0^\infty\frac{y^{\frac{2-2n}{2n}}}{(1+y)^2}dy=$$

$$=\frac{1}{2n}\int_0^\infty\frac{y^{\frac{1}{n}-1}}{(1+y)^2}dy=\frac{1}{2n}\int_0^\infty\frac{y^{\frac{1}{n}-1}}{(1+y)^{\frac{1}{n}+2-\frac{1}{n}}}dy=$$

$$=\frac{1}{2n}B\left(\frac{1}{n};2-\frac{1}{n}\right)=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(2-\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+2-\frac{1}{n}\right)}=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\cdot\Gamma\left(1-\frac{1}{n}\right)}{\Gamma(2)}=$$

$$=\frac{1}{2n}\cdot\frac{\left(2-\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{1}=\frac{\frac{2n-1}{n}}{2n}\cdot\frac{\pi}{\sin\frac{\pi}{n}}=\frac{(2n-1)\pi}{2n^2\sin\frac{\pi}{n}}$$

Corolaries:

4.a.

$$\boldsymbol{\Omega(2)=\int_0^\infty\frac{x}{(1+x^4)^2}dx=\frac{3\pi}{8\sin\frac{\pi}{2}}=\frac{3\pi}{8}}$$

4.b.

$$\boldsymbol{\Omega(3)=\int_0^\infty\frac{x}{(1+x^6)^2}dx=\frac{5\pi}{18\sin\frac{\pi}{3}}=\frac{5\pi}{18\cdot\frac{\sqrt{3}}{2}}=\frac{5\pi}{9\sqrt{3}}}$$

4.c.

$$\boldsymbol{\Omega(4)=\int_0^\infty\frac{x}{(1+x^8)^2}dx=\frac{7\pi}{32\sin\frac{\pi}{4}}=\frac{7\pi}{32\cdot\frac{\sqrt{2}}{2}}=\frac{7\pi}{16\sqrt{2}}}$$

Application 5.

Find:

$$\boldsymbol{\Omega(n)=\int_0^\infty\frac{x^2}{(1+x^{2n})^2}dx;n\in\mathbb{N}^*;n\geq 2}$$

Proof:

$$\Omega(n)=\frac{x^2}{(1+x^{2n})^2}dx\overset{y=x^{2n}}{=}\int_0^\infty\frac{\left(\sqrt[2n]{y}\right)^2}{(1+y)^2}\cdot\frac{1}{2n\sqrt[2n]{y^{2n-1}}}dy=$$

$$=\frac{1}{2n}\int_0^{\infty}\frac{y^{\frac{1}{n}}\cdot y^{-\frac{2n-1}{2n}}}{(1+y)^2}d=\frac{1}{2n}\int_0^{\infty}\frac{y^{\frac{2}{2n}-1+\frac{1}{2n}}}{(1+y)^2}dy=$$

$$=\frac{1}{2n}\int_0^{\infty}\frac{y^{\frac{3}{2n}-1}}{(1+y)^{\frac{3}{2n}+2-\frac{3}{2n}}}dy=\frac{1}{2n}B\left(\frac{3}{2n},2-\frac{3}{2n}\right)=$$

$$=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{3}{2n}\right)\cdot\Gamma\left(2-\frac{3}{2n}\right)}{\Gamma\left(\frac{3}{2n}+2-\frac{3}{2n}\right)}=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{3}{2n}\right)\Gamma\left(1+1-\frac{3}{2n}\right)}{\Gamma(2)}=$$

$$=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{3}{2n}\right)\cdot\Gamma\left(2-\frac{3}{2n}\right)}{\Gamma\left(\frac{3}{2n}+2-\frac{3}{2n}\right)}=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{3}{2n}\right)\Gamma\left(1+1-\frac{3}{2n}\right)}{\Gamma(2)}=$$

$$=\frac{1}{2n}\cdot\frac{\Gamma\left(\frac{3}{2n}\right)\Gamma\left(1-\frac{3}{2n}\right)\cdot\left(2-\frac{3}{2n}\right)}{1}=\frac{2-\frac{3}{2n}}{2n}\cdot\Gamma\left(\frac{3}{2n}\right)\cdot\Gamma\left(1-\frac{3}{2n}\right)=\frac{4n-3}{4n^2}\cdot\frac{\pi}{\sin\left(\frac{3\pi}{2n}\right)}$$

Corolaries:

5.a.

$$\boldsymbol{\Omega(2)=\int_0^{\infty}\frac{x^2}{(1+x^4)^2}dx=\frac{\pi}{\sin\frac{3\pi}{4}}=\frac{\pi}{\frac{\sqrt{2}}{2}}=\pi\sqrt{2}}$$

5.b.

$$\boldsymbol{\Omega(3)=\int_0^{\infty}\frac{x^2}{(1+x^6)^2}dx=\frac{\pi}{\sin\frac{\pi}{2}}=\pi}$$

5.c.

$$\boldsymbol{\Omega(4)=\int_0^{\infty}\frac{x^2}{(1+x^8)^2}dx=\frac{\pi}{\sin\frac{3\pi}{8}}}$$

Application 6.

Find:

$$\boldsymbol{\Omega(n)=\int_0^{1}\sqrt[n]{1-x^n}\,dx;n\in\mathbb{N};n\geq 2}$$

Proof:

$$\Omega(n)=\int_0^1\sqrt[n]{1-x^n}\,dx\overset{x^n=y}{=}\int_0^1\sqrt[n]{1-y}\cdot\frac{1}{n\sqrt[n]{y^{n-1}}}dy=$$

$$=\frac{1}{n}\int_0^1 y^{-\frac{n-1}{n}}\cdot(1-y)^{\frac{1}{n}}dy=\frac{1}{n}\int_0^1 y^{\frac{1}{n}-1}\cdot(1-y)^{1+\frac{1}{n}-1}dy=$$

$$=\frac{1}{n}B\left(\frac{1}{n},1+\frac{1}{n}\right)=\frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1+\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+1+\frac{1}{n}\right)}=$$

$$=\frac{\Gamma\left(\frac{1}{n}\right)\cdot\frac{1}{n}\Gamma\left(\frac{1}{n}\right)}{n\Gamma\left(1+\frac{2}{n}\right)}=\frac{\left(\Gamma\left(\frac{1}{n}\right)\right)^2}{n^2\cdot\frac{2}{n}\cdot\Gamma\left(\frac{2}{n}\right)}=\frac{\left(\sqrt{\pi}\right)^2}{2n\Gamma\left(\frac{2}{n}\right)}=\frac{\pi}{2n\,\Gamma\left(\frac{2}{n}\right)}$$

Corolaries:

6.a.

$$\boldsymbol{\Omega(2)=\int_0^1\sqrt{1-x^2}\,dx=\frac{\pi}{4\Gamma(1)}=\frac{\pi}{4}}$$

6.b.

$$\boldsymbol{\Omega(3)=\int_0^1\sqrt[3]{1-x^3}\,dx=\frac{\pi}{6\Gamma\left(\frac{2}{3}\right)}}$$

6.c.

$$\boldsymbol{\Omega(4)=\int_0^1\sqrt[4]{1-x^4}\,dx=\frac{\pi}{8\Gamma\left(\frac{2}{4}\right)}=\frac{\pi}{8\Gamma\left(\frac{1}{2}\right)}=\frac{\pi}{8\sqrt{\pi}}=\frac{\sqrt{\pi}}{8}}$$

6.d.

$$\boldsymbol{\Omega(5)=\int_0^1\sqrt[5]{1-x^5}\,dx=\frac{\pi}{10\Gamma\left(\frac{2}{5}\right)}}$$

Application 7.

Find:

$$\boldsymbol{\Omega(n)=\int_0^1\sqrt{x-x^n}\,dx;\,n\in\mathbb{N};\,n\geq 2.}$$

Proof:

$$\Omega(n)=\int_0^1\sqrt{x(1-x^{n-1})}\,dx=\int_0^1 x^{\frac{1}{2}}(1-x^{n-1})^{\frac{1}{2}}dx=$$

$$\overset{y=x^{n-1}}{=}\int_0^1\left(\sqrt[n-1]{y}\right)^{\frac{1}{2}}\cdot(1-y)^{\frac{1}{2}}\cdot\frac{1}{(n-1)\cdot\sqrt[n-1]{y^{n-2}}}dy=$$

$$=\frac{1}{n-1}\int_0^1 y^{\frac{1}{2(n-1)}-\frac{n-2}{n-1}}\cdot(1-y)^{\frac{1}{2}}dy=\frac{1}{n-1}\int_0^1 y^{\frac{1-2n+4}{2n-2}}\cdot(1-y)^{\frac{3}{2}-1}dy=$$

$$=\frac{1}{n-1}\int_0^1 y^{\frac{5-2n}{2n-2}}\cdot(1-y)^{\frac{3}{2}-1}dy=\frac{1}{n-1}\int_0^1 y^{1+\frac{5-2n}{2n-2}}\cdot(1-y)^{\frac{3}{2}-1}dy=$$

$$=\frac{1}{n-1}\int_0^1 y^{\frac{3}{2n-2}-1}\cdot(1-y)^{\frac{3}{2}-1}dy=\frac{1}{n-1}B\left(\frac{3}{2n-2},\frac{3}{2}\right)=$$

$$=\frac{1}{n-1}\cdot\frac{\Gamma\left(\frac{3}{2n-2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3}{2n-1}+\frac{3}{2}\right)}=\frac{1}{n-1}\cdot\frac{\Gamma\left(\frac{3}{2n-2}\right)\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3+3n-3}{2n-2}\right)}=\frac{\sqrt{\pi}}{2(n-1)}\cdot\frac{\Gamma\left(\frac{3}{2n-2}\right)}{\Gamma\left(\frac{3n}{2n-2}\right)}$$

Corolaries:

7.a.

$$\boldsymbol{\Omega(2)=\int_0^1\sqrt{x-x^2}\,dx=\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{6}{2}\right)}=\frac{\sqrt{\pi}}{2}\cdot\frac{\frac{3}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}=\frac{\sqrt{\pi}}{2}\cdot\frac{3}{2}\cdot\frac{\sqrt{\pi}}{2!}=\frac{3\pi}{8}}$$

7.b.

$$\boldsymbol{\Omega(3)=\int_0^1\sqrt{x-x^3}\,dx=\frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{9}{4}\right)}}$$

7.c.

$$\boldsymbol{\Omega(4)=\int_0^1\sqrt{x-x^4}\,dx=\frac{\sqrt{\pi}}{6}\cdot\frac{\Gamma\left(\frac{3}{6}\right)}{\Gamma\left(\frac{12}{6}\right)}=\frac{\sqrt{\pi}}{6}\cdot\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}=\frac{\left(\sqrt{\pi}\right)^2}{6\cdot 1}=\frac{\pi}{6}}$$

7.d.

$$\boldsymbol{\Omega(5)=\int_0^1\sqrt{x-x^5}\,dx=\frac{\sqrt{\pi}}{8}\cdot\frac{\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(\frac{15}{8}\right)}=\frac{\sqrt{\pi}}{8}\cdot\frac{\Gamma\left(\frac{3}{8}\right)}{\Gamma\left(1+\frac{7}{8}\right)}=\frac{\sqrt{\pi}\cdot\Gamma\left(\frac{3}{8}\right)}{8\cdot\frac{7}{8}\Gamma\left(\frac{7}{8}\right)}=\frac{\sqrt{\pi}\Gamma\left(\frac{3}{8}\right)}{7\Gamma\left(\frac{7}{8}\right)}}$$

Application 8.

Find: $\boldsymbol{\Omega(a,b,m,n)=\int_a^b(x-a)^m\,(b-x)^n dx};\boldsymbol{a}<b;m,n\in\mathbb{N}.$

Proof:

$$\Omega(a,b,m,n)=\int_a^b (x-a)^m\,(b-x)^n dx \overset{y=\frac{x-a}{b-x}}{=}$$

$$=\int_0^\infty \left(\frac{by+a}{y+1}-a\right)^m\left(b-\frac{by+a}{y+1}\right)^n\cdot\frac{b-a}{(y+1)^2}dy=$$

$$=\int_0^\infty \left(\frac{(b-a)y}{y+1}\right)^m\cdot\left(\frac{b-a}{y+1}\right)^n\cdot\frac{b-a}{(y+1)^2}dy=$$

$$=(b-a)^{m+n+1}\int_0^\infty \frac{y^m}{(y+1)^m}\cdot\frac{1}{(y+1)^n}\cdot\frac{1}{(y+1)^2}dy=$$

$$=(b-a)^{m+n+1}\int_0^\infty \frac{y^{m+1-1}}{(y+1)^{m+1+n+1}}dy=$$

$$=(b-a)^{m+n+1}B(m+1;n+1)=(b-a)^{m+n+1}\cdot\frac{\Gamma(m+1)\Gamma(n+1)}{\Gamma(m+n+2)}=$$

$$=(b-a)^{m+n+1}\cdot\frac{m!\cdot n!}{(m+n+1)!}$$

Corolaries

8.a.

$$\boldsymbol{\Omega(a,b,n,n)=\int_a^b (x-a)^n\,(b-x)^n dx=(b-a)^{2n+1}\cdot\frac{(n!)^2}{(2n+1)!}}$$

8.b.

$$\boldsymbol{\Omega(0,1,n,n)=\int_0^1 x^m(1-x)^n\,dx=\frac{m!\,n!}{(m+n+1)!}}$$

8.c.

$$\boldsymbol{\Omega(a,b,n,n+1)=\int_a^b (x-a)^n(b-a)^{n+1}\,dx=(b-a)^{2n+2}\cdot\frac{n!\,(n+1)!}{(2n+2)!}}$$

8.d.

$$\boldsymbol{\Omega(0,1,n,n+1)=\int_0^1 x^n(1-x)^{n+1}dx=\frac{n!\,(n+1)!}{(2n+2)!}}$$

References: Romanian Mathematical Magazine – www.ssmrmh.ro

36 METRIC RELATIONSHIPS IN A NEW GEOMETRICAL CONFIGURATION

By Arda AYDIN-Üsküdar American College – İstanbul-Turkie

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In this article we worked in a new geometric configuration inspired by Dan Sitaru's work "METRIC RELATIONSHIPS IN ŞAHIN'S TRIANGLE"-www.ssmrmh.ro. We define a similar triangle ourselves and found metric relationships in this triangle.

THEOREM

Let ΔABC be an acute triangle and $X \in Int(\Delta ABC)$ such that $XR \perp BC$; $XQ \perp AC$;

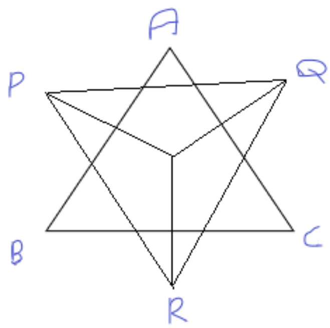
$XP \perp AB$; $|XR| = h_a$; $|XQ| = h_b$; $|XP| = h_c$ (such in figure).

Notations:

$$F = area\ of\ the\ original\ \Delta ABC\ , m_a, m_b, m_c = medians\ in\ the\ original\ \Delta ABC$$

$$\alpha, \beta, \vartheta = angles\ of\ the\ original\ \Delta ABC\ , a, b, c = sides\ of\ original\ \Delta ABC$$

$$a', b', c' = sides\ of\ \Delta PQR, \qquad R^*, R = \ circumradii\ of\ \Delta PQR\ and\ \Delta ABC$$



In these conditions:

$$\mathbf{1.}\ a' = \frac{a.m_a}{R}, b' = \frac{b.m_b}{R}, c' = \frac{c.m_c}{R}$$

Proof: $(a')^2 = \frac{4F^2}{b^2} + \frac{4F^2}{b^2} + \frac{4F^2 cos\alpha}{2bc} = \frac{8F^2c^2+8F^2b^2+8F^2(b^2+c^2-a^2)}{2b^2c^2} = \frac{4F^2(2b^2+2c^2-a^2)}{b^2c^2} = \frac{F^2(m_a)^2}{b^2c^2} = \frac{a^2(m_a)^2}{R^2}$

2. $Area(\Delta XPQ) = F.(\alpha)$

Proof: $\frac{1}{2} h_c h_b\ sin\ sin\ 180 - \alpha\ = \frac{1}{2} h_c h_b\ sin\ sin\ \alpha\ = \frac{1}{2} h_c h_b \frac{h_b}{c} = \frac{4F^3}{b^2c^2} = F(\alpha)$

3. $Area(\Delta XQR) = F.(\beta)$

Proof: It's found similary to 2.

4. $\boldsymbol{Area(\Delta XRP) = F.(\vartheta)}$

Proof: It's found similary to 2.

5. $\boldsymbol{Area(\Delta PQR) = F.(\alpha + \beta + \vartheta)}$

Proof: Equations 2,3,4 are found by adding them side by side.

6. $\boldsymbol{\frac{Ra'b'c'}{R^*abc} = \alpha + \beta + \vartheta)}$

Proof: $Area(\Delta PQR) = \frac{a'b'c'}{4R^*} = F.(\alpha + \beta + \vartheta) = \frac{abc}{4R}(\alpha + \beta + \vartheta)$

$\Rightarrow \frac{Ra'b'c'}{R^*abc} = \alpha + \beta + \vartheta)$

7. $\boldsymbol{\frac{m_a m_b m_c}{R^* R^2} = \alpha + \beta + \vartheta)}$

Proof: In the equation, replace a', b', c' with the equations 1 and the equality is found.

$$\frac{Ra'b'c'}{R^*abc} = \alpha + \beta + \vartheta) \Rightarrow \frac{Rabcm_a m_b m_c}{R^*abcR^3} = \alpha + \beta + \vartheta) \Rightarrow$$

$$\Rightarrow \frac{m_a m_b m_c}{R^* R^2} = \alpha + \beta + \vartheta)$$

8. $\boldsymbol{Area(\Delta PQR) = \frac{4F^3.(a^2+b^2+c^2)}{a^2b^2c^2}}$

Proof: It is used that $\alpha + \beta + \vartheta) = \frac{(a^2+b^2+c^2)}{4R^2}$ and $R = \frac{abc}{4F}$ and equality is found.

9. $\boldsymbol{Area(\Delta PQR) = \frac{F.(m_a^2+m_b^2+m_c^2)}{3R^2}}$

Proof: It is used that $3(a^2 + b^2 + c^2) = 4(m_a^2 + m_b^2 + m_c^2)$ and equality is found.

10. $\boldsymbol{Area(\Delta PQR) = \frac{16F^3.(m_a^2+m_b^2+m_c^2)}{3a^2b^2c^2}}$

Proof: It is found by writing $R = \frac{abc}{4F}$ in equation 9.

11. $\boldsymbol{Area(\Delta PQR) = (3 - \alpha - \beta - \vartheta)F}$

Proof: It is found by writing $\alpha + \beta + \vartheta) = (3 - \alpha - \beta - \vartheta)$ in equation 5.

12. $\boldsymbol{Area(AQBRCP) = 3F}$

Proof: $Area(AQBRCP) = Area(APBX) + Area(BRCX) + Area(CQAX) = \frac{ah_a+bh_b+ch_c}{2} = 3F$

13. $\boldsymbol{Area(\Delta PQR) = 2.R^2 sin\alpha sin\beta sin\vartheta(\alpha + \beta + \vartheta)}$

Proof: It is used that $F = 2R^2 sin\alpha sin\beta sin\vartheta$ and equality is found.

14. $\boldsymbol{Area(\Delta PQR) = (2 + cos\, cos\, \alpha\ \ cos\, cos\, \beta\ \ cos\, cos\, \vartheta\,)F}$

Proof: It is used that $\alpha + \beta + \vartheta = 180 \Rightarrow \ \alpha\ + \beta\ + \vartheta\ = 2 + cos\, cos\, \alpha\ \ cos\, cos\, \beta$ $cos\, cos\, \vartheta$ and equality is found.

15. $\boldsymbol{Area(\Delta PQR) \geq \frac{\sqrt{3}F^2}{R^2}}$

Proof: $Area(\Delta PQR) = F.(\alpha\ + \beta\ + \vartheta\,) = F.\frac{(a^2+b^2+c^2)}{4R^2}$

It used to $a^2 + b^2 + c^2 \geq 4\sqrt{3}F$ and equality is found.

16. $\boldsymbol{|XK|^2 + |XL|^2 + |XM|^2 \geq \frac{4F^2}{9R^2}}$

17. $\boldsymbol{(|AX| + |BX| + |CX|)^2 \geq \frac{16F^2}{9R^2} + 8(|XM||XL| + |XM||XK| + |XK||XL|)}$

18. $\boldsymbol{Area(\Delta PQR) = \frac{Fm_am_bm_c}{R^*R^2}}$

19. $\boldsymbol{\frac{aa'}{R_1} + \frac{bb'}{R_2} + \frac{cc'}{R_3} = \frac{m_am_bm_c}{R^*}}$

20. $\boldsymbol{sin\, sin\, (RPQ)\ = \frac{R^2 sinsin\, \vartheta\, (\alpha\, + \beta\, + \vartheta\,)}{m_am_b}}$

21. $\boldsymbol{(RQP)\ = \frac{R^2 sinsin\, \beta\, (\alpha\, + \beta\, + \vartheta\,)}{m_am_c}}$

22. $\boldsymbol{sin\, sin\, (PRQ)\ = \frac{R^2 sinsin\, \alpha\, (\alpha\, + \beta\, + \vartheta\,)}{m_bm_c}}$

23. $\boldsymbol{sin\, sin\, (RPQ)\ = \frac{sinsin\, \vartheta\, (a^2+b^2+c^2)}{4m_am_b}}$

24. $\boldsymbol{sin\, sin\, (RQP)\ = \frac{sinsin\, \beta\, (a^2+b^2+c^2)}{4m_am_c}}$

25. $\boldsymbol{sin\, sin\, (PRQ)\ = \frac{sinsin\, \alpha\, (a^2+b^2+c^2)}{4m_bm_c}}$

26. $\boldsymbol{(a' + b' + c')^2 \leq \frac{3}{4}(a^2 + b^2 + c^2)^2}$

27. $\boldsymbol{(a' + b' + c')^2 \leq \frac{27(a^2+b^2+c^2)}{4}}$

28. $\boldsymbol{(a' + b' + c')^2 \leq \frac{243R^2}{4}}$

29. $\boldsymbol{\frac{sinsin\, RQP}{sinsin\, RPQ} = \frac{sin\beta.m_b}{sin\vartheta.m_c}}$

30. $\boldsymbol{\frac{sinsin\, RQP}{sinsin\, PRQ} = \frac{sin\beta.m_b}{sin\alpha.m_a}}$

31. $\frac{sinsin\ RPQ}{sinsin\ PRQ}=\frac{sin\vartheta.m_c}{sin\alpha.m_a}$

32. $\frac{sinsin\ RPQ}{sin\vartheta.m_c}=\frac{sinsin\ PRQ}{sin\alpha.m_a}=\frac{sinsin\ RQP}{sin\beta.m_b}=\frac{1}{R^*}=\frac{a^2+b^2+c^2}{4m_am_bm_c}$

33. $\frac{sinsin\ RPQ}{sin\vartheta.m_c}=\frac{sinsin\ PRQ}{sin\alpha.m_a}=\frac{sinsin\ RQP}{sin\beta.m_b}\le\frac{9R^2}{4m_am_bm_c}$

34. $\frac{sinsin\ RPQ}{sin\vartheta.m_c}=\frac{sinsin\ PRQ}{sin\alpha.m_a}=\frac{sinsin\ RQP}{sin\beta.m_b}\ge\frac{\sqrt{3}F}{m_am_bm_c}$

35. $R^*=\frac{4m_am_bm_c}{a^2+b^2+c^2}$

36. $R^*\le\frac{3}{2}.R$

$Reference$:

$Daniel\ Sitaru$ – "$Metric\ relationships\ in\ Șahin's\ triangle$" – $www.ssmrmh.ro$

GAKOPOULOS – NAG THEOREM —– GENERALIZATION OF BUTTERFLY THEOREM USING PLAGIOGONAL SYSTEM

By Athanasios Gakopoulos-Greece, Debabrata Nag-India

Abstract: In this paper we have tried to put forward a theorem generalizing the well-known Butterfly Theorem of Euclidean Geometry using PLAGIOGONAL (Oblique) co-ordinate system of Analytical Geometry. We have also demonstrated two examples of the above theorem.

Keywords: Butterfly Theorem, PLAGIOGONAL, Analytical Geometry

1.0 INTRODUCTION

Butterfly theorem of the Euclidean Geometry is very well-known whose statement goes like: "Let M be the midpoint of a chord ***PQ*** of a circle, through which two other chords ***AB*** and ***CD*** are drawn; ***AD*** and ***BC*** intersect chord ***PQ*** at X and Y correspondingly. Then M is the midpoint of ***XY*** ". [See Figure 1 for the theorem]. Many straight forward geometric poofs of the validity of the theorem do exist in the literature

In this present work, we have presented a more generalized version of the above theorem as shown in Figure 2.

2.0 GENERALIZED BUTTERFLY THEOREM

We refer to figure 2 where ***AD*** and ***BC*** are the two intersecting chords at point S of a circle whose center is at O. Let a line be drawn through S intersecting the chords ***AB*** and ***CD*** at P and Q. If ***AS = a***, ***SB = b***, ***SC = c***, ***SD = d***, ***PS/SQ = k*** (***k*** being any non-zero real number), angle ***ASC = ϑ*** and angle ***OSQ = ρ***. Theorem states that:

$$\cot\rho=\frac{(k-1)(ac-bd)\sin\theta}{a^2+b^2+k\left(c^2+d^2\right)-(k+1)(ad+bc)-\left[(k+1)(ac+bd)-2(ab+k\cdot cd)\right]\cos\theta}$$

And it is obvious from the above that: $\rho=90^0\Leftrightarrow k=1$ or in other words, S will be the middle point of the line ***PQ*** and also it will be the middle point of the chord through P and Q (i.e., when ***PQ*** is produced in either way to intersect the circumference of the circle.

We use the PLAGIOGONAL (i.e., non-orthogonal oblique axes) to prove the above theorem.

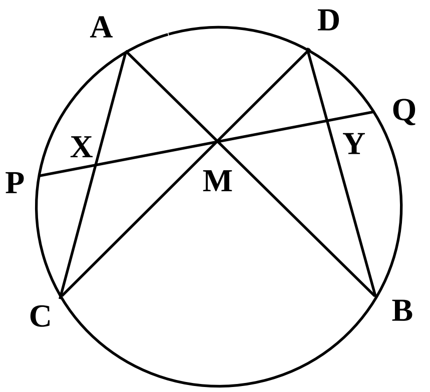


Figure 1: Butterfly Theorem

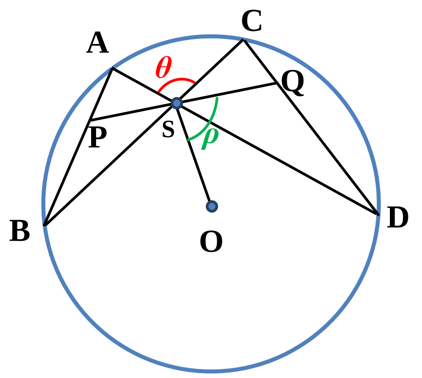


Figure 2: Generalized Butterfly Theorem

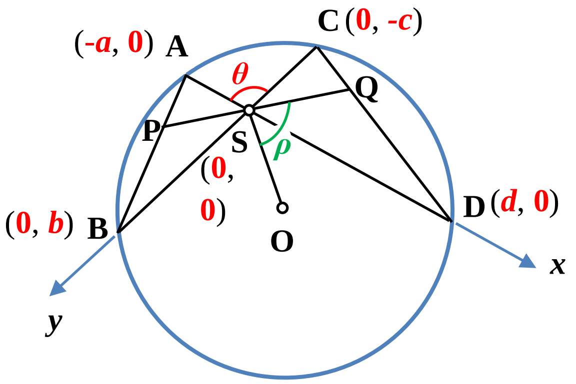


Figure 3: PLAGIOGONAL Co-Ordinate System for Generalized Butterfly Theorem

<u>Proof</u>:- We consider a co-ordinate system whose origin is at the point S and we consider ***SD*** and ***SB*** as the ***x*** and ***y*** – axes respectively. Accordingly co-ordinates of various points are shown in figure 3 above. Now, we note that the equation of the circle containing the points A, B, C and D can be written as:

$x^2+y^2+2xy\cos\theta-(d-a)x-(b-c)y-ad=0,\quad ad=bc\ (POP)\Rightarrow$ Co-ordinates of the center O of this circle are: $\left(\frac{(d-a)-(b-c)\cos\theta}{2\sin^2\theta}, \frac{(b-c)-(d-a)\cos\theta}{2\sin^2\theta}\right)$ and hence

$\Rightarrow m_{OS} = \frac{(b-c)-(d-a)\cos\theta}{(d-a)-(b-c)\cos\theta}$. Further, equations of the chords ***AB*** and ***DC*** can be written as: $\frac{x}{a}-\frac{y}{b}=-1$ and $\frac{x}{d}-\frac{y}{c}=1$. Let the co-ordinates of **P** and **Q** be (p_1, p_2) and (q_1, q_2) and co-ordinates of P will obviously satisfy the equation of ***AB*** and those of Q will satisfy the equation of ***CD***. From the given condition of ***PS/SQ = k***, we can find these co-ordinates as:

$$\left(\frac{ad(b-ck)}{(ac-bd)}, \frac{bc(a-dk)}{(ac-bd)}\right) \text{ and } \left(\frac{ad(ck-b)}{k(ac-bd)}, \frac{bc(dk-a)}{k(ac-bd)}\right) \Rightarrow$$

$$m_{PQ} = \frac{p_2}{p_1} = \frac{q_2}{q_1} = \frac{bc(a-dk)}{ad(b-ck)}. \text{ Thus:}$$

$$\tan\rho = \frac{\left[\frac{(b-c)-(d-a)\cos\theta}{(d-a)-(b-c)\cos\theta} - \frac{bc(a-dk)}{ad(b-ck)}\right]\sin\theta}{1+\frac{(b-c)-(d-a)\cos\theta}{(d-a)-(b-c)\cos\theta}\cdot\frac{bc(a-dk)}{ad(b-ck)} + \left[\frac{bc(a-dk)}{ad(b-ck)} + \frac{(b-c)-(d-a)\cos\theta}{(d-a)-(b-c)\cos\theta}\right]\cos\theta}$$

$$\Rightarrow \boxed{\cot\rho = \frac{[(b-ck)(d-a)+(b-c)(a-dk)]\sin\theta}{(b-ck)(b-c)-(a-dk)(d-a)-[(b-ck)(d-a)-(a-dk)(b-c)]\cos\theta}} \text{ [QED]}$$

Hence, it is obvious that if ***k* = 1**, i.e., when S is the middle point of the line segment ***PQ***, ***OS*** becomes perpendicular to ***PQ*** at point S and hence it is perpendicular to the chord containing the line segment ***PQ*** and thus S becomes the middle of the chord also. This is nothing but the statement of the Butterfly theorem. We note that the converse of the statement is also true. Thus the new theorem becomes the generalised Butterfly theorem in the sense that if when ***OS*** is not perpendicular to ***PQ***, we can set the relation between the angles ***ϑ*** and ***ρ*** as shown above.

3.0 TWO EXAMPLES

<u>**Example 1**</u>: Refer to figure 4(a). If ***FA* = 6**, ***FB* = 10**, ***FC* = 5**, ***PS / SQ = k***, angle ***AFC*** = **60^0**, then we need to prove that: $\cot\theta = \frac{14\sqrt{3}(1-k)}{(2+35k)}$

<u>**Proof:**</u>**-** Following the same line of the generalized theorem, this time taking F as the origin and ***FB*** and ***FC*** as the ***x*** and ***y*** – axes we can easily find the equation of the circle as:

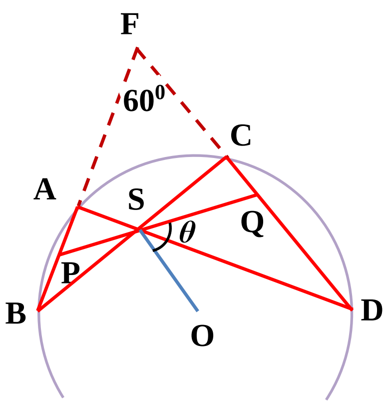
$x^2+y^2+xy-16x-17y+60=0 \Rightarrow$ co-ordinates of O are: $\left(\frac{14}{3},\frac{8}{3}\right) \Rightarrow$ co-ordinates of **P** and **Q** are respectively obtained as: $\left(\frac{14(k+1)}{3},0\right)$ and $\left(0,\frac{8(k+1)}{3k}\right) \Rightarrow m_{PQ}=-\frac{4}{7k}$ and

$$m_{OS}=\frac{6-\frac{8}{3}}{5-\frac{14}{3}}=10 \Rightarrow \tan\theta=\frac{\left(\frac{4}{7k}+10\right)\sin 60^0}{-1+\frac{40}{7k}-\left(10-\frac{4}{7k}\right)\cos 60^0}=\frac{(4+70k)\sqrt{3}}{84(1-k)}$$

$$\Rightarrow \boxed{\cot\theta=\frac{14\sqrt{3}(1-k)}{(2+35k)}}$$ [QED]

Example 2: Refer to figure 4(b). If ***SA* = 7**, ***SB* = 4**, ***SC* = 14**, ***SD* = 8**, ***SQ* = 2 *PS***, $\tan\angle QSO=\frac{9\sqrt{3}}{11}$, then we need to find angle ***BSD***.

Proof:- With the choice of axes as shown in figure 4(b), we find Equation of the circle: $x^2+y^2+2xy\cos\theta-x+10y-56=0 \Rightarrow$ co-ordinates of the center of the circle are: $\left(\frac{1+10\cos\theta}{2\sin^2\theta},-\frac{10+\cos\theta}{2\sin^2\theta}\right) \Rightarrow m_{OS}=-\frac{10+\cos\theta}{1+10\cos\theta}$. Let the co-ordinates of **P** and **Q** be: $\left(\frac{-7}{\lambda+1},\frac{4\lambda}{\lambda+1}\right)$ and $\left(\frac{8}{\mu+1},-\frac{14\mu}{\mu+1}\right),(\lambda,\mu\neq-1)$.



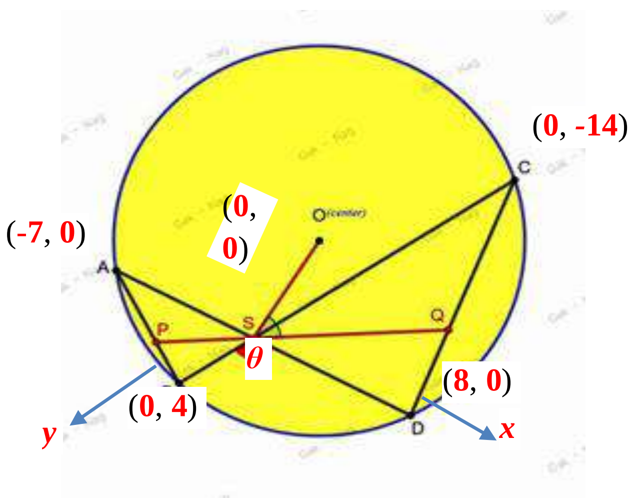


Figure 4(a), (b): Example 1 and Example 2

As ***SQ* = 2 *PS***, therefore: $4\lambda - 7\mu = 3$ and $\frac{4\lambda}{\lambda+1} = \frac{7\mu}{\mu+1} \Rightarrow \lambda\mu = 1 \Rightarrow \lambda = \frac{7}{4} \quad \mu = \frac{4}{7}$

$$\Rightarrow m_{PQ} = -\frac{4\lambda}{7} = -1$$

$$\therefore\ \tan(\angle OSQ) = \frac{\left(\frac{10+\cos\theta}{1+10\cos\theta} - 1\right)\sin\theta}{1 + \frac{10+\cos\theta}{1+10\cos\theta} - \left(1 + \frac{10+\cos\theta}{1+10\cos\theta}\right)\cos\theta} = \frac{9(1-\cos\theta)}{11\sin\theta} = \frac{9\sqrt{3}}{11} \Rightarrow \tan\frac{\theta}{2} = \sqrt{3} \Rightarrow \theta = 120^0$$

$\Rightarrow \boxed{\angle BSD = 120^0}$ [**Ans**]

The above problems can be solved in alternative manner and the truth of the theorem thus can be easily established.

4.0 CONCLUSION:

In this short paper, authors have tried to formulate in the form of a generalized theorem on the geometric shape akin to that of the celebrated and well-known theorem of Euclidean geometry, that is, the Butterfly theorem. This new theorem helps us not only to set a geometric relationship between the angles and various measures of the chords, but also it gives back the Butterfly theorem as a special case of this generalized study. The theorem and the associated work have been applied to solve two exercise problems for illustration.

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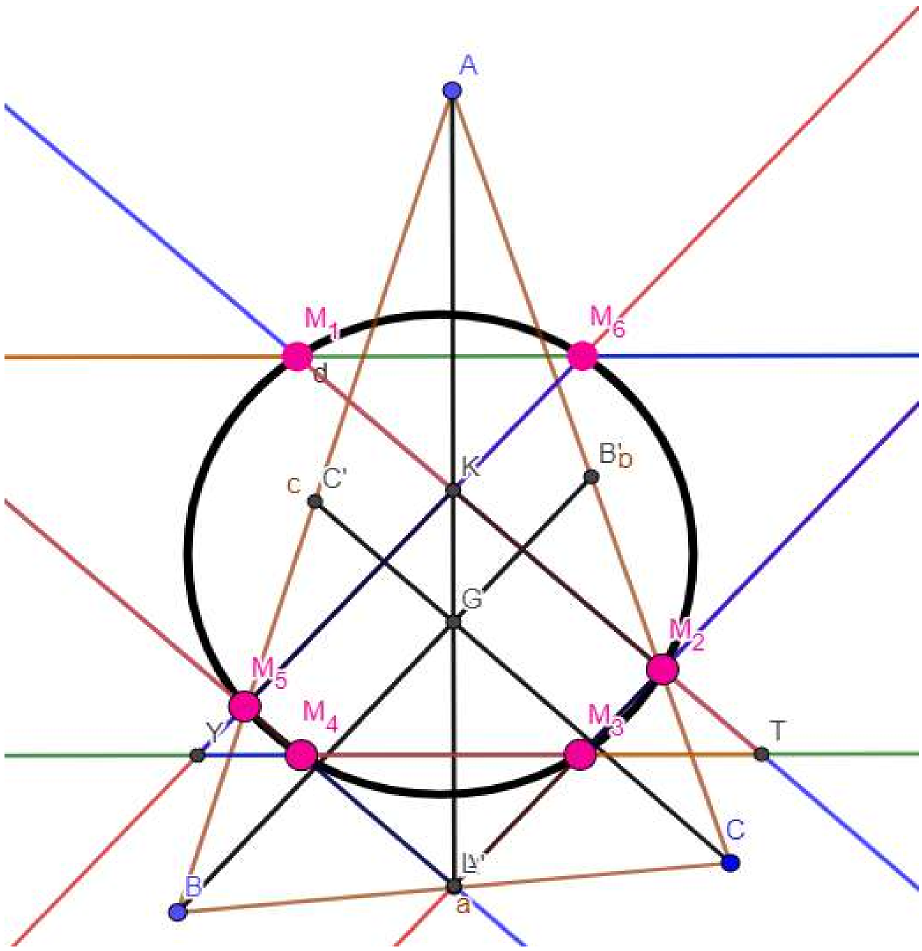
VAN LAMOEN'S CIRCLE REVISITED

By Ahmet ERÇIKDI, Yiğit TÜRK -Turkiye

Abstract: In this article, we give a new proof of the existence of Van Lamoen's circle. Our proof relies on Euclidean methods and offers a more direct understanding of why the Van Lamoen's circle exist in any given triangle.

Problem: If the medians of triangle ABC intersects with sides BC, AC, AB at the points A', B', C' respectively and G is the centroid of ABC, then the circumcenters of six triangles $AB'G$, $CB'G$, $CA'G$, $BA'G$, $BC'G$, $AC'G$ are concylic.



Notations:

Let G be the centroid of the triangle.

Let $A^{'} = AG \cap BC$, $B^{'} = BG \cap AC$ and $C^{'} = CG \cap AB$.

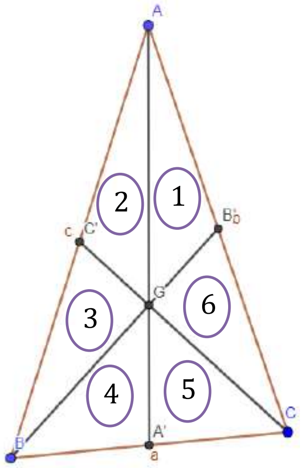
Let A_1 be the midpoint of $[AG]$ and A_2 be the midpoint of $[A^{'}G]$. Also, we define the perpendicular bisector of $[AG]$ as a_1 and the perpendicular bisector of $[A^{'}G]$ as a_2. Similarly we define the midpoint of $[BG]$ as B_1, the midpoint of $[B^{'}G]$ as B_2, the perpendicular bisector of $[BG]$ as b_1, the perpendicular bisector of $[B^{'}G]$ as b_2, the midpoint of $[CG]$ as C_1, the midpoint of $[C^{'}G]$ as C_2, the perpendicular bisector of $[CG]$ as c_1 and the perpendicular bisector of $[C^{'}G]$ as c_2.

We also define the points,

$a_1 \cap b_1 = P_{a1\_b1}$ $\quad$ $a_1 \cap b_2 = P_{a1\_b2}$ $\quad$ $a_1 \cap c_1 = P_{a1\_c1}$ $\quad$ $a_1 \cap c_2 = P_{a1\_c2}$

$a_2 \cap b_1 = P_{a2\_b1}$ $\quad$ $a_2 \cap b_2 = P_{a2\_b2}$ $\quad$ $a_2 \cap c_1 = P_{a2\_c1}$ $\quad$ $a_2 \cap c_2 = P_{a2\_c2}$

$b_1 \cap c_1 = P_{b1\_c1}$ $\quad$ $b_1 \cap c_2 = P_{b1\_c2}$ $\quad$ $b_2 \cap c_1 = P_{b2\_c1}$ $\quad$ $b_2 \cap c_2 = P_{b2\_c2}$



Let's number the triangles which are formed when the medians are drawn as 1, 2, 3, 4, 5 and 6.

$\Delta AGB' \rightarrow 1$	$\Delta AGC' \rightarrow 2$	$\Delta BGC' \rightarrow 3$
$\Delta BGA' \rightarrow 4$	$\Delta CGA' \rightarrow 5$	$\Delta CGB' \rightarrow 6$

We denote the circumcenter of the n-th triangle as O_n. Thus, we can write:

$O_1 = P_{a1\_b2}$	$O_2 = P_{a1\_c2}$	$O_3 = P_{b1\_c2}$
$O_4 = P_{a2\_b1}$	$O_5 = P_{a2\_c1}$	$O_6 = P_{b2\_c1}$

Let's define $|AA_1| = 2a$. Then, we have $|GA_1| = 2a$, $|GA_2| = a$ and $|A'A_2| = a$.

Similarly we define $|BB_1| = 2b$. Then, we have $|GB_1| = 2b$, $|GB_2| = b$ and $|B'B_2| = b$.

Similarly we define $|CC_1| = 2c$. Then, we have $|GC_1| = 2c$, $|GC_2| = c$ and $|C'C_2| = c$.

Lemma: Let x, y,z and α, β, θ be angles of two triangles. If $x + y = \alpha + \beta$ and $\frac{sinx}{siny} = \frac{sin\alpha}{sin\beta}$ then $x = \alpha$ and $y = \beta$.

Proof: If $x + y = \alpha + \beta$ and $\frac{sinx}{siny} = \frac{sin\alpha}{sin\beta}$, then we can write

$$\frac{sinx + siny}{siny} = \frac{sin\alpha + sin\beta}{sin\beta} \Rightarrow \frac{2\sin\left(\frac{x+y}{2}\right).\cos\left(\frac{x-y}{2}\right)}{siny} = \frac{2\sin\left(\frac{\alpha+\beta}{2}\right).\cos\left(\frac{\alpha-\beta}{2}\right)}{sin\beta}$$

If we write $x + y = \alpha + \beta$ and simplify we have:

$$\cos\left(\frac{x-y}{2}\right)sin\beta = \cos\left(\frac{\alpha-\beta}{2}\right)\text{sin}y$$

With the inverse transformation we have

$$\frac{1}{2}\left(\sin\left(\frac{x-y}{2} + \beta\right) + \sin\left(\beta - \frac{x-y}{2}\right)\right) = \frac{1}{2}\left(\sin\left(\frac{\alpha-\beta}{2} + y\right) + \sin\left(y - \frac{\alpha-\beta}{2}\right)\right)$$

If we write $x + y - \alpha = \beta$ then,

$$\sin\left(\frac{3x + y - 2\alpha}{2}\right) + \sin\left(\frac{x + 3y - 2\alpha}{2}\right) = \sin\left(\frac{-x + y + 2\alpha}{2}\right) + \sin\left(\frac{x + 3y - 2\alpha}{2}\right)$$

$$\sin\left(\frac{3x + y - 2\alpha}{2}\right) = \sin\left(\frac{y - x + 2\alpha}{2}\right)$$

$$\frac{3x+y-2\alpha}{2}=\frac{y-x+2\alpha}{2}+2k\pi \quad \vee \quad \frac{3x+y-2\alpha}{2}=\pi-\left(\frac{y-x+2\alpha}{2}\right)+2k\pi \quad \text{where } k\in\mathbb{Z}$$

$$x=\alpha+k\pi \qquad \vee \qquad x+y=\pi+2k\pi$$

$\Downarrow$ $\Downarrow$

Since $|x-\alpha|<\pi$, Since $0<x+y<\pi$,

$k=0$ and $x=\ \alpha$. There are no solutions for x.

Proof: Let's define the following angles: $\angle CGA^{'}=\alpha$, $\angle BGA^{'}=\beta$ and $\angle BGC^{'}=\theta$

thus $+\beta+\theta=\pi$.

It is obvious that if the acute angle between two lines is γ, then the acute angle between the lines which are perpendicular to the initial lines is also γ. Hence, we can obtain that $\angle CGA^{'}=\angle C'GA=\angle P_{a1\_b1}O_2P_{a2\_c2}=\ \alpha$. Thus $\angle P_{a1\_c1}O_2O_3=\pi-\alpha=\beta+\theta$.

Now let's consider AGC triangle. GB' is a median of this triangle and the areas of two triangles that formed when the median intersects with the opposing edge, are equal.

$$Area(\Delta AGB^{'})=Area(\Delta CGB^{'})$$

Using this equation, we can write $2a.\sin\beta=2c.\sin\theta$ and $\frac{\sin\beta}{\sin\theta}=\frac{c}{a}$ by the sine area formula… (1)

Now, draw O_2O_5 and define the length $|O_2O_5|=l$. We define $\angle O_5O_2O_3=x$ and $\angle O_5O_2P_{a1\_c1}=y$. Let's draw a perpendicular from O_5 to c_2 and name the intersection of c_2 and the perpendicular T_1. Here, we can see that $|O_5T_1|=|C_1C_2|=3c$ since $O_5T_1//CC'$. Thus, we can write $\sin x=\frac{3c}{l}$. Similarly, if we draw a perpendicular from O_5 to a_1 we find that $\sin y=\frac{3a}{l}$… (2)

By (1) and (2), we have $\frac{\sin x}{\sin y}=\frac{c}{a}=\frac{\sin\beta}{\sin\theta}$.

We also showed that $\angle P_{a1\_c1}O_2O_3=\beta+\theta=x+y$. Thus, by using the lemma, we can say that $x=\beta$ and $y=\theta$. Therefore points O_1,O_6,O_5,O_4 are concylic since $\angle O_3O_4O_5=\angle O_5O_2O_3$. If the process is repeated with the triangles AGB and BGC, it can be shown that O_1,O_6,O_5,O_4 and O_3,O_2,O_1,O_6 are concylic, respectively.

If we draw O_1O_4 and use the lemma for $\angle P_{a1\_b1}O_1O_4$ and $\angle P_{a2\_b2}O_1O_4$ it can be shown that $\angle P_{a1\_b1}O_1O_4=\ \theta$. We have already shown that $\angle O_5O_2P_{a1c1}=\theta$.

Therefore $\angle P_{a1\_b1}O_1O_4=\angle O_5O_2P_{a1c1}$ and O_1,O_2,O_4,O_5 are concylic.

Points O_2,O_3,O_4,O_5 are on a circle. It is shown that O_1 is on the same circle too.

Since O_1,O_6,O_5,O_4 are conylic, O_6 also lie on the same circle.

TEN APPLICATIONS FOR IONESCU- WEITZENBÖCK'S INEQUALITY

By Neculai Stanciu-Romania

Abstract. This paper presents refinements and new solutions for some problems - published by math journals from all over the world related to Ionescu-Weitzenböck inequality .

INTRODUCTION

Ion Ionescu discovered with 22 years before Weitzenböck the inequality $a^2+b^2+c^2 \ge 4\sqrt{3}S$

The author of this paper along with prof. D. M. Bătineţu-Giurgiu demonstrated in Romanian Mathematical Gazette , No. 1/2013, pp. 1-10, that the Weitzenböck's inequality must be named the Ionescu-Weitzenböck's inequality.

Our proof is based on: Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 2 , 15 October 1897, on page 52, Ion Ionescu, the founder of Romanian Mathematical Gazette, published problem 273: prove that there is no triangle for which the inequality $4S\sqrt{3} > a^2+b^2+c^2$ **can be satisfied. The solution of the problem 273, apperead in Romanian Mathematical Gazette, Vol. III (15 September 1897 – 15 August 1898), No. 12 , 15 August 1898, on pages 281, 282 and 283. In the year 1919, Roland Weitzenböck publishead in Mathematische Zeitschrift, Vol. 5, No. 1-2, pp. 137-146 the article Uber eine Ungleichung in der Dreiecksgeometrie, where he proof that: In any triangle** ABC**, with usual notations holds the inequality:** $a^2+b^2+c^2 \ge 4\sqrt{3}S$ **.**

MAIN RESULTS

Application 1 (D.M. Bătineţu-Giurgiu , N. Stanciu - La Gaceta de la RSME, No. 2/2020, problem 397).

Prove that in any triangle ABC **with the area** F **, the medians** m_a, m_b, m_c **(**m_a **is the mediane from the vertex** A**,** m_b **is the mediane from the vertex** B **,** m_c **is the mediane from the vertex** C**) and usual notations is true the following inequality**

$$\frac{\sqrt{3}}{2}(a^2+b^2+c^2) \ge am_a + bm_b + cm_c \ge 6F \text{ .}$$

Solution. First we prove that $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a$, (1). Indeed,

$a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a \Leftrightarrow (a^2+b^2+c^2)^2 \ge 12\cdot a^2\cdot m_a^2 = 3a^2\cdot 4m_a^2 =$
$3a^2(2b^2+2c^2-a^2)$

$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \ge 6a^2b^2+6c^2a^2-3a^4$

$\Leftrightarrow 4a^4+b^2+c^2-4a^2b^2-4c^2a^2+2b^2c^2 \ge 0$

$\Leftrightarrow (2a^2-b^2-c^2)^2 = (b^2+c^2-2a^2)^2 \ge 0$, true. Therefore, $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a$; $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot b\cdot m_b$ and $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot c\cdot m_c$ which by adding up yielding that $3(a^2+b^2+c^2) \ge 2\sqrt{3}(am_a+bm_b+cm_c)$, (2). Next we have $2\sqrt{3}(am_a+bm_b+cm_c) \ge 2\sqrt{3}(ah_a+bh_b+ch_c) = 12\sqrt{3}F$, (3), where we denote h_a the altitude from the vertex A, h_b the altitude from the vertex B, h_c the altitude from the vertex C. From (2) and (3) we obtain the desired inequality.

Application 2. (D.M. Bătineţu-Giurgiu , N. Stanciu- Refinement of Ionescu-Weitzenböck inequality).

Prove that in any triangle ABC with the area F, the medians m_a, m_b, m_c (m_a is the mediane from the vertex A, m_b is the mediane from the vertex B, m_c is the mediane from the vertex C and usual notations is true the following inequality

$$a^2+b^2+c^2 \ge \frac{2}{\sqrt{3}}(am_a+bm_b+cm_c) \ge 4\sqrt{3}F.$$

Solution. First we prove that $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a$, (1). Indeed,

$a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a \Leftrightarrow (a^2+b^2+c^2)^2 \ge 12\cdot a^2\cdot m_a^2 = 3a^2\cdot 4m_a^2 =$
$3a^2(2b^2+2c^2-a^2)$

$\Leftrightarrow a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 \ge 6a^2b^2+6c^2a^2-3a^4$

$\Leftrightarrow 4a^4+b^2+c^2-4a^2b^2-4c^2a^2+2b^2c^2 \ge 0$

$\Leftrightarrow (2a^2-b^2-c^2)^2 = (b^2+c^2-2a^2)^2 \ge 0$, true.

Therefore, $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot a\cdot m_a$; $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot b\cdot m_b$; $a^2+b^2+c^2 \ge 2\sqrt{3}\cdot c\cdot m_c$

which by adding up yielding that $3(a^2+b^2+c^2) \ge 2\sqrt{3}(am_a+bm_b+cm_c)$, (2).

Next we have $2\sqrt{3}(am_a+bm_b+cm_c) \ge 2\sqrt{3}(ah_a+bh_b+ch_c) = 12\sqrt{3}F$, (3), where we denote h_a the altitude from the vertex A, h_b the altitude from the vertex B, h_c the altitude from the vertex C. From (2) and (3) we obtain the desired inequality.

Application 3. (D.M. Bătineţu-Giurgiu , N. Stanciu - Other refinement of Ionescu-Weitzenböck inequality).

Prove that in any triangle ABC with the area F, the altidudes h_a, h_b, h_c, the interior angles bisectors w_a, w_b, w_c and usual notations is true the following inequality

$$a^2+b^2+c^2 \geq \frac{4F}{\sqrt{3}}\left(\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\right) \geq 4\sqrt{3}F.$$

Solution. WLOG we can assume that $a \leq b \leq c$, then $w_a \geq w_b \geq w_c$ and $h_a \geq h_b \geq h_c$.

By Chebyshev's inequality we get that $\sum_{cyclic}\frac{w_a}{h_a} \leq \frac{1}{3}\left(\sum_{cyclic} w_a\right)\left(\sum_{cyclic}\frac{1}{h_a}\right)$, (1);

$\sum_{cyclic}\frac{1}{h_a} = \sum_{cyclic}\frac{a}{ah_a} = \frac{1}{2F}(a+b+c) = \frac{s}{F}$, (2), and,

$$w_a = \frac{2bc}{b+c}\cos\frac{A}{2} = \frac{2bc}{b+c}\sqrt{\frac{s(s-a)}{bc}} = \frac{2\sqrt{bc}}{b+c}\cdot\sqrt{s(s-a)} \leq \frac{2\sqrt{bc}}{2\sqrt{bc}}\sqrt{s(s-a)} = \sqrt{s(s-a)}, \quad (3)$$

and other two similar. Using the inequality $x^2+y^2+z^2 \geq \frac{(x+y+z)^2}{3}$, (*), and (3) we deduce

that $\sum_{cyclic} s(s-a) \geq \frac{\left(\sum_{cyclic}\sqrt{s(s-a)}\right)^2}{3} \Leftrightarrow 3s^2 \geq \left(\sum_{cyclic}\sqrt{s(s-a)}\right)^2 \Leftrightarrow \sum_{cyclic}\sqrt{s(s-a)} \leq s\sqrt{3} \Leftrightarrow$

$\Leftrightarrow \sum_{cyclic} w_a \leq s\sqrt{3}$, (4). From (1), (2), (4) and (*) we obtain that

$$\sum_{cyclic}\frac{w_a}{h_a} \leq \frac{1}{3}\left(\sum_{cyclic} w_a\right)\left(\sum_{cyclic}\frac{1}{h_a}\right) \leq \frac{1}{3}\cdot s\sqrt{3}\cdot\frac{s}{F} = \frac{(a+b+c)^2}{4F\sqrt{3}} \leq \frac{3(a^2+b^2+c^2)}{4F\sqrt{3}}, \quad (5).$$

Since $w_a \geq h_a$, $w_b \geq h_b$, $w_c \geq h_c$ from (5) we get

$$3 \leq \sum_{cyclic}\frac{w_a}{h_a} \leq \frac{3(a^2+b^2+c^2)}{4F\sqrt{3}} \Leftrightarrow a^2+b^2+c^2 \geq \frac{4F}{\sqrt{3}}\left(\frac{w_a}{h_a}+\frac{w_b}{h_b}+\frac{w_c}{h_c}\right) \geq 4\sqrt{3}F, \text{ Q.E.D.}$$

Application 4. (D.M. Bătineţu-Giurgiu , N. Stanciu - Math Problems, Volume 3, Issue 1, Junior MathProblems, Problem 1, 2013).

Prove that in all triangle ABC, with usual notations, holds:

$$\frac{a^3}{b\cdot R+c\cdot r}+\frac{b^3}{c\cdot R+a\cdot r}+\frac{c^3}{a\cdot R+b\cdot r}\geq\frac{4\sqrt{3}}{R+r}S.$$

Solution. We have $V=\sum_{cyc}\frac{a^3}{b\cdot R+c\cdot r}=\sum_{cyc}\frac{a^4}{abR+acr}\geq 2\cdot\sum_{cyc}\frac{(a^2)^2}{(a^2+b^2)R+(a^2+c^2)r}$,

where we apply *Bergström* 's inequality and we deduce that:

$$V\geq 2\cdot\frac{\left(\sum_{cyc}a^2\right)^2}{\sum_{cyc}R(a^2+b^2)+\sum_{cyc}r(a^2+c^2)}=2\cdot\frac{\left(\sum_{cyc}a^2\right)^2}{2(R+r)\cdot\sum_{cyc}a^2}=\frac{\sum_{cyc}a^2}{R+r}.$$

Then applying Ionescu- Weitzenböck' s inequality, i.e $\sum_{ciclic}a^2\geq 4S\sqrt{3}$, $V\geq\frac{4S\sqrt{3}}{R+r}$, and we are done.

Application 5. (D.M. Bătineţu-Giurgiu , N. Stanciu - Revista Escolar de la Olimpiada Iberoamericana de Matematica, Numero 49 (julio-agosto 2013), Problema 242).

If $m\in R_+$, then in all triangle ABC, with usual notations (i.e. R - circumradius, r - inradius, the lengths of the sides are a,b,c and S - the area of triangle ABC) the following inequality holds:

$$\frac{a^{m+2}}{(b\cdot R+c\cdot r)^m}+\frac{b^{m+2}}{(c\cdot R+a\cdot r)^m}+\frac{c^{m+2}}{(a\cdot R+b\cdot r)^m}\geq\frac{4\sqrt{3}}{(R+r)^m}S.$$

Solution. $W=\sum_{cyc}\frac{a^{m+2}}{(b\cdot R+c\cdot r)^m}=\sum_{cyc}\frac{a^{2(m+1)}}{(abR+acr)^m}\geq 2^m\cdot\sum_{cyc}\frac{(a^2)^{m+1}}{((a^2+b^2)R+(a^2+c^2)r)^m}$,

and applying the inequality of *J. Radon* we obtain

$$W\geq 2^m\cdot\frac{\left(\sum_{ciclic}a^2\right)^{m+1}}{(\sum_{ciclic}R(a^2+b^2)+\sum_{ciclic}r(a^2+c^2))^m}=2^m\cdot\frac{\left(\sum_{ciclic}a^2\right)^{m+1}}{2^m(R+r)^m\cdot(\sum_{ciclic}a^2)^m}=\frac{\sum_{ciclic}a^2}{(R+r)^m}.$$

By *Ionescu- Weitzenböck*'s inequality, i.e. $\sum_{cyc} a^2 \geq 4S\sqrt{3}$, we get $W \geq \frac{4S\sqrt{3}}{(R+r)^m}$, and the proof is complete.

Application 6. (D.M. Bătineţu-Giurgiu , N. Stanciu - Math Problems, Volume 3, Issue 1, Junior MathProblems, Problem 14, 2013) . If $m \in [0,\infty)$, $x,y,z,t \in (0,\infty)$, then in any triangle ABC, with usual notations holds:

$$\sum_{cyclic} \frac{\left(xm_a^2 + ym_b^2\right)^{m+1}}{\left(zw_a^2 + tw_b^2\right)^m} \geq 3 \cdot \sqrt{3} \cdot \frac{(x+y)^{m+1}}{(z+t)^m} S .$$

Solution. $w_a = \frac{2\sqrt{bc}}{b+c}\sqrt{s(s-a)} \Rightarrow w_a \leq \sqrt{s(s-a)} \Rightarrow w_a^2 \leq s(s-a)$, and other two similar.

So, $w_a^2 + w_b^2 + w_c^2 \leq s(s-a+s-b+s-c) = s(3s-2s) = s^2 =$

$= \frac{(a+b+c)^2}{4} \leq \frac{3}{4}(a^2+b^2+c^2) = m_a^2 + m_b^2 + m_c^2$, (1); by $\sum_{cyclic} m_a^2 = \frac{3}{4}\sum_{cyclic} a^2$, J. Radon's inequality and (1), we obtain $\sum_{cyclic} \frac{\left(xm_a^2 + ym_b^2\right)^{m+1}}{\left(zw_a^2 + tw_b^2\right)^m} \overset{RADON}{\geq} \frac{\left(\sum_{cyclic}(xm_a^2 + ym_b^2)\right)^{m+1}}{\left(\sum_{cycclic}(zw_a^2 + tw_b^2)\right)^m} =$

$$= \frac{\left((x+y)\sum_{cyclic} m_a^2\right)^{m+1}}{\left((z+t)\sum_{cycclic} w_a^2\right)^m} \geq \frac{(x+y)^{m+1}\left(\sum_{cyclic} m_a^2\right)^{m+1}}{(z+t)^m\left(\sum_{cyclic} m_a^2\right)^m} = \frac{(x+y)^{m+1}\left(\sum_{cyclic} m_a^2\right)}{(z+t)^m} = \frac{3}{4} \cdot \frac{(x+y)^{m+1}}{(z+t)^m}\sum_{cyclic} a^2 ,$$

(2), by Ion Ionescu – Weitzenböck inequality we have $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$, (3). By (2) and (3) we obtain

$$\sum_{cyclic} \frac{\left(xm_a^2 + ym_b^2\right)^{m+1}}{\left(zw_a^2 + tw_b^2\right)^m} \geq 3 \cdot \sqrt{3} \cdot \frac{(x+y)^{m+1}}{(z+t)^m} S$$, q.e.d.

Application 7. (D.M. Bătineţu-Giurgiu , N. Stanciu - Recreaţii Matematice, 2/2013).

Prove that in all triangle ABC, with usual notations, holds

$$\frac{m_a^3}{R \cdot m_b + r \cdot m_c} + \frac{m_b^3}{R \cdot m_c + r \cdot m_a} + \frac{m_c^3}{R \cdot m_a + r \cdot m_b} \geq \frac{3\sqrt{3}}{R+r} S .$$

Solution. $U=\sum_{cyc}\frac{m_a^3}{R\cdot m_b+r\cdot m_c}=\sum_{cyc}\frac{(m_a^2)^2}{R\cdot m_a\cdot m_b+r\cdot m_a\cdot m_c}\geq$

$\geq 2\cdot\sum_{cyc}\frac{(m_a^2)^2}{R(m_a^2+m_b^2)+r(m_a^2+m_c^2)}$, where we apply *Bergström's* inequality and well-known formula $m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$. We obtain that

$$U\geq 2\cdot\frac{\left(\sum_{cyc}m_a^2\right)^2}{R\cdot\sum_{cyc}(m_a^2+m_b^2)+r\cdot\sum_{cyc}(m_a^2+m_c^2)}=\frac{2\cdot\left(\sum_{cyc}m_a^2\right)^2}{2R\cdot\sum_{cyc}m_a^2+2r\cdot\sum_{cyc}m_a^2}=$$

$=\frac{\sum_{cyc}m_a^2}{R+r}=\frac{3}{4}\cdot\frac{a^2+b^2+c^2}{R+r}$, where we use the *Ionescu – Weitzenböck* inequality, i.e. $a^2+b^2+c^2\geq 4S\sqrt{3}$, and we deduce that $U\geq\frac{3}{4}\cdot\frac{1}{R+r}\cdot 4S\sqrt{3}=\frac{3\sqrt{3}}{R+r}S$, and we are done.

Application 8. (D.M. Bătineţu-Giurgiu , N. Stanciu - The College Mathematics Journal (CMJ), Vol. 45, No. 5, November 2014, Problem 1038).

If $m\in R_+, x,y\in R_+^*$, then in any triangle ABC holds

$$\frac{a^{m+2}}{(xb+yc)^m}+\frac{b^{m+2}}{(xc+ya)^m}+\frac{c^{m+2}}{(xa+yb)^m}\geq\frac{4\sqrt{3}}{(x+y)^m}\cdot areaABC.$$

Solution. $E=\sum_{cyclic}\frac{a^{m+2}}{(xb+yc)^m}=\sum_{cyclic}\frac{a^{2(m+1)}}{(xab+yac)^m}\overset{(RADON)}{\geq}\frac{(a^2+b^2+c^2)^{m+1}}{\left(\sum_{cyclic}(xab+yac)\right)^m}=$

$=\frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(ab+bc+ca)^m}$, and by $a^2+b^2+c^2\geq ab+bc+ca$, we deduce

$$E\geq\frac{(a^2+b^2+c^2)^{m+1}}{(x+y)^m(a^2+b^2+c^2)^m}=\frac{a^2+b^2+c^2}{(x+y)^m}\text{ , (1).}$$

Ionescu-Weitzenböck inequality $a^2+b^2+c^2\geq 4\sqrt{3}\cdot areaABC$, (2)

From (1) and (2) we obtain the desired inequality.

Application 9. (D.M. Bătineţu-Giurgiu , N. Stanciu - The College Mathematics Journal (CMJ), Vol. 46, No. 2, March 2015, Problem 1050).

Let $A_1A_2...A_n$, $n\ge 3$, be a convex polygon with a_k the length of the side $[A_kA_{k+1}]$, $k=\overline{1,n}$, $A_{n+1}=A_1$, and let S be the area of the polygon. Prove that

$$\left(\sum_{k=1}^{n}a_k^{2m+4}\right)\left(\sum_{k=1}^{n}\frac{1}{a_k^{2m}}\right)\ge 16S^2tg^2\frac{\pi}{n},\ \forall m\in R_+.$$

Solution. By *J. Radon's* inequality we deduce that: $\sum_{k=1}^{n}a_k^{2m+4}\ge\frac{1}{n^{m+1}}\left(\sum_{k=1}^{n}a_k^2\right)^{m+2}$, $\forall m\in R_+$, (1),

and also by *J. Radon's* inequality we obtain that $\sum_{k=1}^{n}\frac{1}{a_k^{2m}}\ge\frac{n^{m+1}}{\left(\sum_{k=1}^{n}a_k^2\right)^m}$, $\forall m\in R_+$, (2).

So, (1) and (2) yields that $U_n=\left(\sum_{k=1}^{n}a_k^{2m+4}\right)\left(\sum_{k=1}^{n}\frac{1}{a_k^{2m}}\right)\ge\left(\sum_{k=1}^{n}a_k^2\right)^2$, $\forall m\in R_+$, (3).

By *E. Just and N. Schaumberger'* s inequality (The Problem 1634 from AMM, 70(1963)) we have that $\sum_{k=1}^{n}a_k^2\ge 4S\cdot tg\frac{\pi}{n}$, (4). Therefore, from (3) and (4) we obtain that $U_n\ge 16S^2tg^2\frac{\pi}{n}$, $\forall m\in R_+$ and we are done.

Application 10. (D.M. Bătineţu-Giurgiu , N. Stanciu - Revista Escolar de la Olimpiada Iberoamericana de Matematica, Problem 277, August 2016) . Show that in any triangle ABC (with usual notations) holds the following inequality

$$(ab+bc+ca)^2+2(a^2+b^2+c^2)^2\ge 16\sqrt{3}\cdot s^3\cdot r.$$

Solution. For any $w_1,\ w_2,\ w_3\in R$ we have $w_1^2+w_2^2+w_3^2\ge w_1\cdot w_2+w_2\cdot w_3+w_3\cdot w_1$, (1), with equality iff $w_1=w_2=w_3$. If $m,\ n,\ p\in R_+^*$, then $\frac{m\cdot n}{p}+\frac{n\cdot p}{m}+\frac{p\cdot m}{n}\ge m+n+p$, (2) with equality iff $m=n=p$. Indeed if in (1) we take $w_1=\sqrt{\frac{m\cdot n}{p}}$, $w_2=\sqrt{\frac{n\cdot p}{m}}$, $w_3=\sqrt{\frac{p\cdot m}{n}}$ yields (2).

If $t,\ u,\ v,\ x,\ y, z\in R_+^*$, then $\frac{(tx+uy+vz)(ty+uz+vx)}{tz+ux+vy}+\frac{(ty+uz+vx)(tz+ux+vy)}{tx+uy+vz}+$

$+\frac{(tz+ux+vy)(tx+uy+vz)}{ty+uz+vx}\ge(t+u+v)(x+y+z)$, (3). Indeed if in (2) we take $m=tx+uy+vz$, $n=ty+uz+vx$, $p=tz+ux+vy$ yields (3).

Lemma. For any $x,\ y,\ z\in R_+^*$, holds

$$(xy+yz+zx)^2+2(x^2+y^2+z^2)^2 \geq (x^2+y^2+z^2)(x+y+z)^2, \text{ (4).}$$

Proof. In (3) we take $t=x, u=y, v=z$ and we obtain

$$\frac{(x^2+y^2+z^2)(xy+yz+zx)}{xz+xy+zy}+\frac{(xy+yz+zx)(xz+xy+yz)}{x^2+y^2+z^2}+$$

$$+\frac{(xz+xy+yz)(x^2+y^2+z^2)}{xy+yz+xz} \geq (x+y+z)^2 \Leftrightarrow \frac{(xy+yz+zx)^2}{x^2+y^2+z^2}+2(x^2+y^2+z^2) \geq (x+y+z)^2$$

$$\Leftrightarrow (xy+yz+zx)^2+2(x^2+y^2+z^2)^2 \geq (x+y+z)^2(x^2+y^2+z^2), \text{ q.e.d.}$$

If in (4) we take $x=a, y=b, z=c$ we obtain

$$(ab+bc+ca)^2+2(a^2+b^2+c^2)^2 \geq (a+b+c)^2(a^2+b^2+c^2), \text{ (5).}$$

By *Ionescu-Weitzenböck* 's inequality we have $a^2+b^2+c^2 \geq 4\sqrt{3}\cdot S$, (I-W).

So (5) becomes $(ab+bc+ca)^2+2(a^2+b^2+c^2)^2 \geq 4s^2\cdot 4\sqrt{3}S = 16\sqrt{3}\cdot s^2\cdot sr = 16\sqrt{3}s^3r$.

GEOMETRICAL INEQUALITIES WITH CLASSICAL MEANS

By Bogdan Fuștei-Romania

For positive numbers x,y>0 (with x≠y; the equality case is the common limit), we present the definitions and a simple proof of the chain:

G(x,y) ≤ L(x,y) ≤ I(x,y) ≤ A(x,y)(1)

with equality if and only if x=y.

1.DEFINITIONS

Geometric Mean (G).

Multiplicative average, natural for ratios and growth rates.

$G(x,y)=\sqrt{xy}$ **equivalently $G(x,y)=\exp((\ln x+\ln y)/2)$.**

Logarithmic Mean (L).

The mean attached to the logarithm; it equals the average along the geometric interpolation $x^{1-t}y^t$.

$$\boxed{\mathbf{L(x,y)=\frac{y-x}{\ln y-\ln x}}}\quad \mathbf{(x\neq y),}\qquad \mathbf{L(x,x)=x.}$$

Useful identities:

$$L(x,y)=\int_0^1 x^{1-t}\,y^t\,dt,$$
$$\frac{1}{L(x,y)}=\frac{1}{y-x}\int_x^y \frac{dt}{t}.$$

Identric Mean (I).

A mean between L and A; it arises by averaging lnt over [x,y].

$$\boxed{\mathbf{I(x,y)=\frac{1}{e}\left(\frac{y^y}{x^x}\right)^{\frac{1}{y-x}}}},\qquad \mathbf{\ln I(x,y)=\frac{1}{y-x}\int_x^y \ln t\,dt.}$$

Arithmetic Mean (A).

The familiar additive average:

$$\boxed{\mathbf{A(x,y)=\frac{x+y}{2}}}.$$

2.THE INEQUALITY CHAIN G≤L≤I≤A

Step 1: G≤L.

Using the integral form of L and Jensen's inequality for the concave function ln:

$$\ln L(x,y)=\ln\left(\int_0^1 x^{1-t}\,y^t\,dt\right)\geq\int_0^1 \ln\left(x^{1-t}y^t\right)dt=\frac{\ln x+\ln y}{2}.$$

Exponentiating gives $L(x,y)\geq\exp\big((\ln x+\ln y)/2\big)=\sqrt{xy}=G(x,y)$, with equality iff x=y.

Step 2: L≤I.

Let T be uniformly distributed on [x,y]. Then

$$E[\ln T]=\frac{1}{y-x}\int_x^y \ln t\,dt=\ln I(x,y),$$
$$E\left[\frac{1}{T}\right]=\frac{1}{y-x}\int_x^y \frac{dt}{t}=\frac{1}{L(x,y)}.$$

Apply Jensen to ln (concave) with the positive random variable X=1/T:

$$\ln E[X] \geq E[\ln X] \quad \Rightarrow \quad \ln\left(\frac{1}{L}\right) \geq -\ln I \Rightarrow \ln L \leq \ln I \Rightarrow L \leq I,$$

with equality iff x=y.

Step 3: I≤A (and G≤I).

Hermite–Hadamard for the convex function $f(t)=-\ln t$ on $[x,y]$ yields

$$-\ln(x+y/2) \leq \frac{1}{y-x}\int_x^y (-\ln t)\,dt \leq \frac{-\ln x-\ln y}{2}.$$

Equivalently $-\ln A(x,y) \leq -\ln I(x,y) \leq -\ln\sqrt{xy}$. Since ln is increasing, we obtain

$$I(x,y) \leq A(x,y) \quad \text{and} \quad I(x,y) \geq \sqrt{xy} = G(x,y),$$

with equality iff x=y.

APPLICATIONS

ABC triangle with usual notations; for $x=\frac{b}{c}$; $y=\frac{c}{b} \to G\left(\frac{b}{c}, \frac{c}{b}\right)=1$

$$\mathbf{1 \leq L\left(\frac{b}{c}, \frac{c}{b}\right) \leq I\left(\frac{b}{c}, \frac{c}{b}\right) \leq \frac{1}{2}\left(\frac{b}{c}+\frac{c}{b}\right) \to 2 \leq 2L\left(\frac{b}{c}, \frac{c}{b}\right) \leq 2I\left(\frac{b}{c}, \frac{c}{b}\right) \leq \frac{b}{c}+\frac{c}{b} \ (2)}$$

n_a-Nagel's cevian from A;

Useful results:

$$\mathbf{1+\frac{n_a}{h_a} \geq \frac{b}{c}+\frac{c}{b}} \ \textbf{(and analogous)[1].(3)}$$

Also : $\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2r_ah_a}$ (and analogous)[2].

$\frac{R}{r}-1=\frac{n_a^2+r_a^2}{2r_ah_a} \to r_ah_a\left(\frac{R}{r}-1\right)=\frac{1}{2}(n_a^2+r_a^2) \to \frac{h_a}{n_a}\left(\frac{R}{r}-1\right)=\frac{1}{2}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)$(and analogous)

$G\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right)=1 \leq L\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq I\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq \frac{1}{2}\left(\frac{n_a}{r_a}+\frac{r_a}{n_a}\right)=\frac{h_a}{n_a}\left(\frac{R}{r}-1\right)$

$\frac{n_a}{h_a} \leq \frac{n_a}{h_a}L\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq \frac{n_a}{h_a}I\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq \frac{R}{r}-1$

$$\mathbf{1+\frac{n_a}{h_a} \leq 1+\frac{n_a}{h_a}L\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq 1+\frac{n_a}{h_a}I\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq \frac{R}{r}} \ \textbf{(4)(and analogous)}$$

From (2),(3),(4)→

$$\mathbf{2 \leq 2L\left(\frac{b}{c}, \frac{c}{b}\right) \leq 2I\left(\frac{b}{c}, \frac{c}{b}\right) \leq \frac{b}{c}+\frac{c}{b} \leq 1+\frac{n_a}{h_a} \leq 1+\frac{n_a}{h_a}L\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq 1+\frac{n_a}{h_a}I\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq \frac{R}{r} \ (5)}$$

From (4)→ $\frac{n_an_bn_c}{h_ah_bh_c} \leq \frac{n_an_bn_c}{h_ah_bh_c}\prod L\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq \frac{n_an_bn_c}{h_ah_bh_c}\prod I\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right) \leq \left(\frac{R}{r}-1\right)^3$

$$\mathbf{1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}} \leq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}\prod L\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right)} \leq 1+\sqrt[3]{\frac{n_an_bn_c}{h_ah_bh_c}\prod I\left(\frac{n_a}{r_a}, \frac{r_a}{n_a}\right)} \leq \frac{R}{r} \ (6)}$$

From $\frac{n_a}{h_a}=\frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogous)[2] ;$\sqrt{4r^2+(b-c)^2}\geq 2r$ (and analogous) and (4)

$2\leq 1+\frac{\sqrt{4r^2+(b-c)^2}}{2r}\leq 1+\frac{\sqrt{4r^2+(b-c)^2}}{2r}L\left(\frac{n_a}{r_a},\frac{r_a}{n_a}\right)\leq 1+\frac{\sqrt{4r^2+(b-c)^2}}{2r}I\left(\frac{n_a}{r_a},\frac{r_a}{n_a}\right)\leq\frac{R}{r}$

$$\mathbf{2r\leq r+\frac{\sqrt{4r^2+(b-c)^2}}{2}\leq r+\frac{\sqrt{4r^2+(b-c)^2}}{2}L\left(\frac{n_a}{r_a},\frac{r_a}{n_a}\right)\leq r+\frac{\sqrt{4r^2+(b-c)^2}}{2}I\left(\frac{n_a}{r_a},\frac{r_a}{n_a}\right)\leq R\ (7)}$$

$l_a=\frac{2\sqrt{bc}}{b+c}\sqrt{r_br_c}$ (and analogous)$\rightarrow\frac{\sqrt{r_br_c}}{l_a}=\frac{1}{2}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{c}{b}}\right)$(and analogous)

We obtain:

$$\mathbf{1\leq L\left(\sqrt{\frac{b}{c}},\sqrt{\frac{c}{b}}\right)\leq I\left(\sqrt{\frac{b}{c}},\sqrt{\frac{c}{b}}\right)\leq\frac{\sqrt{r_br_c}}{l_a}\ \text{(and analogous)(8)}}$$

$r_a=\frac{S}{p-a}$ (and analogous);$p=\frac{1}{2}(a+b+c)$;$2S=ah_a=bh_b=ch_c=(a+b+c)r$;

$r_a=\frac{2S}{2p-2a}=\frac{ah_a}{2p-2a}=\frac{ah_a}{b+c-a}\rightarrow\frac{r_a}{h_a}=\frac{a}{b+c-a}\rightarrow\frac{b+c}{a}=1+\frac{h_a}{r_a}$ (and analogous)

$$\mathbf{2\sqrt{\frac{h_a}{r_a}}\leq 2L\left(1,\frac{h_a}{r_a}\right)\leq 2I\left(1,\frac{h_a}{r_a}\right)\leq\frac{b+c}{a}\ \text{(and analogous) (9)}}$$

I-the incenter of the triangle ABC;$AI=\frac{b+c}{2p}l_a$(and analogous); $AI=\frac{r}{\sin\frac{A}{2}}$(and analogous);

$\sin\frac{A}{2}=\sqrt{\frac{r}{2R}\frac{r_a}{h_a}}$(and analogous)$\rightarrow\frac{AI}{r}=\sqrt{\frac{2R}{r}\frac{h_a}{r_a}}$ (and analogous);

$\frac{AI}{r}=\frac{b+c}{2pr}l_a=\frac{b+c}{a}\frac{l_a}{h_a}$ (and analogous);

(9) became:$2\frac{l_a}{h_a}\sqrt{\frac{h_a}{r_a}}\leq 2\frac{l_a}{h_a}L\left(1,\frac{h_a}{r_a}\right)\leq 2\frac{l_a}{h_a}I\left(1,\frac{h_a}{r_a}\right)\leq\frac{b+c}{a}\frac{l_a}{h_a}=\sqrt{\frac{2R}{r}\frac{h_a}{r_a}}$

$2\frac{l_a}{h_a}\leq 2\frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}}L\left(1,\frac{h_a}{r_a}\right)\leq 2\frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}}I\left(1,\frac{h_a}{r_a}\right)\leq\sqrt{\frac{2R}{r}}$;

$$\mathbf{\frac{l_a}{h_a}\leq\frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}}L\left(1,\frac{h_a}{r_a}\right)\leq\frac{l_a}{h_a}\sqrt{\frac{r_a}{h_a}}I\left(1,\frac{h_a}{r_a}\right)\leq\sqrt{\frac{R}{2r}}\ \text{(and analogous) (10)}}$$

Useful results:

$\frac{b+c}{a}=\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}-1$ (and analogous)[3]. and (9) obtain:

$$\mathbf{2\sqrt{\frac{h_a}{r_a}}\leq 2L\left(1,\frac{h_a}{r_a}\right)\leq 2I\left(1,\frac{h_a}{r_a}\right)\leq\frac{2n_a}{\sqrt{4r^2+(b-c)^2}}-1\ \text{(and analogous)(11)}}$$

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NAGEL'S CEVIANS REVISITED (II)

By Bogdan Fuștei-Romania

Let ABC be a triangle with sides a, b, c, semiperimeter

$$s = \frac{a+b+c}{2},$$

and area Δ. Define: $x = \sqrt{s^2 - 2r_a h_a}$, $\quad y = \sqrt{s^2 - 2r_b h_b}$.

We have

$$r_a = \frac{\Delta}{s-a}, \qquad h_a = \frac{2\Delta}{a},$$

hence

$$2r_a h_a = \frac{4\Delta^2}{a(s-a)}.$$

By Heron's formula $\Delta^2 = s(s-a)(s-b)(s-c)$, we obtain

$$x^2 = s^2 - \frac{4s(s-b)(s-c)}{a}.$$

Since

$$(s-b)(s-c) = \frac{a^2-(b-c)^2}{4},$$

it follows that

$$x^2 = s(s-a) + \frac{s(b-c)^2}{a}.$$

Similarly,

$$y^2 = s(s-b) + \frac{s(a-c)^2}{b}.$$

Set

$$p = s-a, \qquad q = s-b, \qquad t = s-c.$$

Then

$$a = q+t, \quad b = p+t, \quad c = p+q, \quad s = p+q+t,$$

$$b-c = t-q, \qquad a-c = t-p.$$

Therefore

$$x^2 = s\left(p + \frac{(t-q)^2}{q+t}\right), \qquad y^2 = s\left(q + \frac{(t-p)^2}{p+t}\right).$$

Hence

$$x \geq \sqrt{sp}, \qquad y \geq \sqrt{sq}, \qquad xy \geq s\sqrt{pq}.$$

Thus

$$(x+y)^2 \geq s\left(p + q + \frac{(t-q)^2}{t+q} + \frac{(t-p)^2}{t+p} + 2\sqrt{pq}\right).$$

Lemma. For $\mathbf{u}, \mathbf{v} > 0$,

$$\frac{(\mathbf{u}-\mathbf{v})^2}{\mathbf{u}+\mathbf{v}} \geq \left(\sqrt{\mathbf{u}} - \sqrt{\mathbf{v}}\right)^2.$$

Applying it with $(u, v) = (t, q)$ and (t, p) gives

$$\frac{(t-q)^2}{t+q} + \frac{(t-p)^2}{t+p} \geq \left(\sqrt{t} - \sqrt{q}\right)^2 + \left(\sqrt{t} - \sqrt{p}\right)^2.$$

Hence

$$p + q + \frac{(t-q)^2}{t+q} + \frac{(t-p)^2}{t+p} + 2\sqrt{pq} \geq p + q + t + \left(\sqrt{p} + \sqrt{q} - \sqrt{t}\right)^2 > p + q + t.$$

Therefore

$$(x+y)^2 > s(p+q+t) = s^2.$$

$$\boxed{x + y > s}$$

But $n_a=\sqrt{s^2 - 2r_a h_a}$ (and analogous)[1] $\rightarrow$

$n_a+n_b > s$ (and analogous)(1)

From $n_a=\sqrt{s^2 - 2r_a h_a}$ (and analogous)$\rightarrow$

$s > n_a$ (and analogous)(2)

n_a, n_b, n_c-Nagel cevians, from (1) and (2)$\rightarrow$

$n_a+n_b > n_c$ (and analogous) (3)

n_a, n_b, n_c-can be sides of a triangle (4)

From $n_a=\sqrt{s^2 - 2r_a h_a}$ (and analogous)$\rightarrow 2r_a h_a = s^2 - n_a^2 = (s-n_a)(s+n_a)$

$\frac{2r_a}{s+n_a} = \frac{s}{h_a} - \frac{n_a}{h_a}$ (and analogous); $ah_a=2sr \rightarrow \frac{2r_a}{s+n_a} = \frac{a}{2r} - \frac{n_a}{h_a}$

$\frac{a}{2r} = \frac{n_a}{h_a} + \frac{2r_a}{s+n_a}$ (and analogous)(5)

From (3) and (5):

$$\frac{a}{2r} > \frac{n_a}{h_a} + \frac{2r_a}{n_a+n_b+n_c} \textbf{ (and analogous)(6)}$$

$$\frac{a}{2r} > \frac{n_a}{h_a} + \frac{2r_a}{2n_a+n_c} \textbf{(and analogous)(7)}$$

$$\frac{a}{2r} > \frac{n_a}{h_a} + \frac{2r_a}{2n_a+n_b} \textbf{(and analogous)(8)}$$

$2r_ah_a=s^2 - n_a^2=(s-n_a)(s+n_a)\rightarrow\frac{2r_ah_a}{s-n_a}=s+n_a$

$$\frac{2r_a}{s-n_a} = \frac{a}{2r} + \frac{n_a}{h_a} \textbf{(and analogous)(9)}$$

$s-n_a<n_b + n_c - n_a \rightarrow \frac{1}{s-n_a} > \frac{1}{n_b+n_c-n_a} \rightarrow \frac{2r_a}{s-n_a} > \frac{2r_a}{n_b+n_c-n_a}$

$$\rightarrow\frac{a}{2r} + \frac{n_a}{h_a} > \frac{2r_a}{n_b+n_c-n_a} \textbf{ (and analogous)(10)}$$

$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r}$ (and analogous)[2] and from (6),(7),(8) :

$$\frac{n_a+n_b+n_c}{r} > \frac{4r_a}{a-\sqrt{4r^2+(b-c)^2}} \textbf{ (and analogous) (11)}$$

$$\frac{2n_a+n_c}{r} > \frac{4r_a}{a-\sqrt{4r^2+(b-c)^2}} \textbf{(and analogous) (12)}$$

$$\frac{2n_a+n_b}{r} > \frac{4r_a}{a-\sqrt{4r^2+(b-c)^2}} \textbf{ (and analogous) (13)}$$

From $\frac{s}{h_a} = \frac{a}{2r} \rightarrow$

$$\frac{s}{h_a} = \frac{a}{2r} = \frac{s-a}{h_a-2r} \textbf{ (and analogous)(14)}$$

$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} \rightarrow$

$$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} = \frac{n_a-\sqrt{4r^2+(b-c)^2}}{h_a-2r} \textbf{ (and analogous)(15)}$$

$$\frac{s}{h_a} = \frac{a}{2r} \rightarrow \frac{s}{h_a} = \frac{a}{2r} = \frac{s+a}{h_a+2r} \textbf{ (and analogous)(16)}$$

$$\frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} \rightarrow \frac{n_a}{h_a} = \frac{\sqrt{4r^2+(b-c)^2}}{2r} = \frac{n_a+\sqrt{4r^2+(b-c)^2}}{h_a+2r} \textbf{ (and analogous)(17)}$$

From (14) and (5) :

$$\frac{2r_a}{s+n_a} = \frac{s+\sqrt{4r^2+(b-c)^2}-a-n_a}{h_a-2r} \textbf{ (and analogous) (18)}$$

From (5) and (16):

$$\frac{2r_a}{s+n_a} = \frac{s+a-\sqrt{4r^2+(b-c)^2}-n_a}{h_a+2r} \textbf{ (and analogous)(19)}$$

From (18) and (1):

$$\frac{n_b+n_c+\sqrt{4r^2+(b-c)^2}-a-n_a}{h_a-2r} > \frac{2r_a}{s+n_a} \textbf{ (and analogous)(20)}$$

From (19) and (1):

$$\frac{n_b+n_c+a-\sqrt{4r^2+(b-c)^2}-n_a}{h_a+2r} > \frac{2r_a}{s+n_a} \textbf{ (and analogous) (21)}$$

We use :$\frac{r_a}{r} = \frac{h_a}{h_a-2r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}}$ (and analogous) [3]

$$\rightarrow \frac{r_a}{r} = \frac{n_a+h_a}{n_a-\sqrt{4r^2+(b-c)^2}+h_a-2r} \textbf{ (and analogous)(22)}$$

Also :$\frac{r_a}{r} = \frac{h_a}{h_a-2r} = \frac{n_a}{n_a-\sqrt{4r^2+(b-c)^2}} = \frac{s}{s-a}$ (and analogous) [3]

$$\rightarrow \frac{r_a}{r} = \frac{n_a+s}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} \textbf{ (and analogous) (23)}$$

From (1) and (23) :

$$\frac{2n_a+n_b}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} > \frac{r_a}{r} \textbf{ (and analogous) (24)}$$

$$\frac{n_a+n_b+n_c}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} > \frac{r_a}{r} \textbf{ (and analogous)(25)}$$

$$\frac{2n_a+n_c}{s-a+n_a-\sqrt{4r^2+(b-c)^2}} > \frac{r_a}{r} \textbf{ (and analogous) (26)}$$

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EQUATIONS AND INEQUATIONS INVOLVING GCD AND LCM

By Carmen – Victorița Chirfot – Romania

In this article, I will present a set of problems that use the properties of the greatest common divisor and the least common multiple for two non-zero natural numbers. In this regard, we use the usual notation. Thus, (a, b) represents the greatest common divisor of the non-zero natural numbers a and b, and $[a, b]$ represents the least common multiple of the non-zero natural numbers a and b.

Regarding inequality, for the greatest common divisor of the nonzero natural numbers a and b, we know that $(a, b) \leq a$ and $(a, b) \leq b$. For the least common multiple of the natural numbers a and b, we have $[a, b] \geq a$ and $[a, b] \geq b$.

Problem 1: Determine the non-zero natural number n, knowing that $(3n, 2025) = n + 6$.

Solution: From the relationship, it follows that $n + 6|3n$ and $n + 6|2025$. We have that

$$n+6|n+6 \Rightarrow n+6|3n+18 \Rightarrow n+6|3n+18-3n \Rightarrow n+6|18 \Rightarrow$$

$\Rightarrow n+6 \in \{1,2,3,6,9,18\}$. But n is natural non-zero, so $n \in \{3,12\}$. If $n=3 \Rightarrow (9,2025)=9$

true. So, $n=3$ it checks. If $n=12 \Rightarrow (36,2025)=18$, false. So, $n=12$ it does not check. In conclusion, $n=3$.

Problem 2: Determine the non-zero natural number n, knowing that $[n+6,3]=2n$.

Solution: From the relationship, it follows that $n+6|2n$. We have that

$$n+6|n+6 \Rightarrow n+6|2n+12 \Rightarrow n+6|2n+12-2n \Rightarrow n+6|12 \Rightarrow$$

$\Rightarrow n+6 \in \{1,2,3,4,6,12\}$. But n is non-zero natural, so $n=6$. We check, namely $[6+6,3]=2\cdot 6$, equivalent with $[12,3]=12$, which is true. So, $n=6$.

Problem 3: Determine the values of non-zero natural number n such that

$$(2n+4,3n+8)=n$$

Solution: From equation, it follows that $n|2n+4$ and $n|3n+8 \Rightarrow n|3n+8-2n-4 \Rightarrow$

$\Rightarrow n|4 \Rightarrow n \in \{1,2,4\}$. If $n=1 \Rightarrow (6,11)=1$, true. If $n=2 \Rightarrow (8,14)=2$, true. If

$n=3 \Rightarrow (10,17)=3$, false. So, $n \in \{1,2\}$.

Problem 4: Find the non-zero natural number n, knowing that $[2n,3]=n+6$.

Solution: From relationship, it follows that $2n|n+6 \Rightarrow 2n|2n+12 \Rightarrow 2n|2n+12-2n \Rightarrow$

$\Rightarrow 2n|12 \Rightarrow n|6 \Rightarrow n \in \{1,2,3,6\}$. But $3|n+6 \Rightarrow 3|n \Rightarrow n \in \{3,6\}$. If $n=3 \Rightarrow [6,3]=9$, false. If $n=6 \Rightarrow [12,3]=12$, true. So, $n=6$.

Problem 5: Determine the natural number n, such that $(2,n^2+7) \geq n^2-8$.

Solution: If n is even $(2,n^2+7)=1 \Rightarrow 1 \geq n^2-8 \Rightarrow n^2 \leq 9 \Rightarrow n \in \{0,1,2,3\}$. If n is even, so $n \in \{0,2\}$. If n is odd $(2,n^2+7)=2 \Rightarrow 2 \geq n^2-8 \Rightarrow n^2 \leq 10 \Rightarrow n \in \{0,1,2,3\}$. If n is even, so $n \in \{1,3\}$. So, $n \in \{0,1,2,3\}$.

Problem 6: Determine the natural number n, such that $(3,n^2+9) \geq 2n^2-15$.

Solution: If n is multiple of 3, $(3,n^2+9)=3 \Rightarrow 3 \geq 2n^2-15 \Rightarrow 2n^2 \leq 18 \Rightarrow n^2 \leq 9 \Rightarrow$

$\Rightarrow n \in \{0,1,2,3\}$. If n is multiple of 3, it follows $n \in \{0,3\}$. If n isn't multiple of 3,

$(3,n^2+9)=1 \Rightarrow 1 \geq 2n^2-15 \Rightarrow 2n^2 \leq 16 \Rightarrow n^2 \leq 8 \Rightarrow n \in \{0,1,2\}$. If n isn't multiple of 3 it follows that $n \in \{1,2\}$. So, $n \in \{0,1,2,3\}$.

Problem 7: Determine the natural number n, such that $[2,n^2+5] \geq 3n^2-4$.

Solution: If n is even $[2,n^2+5]=2(n^2+5) \Rightarrow 2n^2+10 \geq 3n^2-4 \Rightarrow n^2 \leq 14 \Rightarrow$

$\Rightarrow n \in \{0,1,2,3\}$. But n is even, so $n \in \{0,2\}$. If n is odd $[2,n^2+5]=n^2+5=n^2+5 \Rightarrow$

$\Rightarrow n^2+5 \geq 3n^2-4 \Rightarrow 2n^2 \leq 9 \Rightarrow n^2 \leq 4 \Rightarrow n \in \{0,1,2\}$. If n is odd, so $n=1$. In conclusion, $n \in \{0,1,2\}$.

Problem 8: Determine the values of the natural number n, such that $\left[4, n^2+8\right] \geq 6n^2-6$.

Solution: If n is even $[4, n^2+8] = n^2+8 \Rightarrow n^2+8 \geq 6n^2-6 \Rightarrow 5n^2 \leq 14 \Rightarrow n^2 \leq 2 \Rightarrow$

$\Rightarrow n \in \{0,1\}$. But n is odd, so $n=0$. If n is even $[4, n^2+8] = 4(n^2+8) \Rightarrow$

$\Rightarrow 4n^2+32 \geq 6n^2-6 \Rightarrow 2n^2 \leq 38 \Rightarrow n^2 \leq 19 \Rightarrow n \in \{0,1,2,3,4\}$. If n is odd, so $n \in \{1,3\}$. In conclusion $n \in \{0,1,3\}$.

Problem 9: Determine the values of the non-zero natural number n, such that

$$[2n-4, 3n-8] \leq n$$

Solution: We have that $2n-4 \leq [2n-4, 3n-8] \leq n \Rightarrow 2n-4 \leq n \Rightarrow n \leq 4$. But,

$2n-4 \geq 1$ and $3n-8 \geq 1 \Rightarrow n \geq 3$. So, $n \in \{3,4\}$. If $n=3 \Rightarrow [2,1] \leq 3$, true. If

$n=4 \Rightarrow [4,4] \leq 4$, true. In conclusion, $n \in \{3,4\}$.

Problem 10: Determine that it doesn't exist the natural number n, such that

$$(2n+16, n+7) \geq 3n.$$

Solution: We have that $n+7 \geq (2n+16, n+7) \geq 3n \Rightarrow n+7 \geq 3n \Rightarrow n \leq 3$. If

$n=1 \Rightarrow (18,8) \geq 3$, false. If $n=2 \Rightarrow (20,9) \geq 6$, false. If $n=4 \Rightarrow (22,10) \geq 9$, false. So, it doesn't exist n with the given property.

Problem 11: Prove that it doesn't exist the natural number n, such that

$$[4n, 6n+7] \leq 2n+19.$$

Solution: We have that $6n+7 \leq [4n, 6n+7] \leq 2n+19 \Rightarrow 6n+7 \leq 2n+19 \Rightarrow$

$$4n \leq 12 \Rightarrow n \leq 3$$

If $n=1 \Rightarrow [4,13] \leq 21$, false. If $n=2 \Rightarrow [8,19] \leq 23$, false. If $n=3 \Rightarrow [12,25] \leq 25$, false. So, it doesn't exist n with the given property.

Problem 12: If $b \neq 0$ and $(a+b, a-b) \geq a-b^2$, prove that a is a prime number.

Solution: We know that $a-b \geq (a+b, a-b) \Rightarrow a-b \geq a-b^2 \Rightarrow b^2 \geq b \Rightarrow b \in \{0,1\}$. If $b \neq 0 \Rightarrow b=1 \Rightarrow (a+1, a-1) \geq a-1$. We have and $a-1 \geq (a+1, a-1) \Rightarrow$

$\Rightarrow (a+1, a-1) = a-1 \Rightarrow a-1|a+1$. As $a-1|a-1 \Rightarrow a-1|2 \Rightarrow a-1 \in \{1,2\} \Rightarrow$

$\Rightarrow a \in \{2,3\}$. So, a is a prime number.

Problem 13: If $a \neq 1$ and $\left[a, 2a+b^2\right] \leq b^2+4$, prove that a and b are even numbers.

Solution: From $2a+b^2 \leq [a, 2a+b^2] \Rightarrow 2a+b^2 \leq b^2+4 \Rightarrow a \leq 2$. If a is a non-zero natural number $a \neq 1 \Rightarrow a=2 \Rightarrow [2, 4+b^2] \leq b^2+4 \Rightarrow [2, 4+b^2] = b^2+4 \Rightarrow$

$\Rightarrow 2|b^2+4 \Rightarrow 2|b^2 \Rightarrow 2|b$. So, a and b are even numbers.

Problem 14: Determine the non-zero natural numbers a and b for which

$$\mathbf{4a+b^2-8\leq(a,2a+b^2)}.$$

Solution: From $4a+b^2-8\leq(a,2a+b^2)\leq 2a+b^2\Rightarrow 4a+b^2-8\leq 2a+b^2\Rightarrow$

$\Rightarrow 2a\leq 8\Rightarrow a\leq 4$. If $a=1\Rightarrow b^2-4\leq(1,2+b^2)\Rightarrow b^2-4\leq 1\Rightarrow b^2\leq 5\Rightarrow b\in\{1,2\}$.

If $a=2\Rightarrow b^2\leq(2,4+b^2)$. If b is even number, it follows that $b^2\leq 2\Rightarrow b=0$, false. If b is odd, $b^2\leq 1\Rightarrow b=1$. So, if $a=2\Rightarrow b=1$.

If $a=3\Rightarrow b^2+8\leq(4,8+b^2)$. If b is even, it follows that $b^2+8\leq 4$, false. If b is odd, it follows that $b^2+8\leq 1$, false. So, if $a=4$ we don't have solutions.

In conclusion, if $a=1\Rightarrow b\in\{1,2\}$. If $a=2\Rightarrow b=1$.

Problem 15: If $c\leq(a,b)$ and $(a,b)\leq(a,c)$, prove that $(a,b,c)=c$ and $(b,c)=c$.

Solution: $c\leq(a,b)\leq(a,c)\leq\min\{a,c\}\Rightarrow(a,b)=(a,c)=c\Rightarrow c|a$ and

$c|b\Rightarrow(a,b,c)=c$ and $(b,c)=c$.

Proposed problems:

1. Determine $n\in\mathbb{N}^*$ such that $(n+8,4)=n$.

2. Determine $n\in\mathbb{N}^*$ such that $[2n,7]=n+7$.

3. Determine $n\in\mathbb{N}^*$ such that $(n+3,n+5)=n$.

4. Determine $n\in\mathbb{N}^*$ such that $[3n,10]=10n+10$.

5. Solve the inequation $(3,n^2+6)\geq n^2-10$.

6. Solve the inequation $(3,2n+6)\geq n^2+6$.

7. Solve the equation $[3,n^2]=3n$.

8. Solve the inequation $[4,n^2]\leq 3n$.

9. Find the non – zero natural numbers a and b for which $[a,2a+b^2]\leq a+b^2+4$.

10. Find the non-zero natural numbers a and b for which $2a+b^2-4\leq(a,a+b^2)$.

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ABOUT THE RMM PROBLEM J.3088

By Marin Chirciu-Romania

J.3088. In ΔABC holds:

$$\boldsymbol{R^4 \geq \frac{1}{81}\left(\sum a^2\right)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Marin Chirciu – Romania

Using $\sum a^2 = 2(p^2 - r^2 - 4Rr)$ the inequality can be written:

$R^4 \geq \frac{1}{81}\left(2(p^2 - r^2 - 4Rr)\right)^2 \Leftrightarrow 81R^4 \geq 4(p^2 - r^2 - 4Rr)^2$, which follows from Gerretsen's inequality: $p^2 \leq 4R^2 + 4Rr + 3r^2$.It remains to prove that:

$$81R^4 \geq 4(4R^2 + 4Rr + 3r^2 - r^2 - 4Rr)^2 \Leftrightarrow 81R^4 \geq 4(4R^2 + 2r^2)^2 \Leftrightarrow$$

$\Leftrightarrow 9R^2 \geq 2(4R^2 + 2r^2) \Leftrightarrow R^2 \geq 4r^2$, see $R \geq 2r$, (Euler).

Equality holds if and only if the triangle is equilateral.

Remark: In ΔABC holds

$$\boldsymbol{R^{2n} \geq \frac{1}{9^n}\left(\sum a^2\right)^n, n \in \mathbb{N}.}$$

Marin Chirciu

Solution: For $n = 0$ we obtain the equality $1 = 1$.Next, let be $n \in \mathbb{N}^*$.

Using $\sum a^2 = 2(p^2 - r^2 - 4Rr)$ the inequality can be written:

$R^{2n} \geq \frac{1}{9^n}(\sum a^2)^n \Leftrightarrow 9R^2 \geq 2(p^2 - r^2 - 4Rr)$, which follows from Gerretsen inequality:

$p^2 \leq 4R^2 + 4Rr + 3r^2$.It remains to prove that:

$$9R^2 \geq 2(4R^2 + 4Rr + 3r^2 - r^2 - 4Rr) \Leftrightarrow 9R^2 \geq 2(4R^2 + 2r^2) \Leftrightarrow R^2 \geq 4r^2,$$

see $R \geq 2r$, (Euler). Equality holds if and only if the triangle is equilateral.

ABOUT THE RMM PROBLEM J.3094

By Marin Chirciu-Romania

J.3094. In ΔABC holds:

$$\boldsymbol{\sum \frac{a^3}{b+c} \geq 2\sqrt{3}F}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

Solution by Marin Chirciu – Romania

$$LHS = \sum \frac{a^3}{b+c} \overset{Holder}{\geq} \frac{(\sum a)^3}{3\sum(b+c)} = \frac{(\sum a)^3}{3 \cdot 2\sum a} = \frac{1}{6}\left(\sum a\right)^2 = \frac{(2p)^2}{6} = \frac{4p^2}{6} =$$

$$=\frac{2p^2}{3}\overset{Hodwiger}{\geq}\frac{2\cdot 3F\sqrt{3}}{3}=2\sqrt{3}F=RHS$$

We have used above Hadwiger inequality $p^2 \geq 3F\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

Remark: In ΔABC holds:

$$\sum\frac{a^{2n+1}}{b+c}\geq\left(\frac{2}{3}\right)^{2n-1}\left(3F\sqrt{3}\right)^n, n\in\mathbb{N}$$

Marin Chirciu

Solution

$$LHS=\sum\frac{a^{2n+1}}{b+c}\overset{Holder}{\geq}\frac{(\sum a)^{2n+1}}{3^{2n-1}\sum(b+c)}=\frac{(\sum a)^{2n+1}}{3^{2n-1}\cdot 2\sum a}=\frac{(\sum a)^{2n}}{3^{2n-1}\cdot 2}=\frac{(2p)^{2n}}{3^{2n-1}\cdot 2}=$$

$$=\frac{2^{2n}(p^2)^n}{3^{2n-1}\cdot 2}=\frac{2^{2n-1}(p^2)^n}{3^{2n-1}}\overset{Hadwiger}{\geq}\frac{2^{2n-1}\left(3F\sqrt{3}\right)^n}{3^{2n-1}}=\left(\frac{2}{3}\right)^{2n-1}\left(3F\sqrt{3}\right)^n=RHS$$

We have used above Hadwiger inequality $p^2 \geq 3F\sqrt{3}$. Equality holds if and only if the triangle is equilateral.

Note: For $n=1$ we obtain Problem J.3094 from RMM 47/2025.

ABOUT THE PROBLEM J.3095

By Marin Chirciu-Romania

J.3095 In ΔABC holds:

$$\sum\frac{a^3}{bx+cy}\geq\frac{4\sqrt{3}}{x+y}F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

Solution by Marin Chirciu – Romania

$$LHS=\sum\frac{a^{23}}{xb+yc}\overset{Holder}{\geq}\frac{(\sum a)^3}{3\sum(xb+yc)}=\frac{(\sum a)^3}{3\cdot(x+y)\sum a}=\frac{(\sum a)^2}{3\cdot(x+y)}=$$

$$=\frac{(2p)^2}{3\cdot(x+y)}=\frac{4p^2}{3\cdot(x+y)}\overset{Hadwiger}{\geq}\frac{4\cdot 3F\sqrt{3}}{3(x+y)}=\frac{4F}{x+y}=RHS.$$

We have used above Hadwiger inequality $p^2 \geq 3F\sqrt{3}$. Equality holds if and only if the triangle is equilateral.

Remark: If $x, y > 0$ then in ΔABC holds:

$$\sum\frac{a^{2n+1}}{xb+yc}\geq\frac{3}{x+y}\left(\frac{2}{3}\right)^{2n}\left(3F\sqrt{3}\right)^{n}, n\in\mathbb{N}.$$

Marin Chirciu

Solution

$$LHS=\sum\frac{a^{2n+1}}{xb+yc}\overset{Holder}{\geq}\frac{(\sum a)^{2n+1}}{3^{2n-1}\sum(xb+yc)}=\frac{(\sum a)^{2n+1}}{3^{2n-1}\cdot(x+y)\sum a}=\frac{(\sum a)^{2n}}{3^{2n-1}(x+y)}=$$

$$=\frac{(2p)^{2n}}{3^{2n-1}\cdot(x+y)}=\frac{2^{2n}(p^2)^n}{3^{2n-1}\cdot(x+y)}\overset{Hadwiger}{\geq}\frac{2^{2n}\left(3F\sqrt{3}\right)^n}{3^{2n-1}(x+y)}=\frac{3}{x+y}\left(\frac{2}{3}\right)^{2n}\left(3F\sqrt{3}\right)^n=RHS$$

We have used above Hadwiger inequality $p^2\geq 3F\sqrt{3}$.

Equality holds if and only if the triangle is equilateral.

Note.

For $n=1$ we obtain Problem J.3094 from RMM 47/2025.

J.3094. In ΔABC holds:

$$\sum\frac{a^3}{b+c}\geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu-Giurgiu, Claudia Nănuți – Romania

A CLASS OF SEQUENCES WITH THE RELATIONSHIP:

$$\sum_{k=1}^{n}p^{k-1}a_k=\prod_{k=1}^{n}a_k$$

By Bela Kovacs-Romania

The relationship given in the title of the present paper hides a special, elegant, mysterious and very interesting sequence. It is worthy to deal with it. We may draft new open questions. I have been working on this subject for several years, and now I would like to present some features, by examining general and proper cases.

1. General definition and examinaton

1.1. We define the (a_n) sequence as follows: $a_1=a\neq p$, $p>0$, $a_2=\dfrac{a}{a-p}$,

$a_{n+1} = \dfrac{a_n^2}{a_n^2 - pa_n + p}$, for $n \geq 2$. For a = 0 each term of the (a_n) sequence is 0, therefore we will deal with the cases when $a \neq 0$. If we have for p: $0 < p < 4$ then remains the only one interpretational condition. For $p \geq 4$ we should give some new conditions apart from the given one. We do not deal with this now.

1.2. We have valid the next context in the domain of interpretation of the sequence:

$a_1 + pa_2 + p^2a_3 + \ldots + p^{n-1}a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n$ for any $n \geq 1$.

Proof: In case of n = 1 the equality is obvious. For n = 2 : $a_1 + pa_2 = a + \dfrac{pa}{a-p} = \dfrac{a^2}{a-p} = a_1 \cdot a_2$. results from this $a_2 = \dfrac{a_1}{a_1 - p} \Leftrightarrow a_1 = \dfrac{pa_2}{a_2 - 1}$.

For any $n \geq 2$ we have the recursive context: $a_{n+1} = \dfrac{a_n^2}{a_n^2 - pa_n + p}$. Transforming this we will get: $\dfrac{a_{n+1}}{a_{n+1} - 1} = \dfrac{a_n^2}{p(a_n - 1)}$ From this results, on the one hand: $a_n = p \cdot \dfrac{a_{n+1}}{a_n} \cdot \dfrac{a_n - 1}{a_{n+1} - 1}$, on the other hand: $a_n = p \cdot \dfrac{a_{n+1}}{a_{n+1} - 1} - \dfrac{a_n}{a_n - 1}$. Writing this results for the n = 2 , 3 , . . . etc. values, then multiply in the first case, or let us multiplied with the p , p^2 , p^3 , . . . etc factors in the second case. After that we will add the terms taking into consideration that $\dfrac{pa_2}{a_2 - 1} = a_1$.

It follows that on the one hand: $a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n = p^n \cdot \dfrac{a_{n+1}}{a_{n+1} - 1}$,

on the other hand: $a_1 + pa_2 + p^2a_3 + \ldots + p^{n-1}a_n = p^n \cdot \dfrac{a_{n+1}}{a_{n+1} - 1}$. From this we obtain the above given relation.

1.3. For $0 < p < 4$, then based on $a_{n+1} - 1 = \dfrac{p(a_n - 1)}{a_n^2 - pa_n + p}$, $n \geq 2$ relation, if any term of the sequence is greater than 1, then all the following terms will be greater than 1 too; if any term of the sequence is less than 1, then all the following terms will be less than 1 too. The result is valid for cases of $p \geq 4$ inside the domain of interpretation.

1.4. We can not draw a general conclusion either from the $a_{n+1} - p = \frac{-(a_n - p)((p-1)a_n - p)}{a_n^2 - pa_n + p}$ relation, nor from the $a_{n+1} - a_n = \frac{-a_n(a_n - 1)(a_n - p)}{a_n^2 - pa_n + p}$ and $a_{n+2} - a_n = \frac{-a_n(a_n - 1)(a_n - p)(a_n^2 - p(p-1)a_n + p^2)}{a_n^4 - p^2(a_n - 1)(a_n^2 - pa_n + p)}$, $n \geq 2$ relations. But later we can make use of them in particular cases. If the sequence is convergent, then taking the limit of the recursive relation we find that the limit value can be only the 0, 1 or the p.

2. Examinaton of particular cases

2.1. Let it be p = 1. It is very interesting and elegant. We refound a simple classic case.

In this instance our sequence looks like: $a_1 = a \neq 1$, $a_2 = \frac{a}{a-1}$, $a_{n+1} = \frac{a_n^2}{a_n^2 - a_n + 1}$, for any $n \geq 2$. Always defined, having all the terms positive numbers from the 3rd term. Is valid the next relation: $a_1 + a_2 + a_3 + ... + a_n = a_1 \cdot a_2 \cdot a_3 \cdot ... \cdot a_n$ for any $n \geq 1$. We have $a_{n+1} - a_n = \frac{-a_n(a_n - 1)^2}{a_n^2 - a_n + 1} < 0$, which means that the sequence is severely decreasing. If $a > 1$, then $a_2 > 1$ and the sequence has all terms greater than 1, is severely decreasing, convergent, the limit value being 1 . For $a < 1$, then a_2 can be negativ, or a value between 0 and 1 , but the a_3 value surely is less than 1, being a positive number and similarly the following terms of the sequence. In this case the sequence is severely decreasing, convergent, and the limit value is 0 .

The examination of this sequence was set repeatedly task on mathematics competitions like VII. NMMV Szabadka, 1998, ([3], [9]), Mathematical Olympic , local section, Szatmár country, 2005, [14], and it appeared in mathematical scientific reviews and mathematical collections of examples. It was edited by: Kovács Béla.

It is an open question the determaination of the general terms of the sequence. This was formulated repeatedly already, and experiments are proceeding about this direction: ([2], [5], [6], [7], [8], [10], [11]).

Comment: Introducing the g(a) = $\frac{a}{a-1}$ and f(a) = $\frac{a^2}{a^2 - a + 1}$ functions, the sequence can be written as follows: $a_1 = a \neq 1$, a_2 = g(a) and

$a_n = \underbrace{f \circ f \circ \ldots \circ f}_{n-2}(g(a))$, $n \geq 3$, but this cannot be considered as the general member's closed form.

2.2. In the second case let us take p = 2. It is a very interesting and special case. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2 = \frac{a}{a-2}$,

$a_{n+1} = \frac{a_n^2}{a_n^2 - 2a_n + 2}$, for each $n \geq 2$. Always defined, having all the terms positive numbers from the 3rd term. Is valid the next relation:

$a_1 + 2a_2 + 2^2 a_3 + ... + 2^{n-1} a_n = a_1 \cdot a_2 \cdot a_3 \cdot ... \cdot a_n$ for each $n \geq 1$.

Based on the $a_{n+1} - 2 = \frac{-(a_n - 2)^2}{a_n^2 - 2a_n + 2} \leq 0$ relation each term of the sequence is less or equal than 2 starting from the 3rd term. The equality stands for the a=4 initial value. Furthermore since $a_{n+1} - a_n = \frac{-a_n(a_n - 1)(a_n - 2)}{a_n^2 - 2a_n + 2}$, if $a_2 > 1$, $a > 2$, then the sequence has all terms numbers between 1 and 2 starting from the 3rd term, severely incresing, convergent, the limit value being 2.

For $a < 2$, then a_2 can be negativ, or a value between 0 and 1, but the a_3 value surely is less than 1, being a positive number and similarly the following terms of the sequence. In this case the sequence is severely decreasing, convergent and the limit value is 0.

We can determinate the general term of the sequence. The recursive relation can be written as $\frac{1}{a_{n+1}} = 1 - \frac{2}{a_n} + \frac{2}{a_n^2}$ and multiplied by 2 and transforming it we get: $\frac{2}{a_{n+1}} - 1 = \left(\frac{2}{a_n} - 1\right)^2$,

from that results $\frac{2}{a_n} - 1 = \left(\frac{2}{a_2} - 1\right)^{2^{n-2}}$, for each $n \geq 3$.

Replacing the $a_2 = \frac{a}{a-2}$ value, we found: $a_n = \frac{2a^{2^{n-2}}}{a^{2^{n-2}} + (a-4)^{2^{n-2}}}$, for each $n \geq 2$. The examination of this sequence was setting as task on the XVII Transilvanien Hungarien Mathematics Competition in Miercurea Ciuc, 2007, suggested by Kovács Béla, presented in : Matlap 2007/5, page: 32. [9].

2.3. Thirdly let us take the p=3 case. It seems to be a more complicated and mysterious case.

Now the sequence is : $a_1 = a \neq 3$, $a_2 = \frac{a}{a-3}$, $a_{n+1} = \frac{a_n^2}{a_n^2 - 3a_n + 3}$, fo each $n \geq 2$. Always defined, having all the terms positive numbers from the 3rd term. Is valid the next relation :

$a_1+3a_2+3^2a_3+...+3^{n-1}a_n = a_1\cdot a_2\cdot a_3\cdot...\cdot a_n$ for each $n \geq 1$. In this case , since $a_{n+1} - 4 = \frac{-3(a_n-2)^2}{a_n^2-3a_n+3} \leq 0$ all members of the sequence is less or equal than 4 , for each $n > 3$. In addition $a_{n+1} - 2 = \frac{3-(a_n-3)^2}{a_n^2-3a_n+3}$ and if $2 < a_n < 4$, then $2 < a_{n+1} < 4$, (mathematical induction). If $a > 3$, then $a_2 > 1$, and each term of the sequence is greater than 1. Since $a_{n+1} - a_n = \frac{-a_n(a_n-1)(a_n-3)}{a_n^2-3a_n+3}$, $n \geq 2$, we can remark, that the sequence is severely increasing by the time one of its members reach the 3 value, or a greater value. Based on the $a_{n+1} - 3 = \frac{-(a_n-3)(2a_n-3)}{a_n^2-3a_n+3}$ relation, if $a_n > 3$, then $a_{n+1} < 3$, and vice versa (starting from any term).

For the $a = \frac{9}{2}$ and $a = 9$ initial values each term of the sequence is equal with 3 staring from the 2.nd respective 3.rd member. Using another initial values we found that each member of the sequence is equal with 3 starting from a certain member.

The general definition of this values is an open-question till now. Finally it results from the $a_{n+2} - a_n = \frac{-a_n(a_n-1)(a_n-3)^3}{a_n^4-9(a_n-1)(a_n^2-3a_n+3)}$, $n \geq 2$ context – since the denominator is always positive- that the the members of the sequence with an even rank are growing strictly and the members of a sequence with an odd rank decrease strictly, or reverse (starting from one of the members). Both part sequence are convergent, their common limit value being 3., which is the limit value of the sequence too. If we have $a < 3$, then the terms of the sequence are numbers between 0 and 1 beginning from the 3rd member, in this case the sequence decrease strictly, is convergent, and the limit value is 0.

The definition of the general member of the sequence is an open question.

2.4 In the fourth case let us take the p = 4. This already is a very complicated, special and mysterious case.

Now the sequence is: $a_1 = a \neq 4$, $a_2 = \frac{a}{a-4}$, $a_{n+1} = \frac{a_n^2}{a_n^2-4a_n+4}$, for each $n \geq 2$, not always defined, but having all his terms strictly positive beginning with the 3rd term.

The next relation is valid in the domain of interpretation of the sequence $a_1+4a_2+4^2a_3+...+4^{n-1}a_n = a_1\cdot a_2\cdot a_3\cdot...\cdot a_n$ for each $n \geq 1$. If a = 8 , then the 2nd term of the sequence is 2 and the 3rd is not defined. For $a = 8(2\pm\sqrt{2})$, we have the 3rd term being 2, but the 4th term alredy is not defined. Based on the recursive relation: $a_{n+1} = \frac{a_n^2}{(a_n-2)^2}$, if one member of the sequence has the value 2, then the next members are already not defined. This exists on the case of very much initial value. We deal with this later.

For $a < 4$ the a_2 can be negative, or a number between the 0 and 1 values, but the a_3 is already certainly less then 1, being a positive number, and in the same way the additional members of the sequence. Since $a_{n+1} - a_n = \frac{-a_n(a_n-1)(a_n-4)}{a_n^2-4a_n+4} < 0$ the sequence decrease strictly, is convergent, and the limit value is 0.

When $a > 4$ then we define he general term of the sequence using a bold, special undertaking. In this case $a_2 > 1$ and so are the other terms of the sequence. Transforming the recursive relation we get: $\frac{1}{a_{n+1}} = 1 - \frac{4}{a_n} + \frac{4}{a_n^2} \Leftrightarrow 1 - \frac{2}{a_{n+1}} = 1 - 2\left(1-\frac{2}{a_n}\right)^2$. For $n = 2$ results $1 - \frac{2}{a_3} = 1 - 2\left(1-\frac{2}{a_2}\right)^2$. Since $a_2 = \frac{a}{a-4} > 1$, we have that $-1 < 1 - \frac{2}{a_2} < 1$, so it exist an $\alpha \in [0, \pi)$, so that $\cos\alpha = 1 - \frac{2}{a_2}$, that is $\cos\alpha = \frac{8-a}{a} \Leftrightarrow \alpha = \arccos\frac{8-a}{a}$. In this case, based on the well known $2\cos^2\alpha - 1 = \cos 2\alpha$ relation we get $a_3 = \frac{2}{1+\cos 2\alpha}$ and and we receive applying of the mathematical induction, that $a_n = \frac{2}{1+\cos 2^{n-2}\alpha}$ for each $n \ge 3$. So the general term of the sequence is $a_n = \frac{2}{1+\cos(2^{n-2}\arccos\frac{8-a}{a})}$, which can be witten as $a_n = \frac{1}{\cos^2(2^{n-3}\arccos\frac{8-a}{a})}$, or as $a_n = 1 + \operatorname{tg}^2(2^{n-3}\arccos\frac{8-a}{a})$ being valid for each $n \ge 3$. From this we may establish those initial values already, for which the sequence is not defined. These will be definied by the solution of the $1 + \cos(2^{n-2}\arccos\frac{8-a}{a}) = 0$ equation. Results : $\arccos\frac{8-a}{a} = \frac{(2k+1)\pi}{2^{n-2}} \Rightarrow$

$\frac{8-a}{a} = \cos\frac{(2k+1)\pi}{2^{n-2}} \Rightarrow a = \frac{8}{1+\cos\frac{(2k+1)\pi}{2^{n-2}}}$, where $k \in Z$ és $0 \le k < 2^{n-2}$.

Depending by the initial value, the sequence may be a constant, from a certain member starting or may be periodic, but also it may be terms mysteriounly random in sequences with an optional member, what is not convergent naturally.

For instance: for a = $\frac{16}{3}$ the sequence $\frac{16}{3}$, 4 , 4 , 4 , 4 , 4 , ... , is constant

For a = 16 the sequence: 16 , $\frac{4}{3}$, 4 , 4 , 4 , 4 , ,. . . , is constant,

For a = 16(2 – $\sqrt{3}$) the sequence looks like: 16(2 – $\sqrt{3}$) , 4(2 + $\sqrt{3}$) , $\frac{4}{3}$, 4, 4, 4, . . .

In case of a_1 = a = $\frac{8}{1+\cos\frac{\pi}{5}}$ = $\frac{32}{5+\sqrt{5}}$ the terms of the sequence are 2(3 $\pm\sqrt{5}$) , alternate, periodic sequence. We can receive other periodic sequence on the case of other initial values

For a = 5 we have: $a_1 = a_2$ = 5 and a_n = 1 + $tg^2(2^{n-3}\arccos\frac{3}{5})$, for each n $\geq$ 3 , a special and mysterious sequence.

For a = 6 the sequence: 6 , 3 , 9 , $\frac{81}{49}$, $\frac{6561}{289}$, $\frac{43046721}{35796289}$, . . . , with approaching values : 6 , 3 , 9 , 1.653 , 22.7 , 1.2 , 2.274 , 68.869 , 1.06 , 1.275 , . . . etc. The general member is: a_n = 1 + $tg^2(2^{n-3}\arccos\frac{1}{3})$, for each n$\geq$ 3 .

The sequence can be given in trigonometrical form with the a_1 = $\frac{8}{1+\cos\alpha}$ initial value as follows: $\frac{8}{1+\cos\alpha}$, $\frac{2}{1-\cos\alpha}$, $\frac{1}{\cos 2\alpha}$, $\frac{1}{\cos 4\alpha}$, $\frac{1}{\cos 8\alpha}$, . . . etc. Tha valid relation is: $\frac{16}{\sin^2\alpha} + \sum_{k=3}^{n}\frac{4^{k-1}}{\cos^2 2^{k-3}\alpha} = \frac{16}{\sin^2\alpha}\cdot\prod_{k=3}^{n}\frac{1}{\cos^2 2^{k-3}\alpha} = \frac{4^n}{\sin^2 2^{n-2}\alpha}$, for each n $\geq$ 3 , in the domain of interpretation of the sequence. (for $\alpha \neq \frac{(2k+1)\pi}{2^{n-3}}$)

The definition of the general member of the sequence for a $<$ 4 remain an open question.

3. Comments and open, undetermined questions

It is remarkable that depending on the p parameter and on the initial value on the case of his different values what kind of special and interesting, maybe mysterious sequence we get. It would be interesting to examinate some similar cases. We may get interesting sequence in case of p being a negative value. When our sequence is convergent, constant, or periodic sequence? Is definable the general member of the sequence in other specific cases? How the members of the sequence are scattered when he is not periodic and not convergent, or his limit values not concerned? On the case of what kind of initial values are not defined the sequence? To answer this questions the additional examination of the sequence is necessary. I am sure, that very many opportunities not exploited offer themselves yet. I draw the attention of interested ones to it for the topic.

4. Finally, a few other similar cases, the prof of which I leave to interested readers.

4.1. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2 = \frac{a}{a-2}$ and

$$a_{n+1} = \frac{na_n^2}{na_n^2-(n+1)a_n+n+1}, \text{ for each } n \geq 2.$$

Is valid the next relation: $a_1 + 2a_2 + 3a_3 + \ldots + na_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \quad \forall\, n \geq 1$.

4.2. Now the sequence has the following form: $a_1 = a \neq \frac{1}{2}$, $a_2 = \frac{2a}{2a-1}$ and

$$a_{n+1} = \frac{(n+1)a_n^2}{(n+1)a_n^2-na_n+n}, \text{ for each } n \geq 2$$

Is valid the next relation: $a_1 + \frac{1}{2}a_2 + \frac{1}{3}a_3 + \ldots + \frac{1}{n}a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \quad \forall\, n \geq 1$.

4.3. Now the sequence has the following form: $a_1 = a \in Q - \{2\}$, $a_2 = \frac{a}{a-2}$ and

$$a_{n+1} = \frac{a_n^2}{a_n^2-(n+1)a_n+n+1}, \text{ for each } n \geq 2.$$

Is valid the next relation: $a_1 + 2!\cdot a_2 + 3!\cdot a_3 + \ldots + n!\cdot a_n = a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n \quad \forall\, n \geq 1$.

4.4. Now the sequence has the following form: $a_1 = a \neq \frac{1}{2}$, $a_2 = \frac{2a}{2a-1}$ and

$$a_{n+1} = \frac{(n+1)a_n^2}{(n+1)a_n^2-a_n+1}, \text{ for each } n \geq 2$$

Is valid the next relation: $a_1+\frac{1}{2!}a_2+\frac{1}{3!}a_3+\ldots+\frac{1}{n!}a_n = a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_n \quad \forall\, n\geq 1$.

4.5. Now the sequence has the following form: $a_1 = a \neq \frac{1}{2}$, $a_2=\frac{a}{2a-1}$ and

$$a_{n+1}=\frac{na_n^2}{(n+1)a_n^2-na_n+n-1}\ ,\ \text{for each } n\geq 2$$

Is valid the next relation: $a_1+a_2+a_3+\ldots+a_n = n\cdot a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_n \quad \forall\, n\geq 1$.

4.6. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2=\frac{2a}{a-2}$ and

$$a_{n+1}=\frac{(n^2-1)a_n^2}{n(n-1)a_n^2-(n^2-1)a_n+n(n+1)}\ ,\ \text{for each } n\geq 2$$

Is valid the next relation: $n\cdot(a_1+a_2+a_3+\ldots+a_n) = a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_n \quad \forall\, n\geq 1$.

4.7. Now the sequence has the following form: $a_1 = a \neq 2$, $a_2=\frac{2a}{a-2}$ and

$$a_{n+1} = \frac{2a_n^2}{a_n^2-2a_n+4}\ ,\ \text{for each } n\geq 2$$

Is valid the next relation: $2^{n-1}\cdot(a_1+a_2+a_3+\ldots+a_n) = a_1\cdot a_2\cdot a_3\cdot\ldots\cdot a_n \quad \forall\, n\geq 1$.

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ABOUT AN IDENTITY BY BOGDAN FUŞTEI-IV

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a^2}{r_br_c}+\frac{\sum r_a^2}{\sum r_br_c}=\sum_{cyc}\frac{bc}{w_a^2}$$

Proposed by Bogdan Fuștei-Romania

Solution. **Lemma1. 2) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{m_a^2}{r_br_c}=\frac{R}{r}+4-\left(\frac{4R+r}{s}\right)^2$$

Proof. Using $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ and $r_a=\frac{F}{s-a}$ we get:

$$\sum_{cyc}\frac{m_a^2}{r_br_c}=\sum_{cyc}\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{F}{s-b}\cdot\frac{F}{s-c}}=\frac{1}{4F^2}\sum_{cyc}(2b^2+2c^2-a^2)(s-b)(s-c)=\frac{R}{r}+4-\left(\frac{4R+r}{s}\right)^2$$

Lemma 2. 3) In ΔABC the following relationship holds:

$$\frac{\sum r_a^2}{\sum r_br_c}=\left(\frac{4R+r}{s}\right)^2-2$$

Proof. Using $\sum r_a=4R+r$ and $\sum r_br_c=s^2$, it follows that:

$$\frac{\sum r_a^2}{\sum r_br_c}=\frac{(\sum r_a)^2-2\sum r_br_c}{\sum r_br_c}=\frac{(\sum r_a)^2}{\sum r_br_c}-2=\frac{(4R+r)^2}{s^2}-2=\left(\frac{4R+r}{s}\right)^2-2$$

Lemma 3. 4) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{bc}{w_a^2}=\frac{R}{r}+2$$

Proof. Using identity $w_a=\frac{2bc}{b+c}cos\frac{A}{2}$, it follows that:

$$\sum_{cyc}\frac{bc}{w_a^2}=\sum_{cyc}\frac{bc}{\left(\frac{2bc}{b+c}cos\frac{A}{2}\right)^2}=\frac{1}{2s}\sum_{cyc}\frac{(b+c)^2}{s-a}=\frac{1}{4s}\cdot 4s\left(\frac{R}{r}+2\right)=\frac{R}{r}+2$$

-which follows from $\sum_{cyc}\frac{(b+c)^2}{s-a}=4s\left(\frac{R}{r}+2\right)$. Let's get back to the main problem.

Using these up Lemma's, it follows that:

$$\sum_{cyc}\frac{m_a^2}{r_br_c}+\frac{\sum r_a^2}{\sum r_br_c}=\sum_{cyc}\frac{bc}{w_a^2}$$

5) In ΔABC the following relationship holds:

$$3\le\sum_{cyc}\frac{m_a^2}{r_br_c}\le\frac{R}{r}+\frac{2r}{R}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 6) In ΔABC the following relationship holds:**

$$\sum_{cyc}\frac{m_a^2}{r_br_c}=\sum_{cyc}\frac{m_a^2}{r_br_c}=\frac{R}{r}+4-\left(\frac{4R+r}{s}\right)^2$$

Proof. Using $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ and $r_a=\frac{F}{s-a}$ we get:

$$\sum_{cyc}\frac{m_a^2}{r_br_c}=\sum_{cyc}\frac{\frac{2b^2+2c^2-a^2}{4}}{\frac{F}{s-b}\cdot\frac{F}{s-c}}=\frac{1}{4F^2}\sum_{cyc}(2b^2+2c^2-a^2)(s-b)(s-c)=$$

$=\frac{R}{r}+4-\left(\frac{4R+r}{s}\right)^2$. Let's get back to the main problem.

Using Lemma, inequality can be written as: $4\le\frac{R}{r}+4-\left(\frac{4R+r}{s}\right)^2\le\frac{2R}{r}$, which follows from

Blundon-Gerretsen inequality: $\frac{r(4R+r)^2}{R+r}\le s^2\le\frac{R(4R+r)^2}{2(2R-r)}$.

Equality holds if and only if triangle is equilateral.

7) In ΔABC the following relationship holds:

$$4 \leq \sum_{cyc} \frac{bc}{w_a^2} \leq \frac{2R}{r}$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 8) In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{bc}{w_a^2} = \frac{R}{r} + 2$$

Proof. Using identity $w_a = \frac{2bc}{b+c} cos \frac{A}{2}$, it follows that:

$$\sum_{cyc} \frac{bc}{w_a^2} = \sum_{cyc} \frac{bc}{\left(\frac{2bc}{b+c} cos \frac{A}{2}\right)^2} = \frac{1}{2s} \sum_{cyc} \frac{(b+c)^2}{s-a} = \frac{1}{4s} \cdot 4s\left(\frac{R}{r} + 2\right) = \frac{R}{r} + 2$$

-which follows from $\sum_{cyc} \frac{(b+c)^2}{s-a} = 4s\left(\frac{R}{r} + 2\right)$. Let's get back to the main problem.

Using Lemma, inequality can be written as: $4 \leq \frac{R}{r} + 2 \leq \frac{2R}{r}$, which follows from $R \geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

9) In ΔABC the following relationship holds:

$$2 - \frac{2r}{R} \leq \frac{\sum r_a^2}{\sum r_b r_c} \leq \frac{R}{r} - 1$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 10) In ΔABC the following relationship holds:**

$$\frac{\sum r_a^2}{\sum r_b r_c} = \left(\frac{4R+r}{s}\right)^2 - 2$$

Proof. Using $\sum r_a = 4R + r$ and $\sum r_b r_c = s^2$, it follows that:

$$\frac{\sum r_a^2}{\sum r_b r_c} = \frac{(\sum r_a)^2 - 2\sum r_b r_c}{\sum r_b r_c} = \frac{(\sum r_a)^2}{\sum r_b r_c} - 2 = \frac{(4R+r)^2}{s^2} - 2 = \left(\frac{4R+r}{s}\right)^2 - 2$$

Let's get back to the main problem. Using Lemma, inequality can be written as:

$2 - \frac{2r}{R} \leq \left(\frac{4R+r}{s}\right)^2 - 2 \leq \frac{R}{r} - 1$, which follows from

Blundon-Gerretsen inequality: $\frac{r(4R+r)^2}{R+r} \leq s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$.

Equality holds if and only if triangle is equilateral

REFERENCE: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALIY BY ADIL ABDULLAYEV-XVII

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$\frac{R}{2r} \geq \sqrt{\frac{1}{8}\prod_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution.

Lemma. 2) In ΔABC the following relationship holds:

$$\prod_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3}{8s^2r^2R^2}$$

Proof. We have:

$$\prod_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)=\prod_{cyc}\left(\frac{b^2+c^2}{bc}\right)=\frac{\prod(b^2+c^2)}{(abc)^2}=$$

$$=\frac{2[s^6+s^4(r^2-12Rr)+s^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3]}{(4Rrs)^2}=$$

$$=\frac{s^6+s^4(r^2-12Rr)+s^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3}{8s^2r^2R^2}$$

which follows from $\prod(b^2+c^2)=2[s^6+s^4(r^2-12Rr)+s^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3]$

Let's get back to the main problem. Using Lemma, inequality can be written as:

$$\frac{R}{2r} \geq \sqrt{\frac{1}{8}\prod_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)} \Leftrightarrow$$

$$\frac{R}{2r} \geq \sqrt{\frac{1}{8}\frac{s^6+s^4(r^2-12Rr)+s^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3}{8s^2r^2R^2}} \Leftrightarrow$$

$$\frac{R^2}{4r^2} \geq \frac{s^6+s^4(r^2-12Rr)+s^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3}{64s^2r^2R^2} \Leftrightarrow$$

$$16s^2R^4 \geq s^6+s^4(r^2-12Rr)+s^2r^2(40R^2+8Rr-r^2)-r^3(4R+r)^3 \Leftrightarrow$$

$$s^2[s^2(12Rr-r^2)+(16R^4-40R^2r^2-8Rr^3+r^4)-s^4]+r^3(4R+r)^3 \geq 0$$

Distinguish the cases:

$i)$ $[s^2(12Rr-r^2)+(16R^4-40R^2r^2-8Rr^3+r^4)-s^4]\geq 0$, inequality is obviously true.

$ii)$ $[s^2(12Rr-r^2)+(16R^4-40R^2r^2-8Rr^3+r^4)-s^4]<0$, inequality can be written as:

$$r^3(4R+r)^3\geq s^2[s^4-s^2(12Rr-r^2)-(16R^4-40R^2r^2-8Rr^3+r^4)]\Leftrightarrow$$

$r^3(4R+r)^3\geq s^2[s^2(s^2+r^2-12Rr)-(16R^4-40R^2r^2-8Rr^3+r^4)]$, which follows from

$$16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}\leq 4R^2+4Rr+3r^2(B-Gerretsen)$$

$$16Rr-5r^2+r^2\frac{R-2r}{R-r}\leq s^2\leq 4R^2+4Rr+3r^2-r^2\frac{R-2r}{R-r}(Yang\ X.Z.)$$

Using Yang Xue Zhi inequality, we get:

$$s^2(s^2+r^2-12Rr)\leq$$
$$\leq\left(4R^2+4Rr+3r^2-r^2\frac{R-2r}{R-r}\right)(4R^2+4Rr+3r^2+r^2-12Rr)=$$
$$=\left(4R^2+4Rr+3r^2-r^2\frac{R-2r}{R-r}\right)(4R^2-2Rr+r^2)=$$
$$=\frac{4(4R^5-8R^4-6R^3r^2+3R^2r^3-r^5)}{R-r}$$

Remains to prove that:

$$r^3(4R+r)^2\geq$$
$$\geq\frac{R(4R+r)^2}{2(2R-r)}\left[\frac{4(4R^5-8R^4+2R^3r^2+3R^2r^3-r^5)}{R-r}-(16R^4-40R^2r^2-8Rr^3+r^4\right]$$

$$2(2R-r)r^3(4R+r)^2\geq R(4R+r)^2\left[\frac{4(4R^5-8R^4+2R^3r^2+3R^2r^3-r^5)}{R-r}-(16R^4-40R^2r^2-8Rr^3+r^4\right]$$

$$2(2R-r)r^3(4R+r)\geq R\frac{-16R^4r+48R^3r^2-20R^2r^3-9Rr^4-3r^5}{R-r}$$
$$2r^2(8R^202Rr-r^2)(R-r)\geq -16R^5+48R^4r-20R^3r^2-9R^2r^3-3Rr^4$$
$$16R^3r^2-20R^2r^3+2Rr^4+2r^5\geq -16R^5+48R^4r-20R^3r^2-9R^2r^3-3Rr^4$$
$$16R^5-48R^4r+36R^3r^2-11R^2r^3+5Rr^4+2r^5\geq 0$$

$(R-2r)(16R^4-16R^3r+4R^2r^2-3Rr^3-r^4)\geq 0$, which is obviously true from $R\geq 2r(Euler)$. Equality holds if and only if triangle is equilateral.

2) In ΔABC the following relationship holds:

$$\mathbf{1\leq\sqrt{\frac{1}{8}\prod_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)}\leq\frac{R}{2r}}$$

Solution. See up these inequalities and $1\leq\sqrt{\frac{1}{8}\prod_{cyc}\left(\frac{b}{c}+\frac{c}{b}\right)}\Leftrightarrow\frac{b}{c}+\frac{c}{b}\geq 2$ true from AM-GM inequality. Equality holds if and only if triangle is equilateral.

Refference: ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY ELDENIZ HESENOV-I

By Marin Chirciu – Romania

1) I_a, I_b, I_c – excenters in ΔABC. Prove that:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\geq\frac{3}{S}$$

Eldeniz Hesenov – Georgia

Solution We prove:

Lemma.

2) If I_a, I_b, I_c – excenters in ΔABC, then:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

Proof.

Using $[BCI_a]=\frac{a\cdot r_a}{2}$ we obtain:

$$E=\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\sum\frac{1}{[BCI_a]}=\sum\frac{1}{\frac{a\cdot r_a}{2}}=2\sum\frac{1}{a\cdot r_a}=2\cdot\sum\frac{1}{a\cdot\frac{S}{s-a}}=$$

$$=\frac{2}{S}\sum\frac{s-a}{a}=\frac{2}{sr}\cdot\frac{s^2+r^2-8Rr}{4Rr}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

Let's get back to the main problem. Using the Lemma the inequality can be written:

$\frac{s^2+r^2-8Rr}{2sRr^2}\geq\frac{3}{sr}\Leftrightarrow s^2\geq 14Rr-r^2$, which follows from Gerretsen's inequality

$s^2\geq 16Rr-5r^2$. It remains to prove that:

$16Rr-5r^2\geq 14Rr-r^2\Leftrightarrow R\geq 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

Remark. The inequality can be strengthened.

3) I_a, I_b, I_c – excenters in ΔABC. Prove that:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\geq\frac{2}{S}\left(2-\frac{r}{R}\right)$$

Marin Chirciu – Romania

Proof. Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{8Rr-4r^2}{2sRr^2}=\frac{4r(2R-r)}{2sRr^2}=\frac{2(2R-r)}{sRr}=\frac{2}{S}\left(2-\frac{r}{R}\right)$$

Equality holds if and only if the triangle is equilateral.

Remark. Inequality is stronger than the inequality

4) I_a, I_b, I_c – excenters in ΔABC. Prove that:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\geq\frac{2}{S}\left(2-\frac{r}{R}\right)\geq\frac{3}{S}$$

Solution.

See inequality 3) and $\frac{2}{S}\left(2-\frac{r}{R}\right)\geq\frac{3}{S}\Leftrightarrow R\geq 2r$, (Euler's inequality).

Equality holds if and only if the triangle is equilateral.

Remark. Let's find an inequality of opposite sense.

5) I_a, I_b, I_c – excenters in ΔABC. Prove that:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\leq\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Marin Chirciu – Romania

Proof. Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{4R^2-4Rr+4r^2}{2sRr^2}=\frac{2(R^2-Rr+r^2)}{sRr^2}=\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Equality holds if and only if the triangle is equilateral.

Remark. We can write the double inequality:

6) I_a, I_b, I_c – excenters in ΔABC. Prove that:

$$\frac{2}{S}\left(2-\frac{r}{R}\right)\leq\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}\leq\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Marin Chirciu – Romania

Solution

We prove:

Lemma.

7) If I_a, I_b, I_c – excenters in ΔABC, then:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

Proof.

Using $[BCI_a]=\frac{a\cdot r_a}{2}$ we obtain:

$$E=\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\sum\frac{1}{[BCI_a]}=\sum\frac{1}{\frac{a\cdot r_a}{2}}=2\sum\frac{1}{a\cdot r_a}=2\cdot\sum\frac{1}{a\cdot\frac{S}{s-a}}=$$

$$=\frac{2}{S}\sum\frac{s-a}{a}=\frac{2}{sr}\cdot\frac{s^2+r^2-8Rr}{4Rr}=\frac{s^2+r^2-8Rr}{2sRr^2}$$

Let's get back to the main problem.

RHS inequality.

Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\leq}\frac{4R^2+4Rr+3r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{4R^2-4Rr+4r^2}{2sRr^2}=\frac{2(R^2-Rr+r^2)}{sRr^2}=\frac{2}{S}\left(\frac{R}{r}+\frac{r}{R}-1\right)$$

Equality holds if and only if the triangle is equilateral.

LHS inequality.

Using the Lemma we obtain:

$$\frac{1}{[BCI_a]}+\frac{1}{[CAI_b]}+\frac{1}{[ABI_c]}=\frac{s^2+r^2-8Rr}{2sRr^2}\overset{Gerretsen}{\geq}\frac{16Rr-5r^2+r^2-8Rr}{2sRr^2}=$$

$$=\frac{8Rr-4r^2}{2sRr^2}=\frac{4r(2R-r)}{2sRr^2}=\frac{2(2R-r)}{sRr}=\frac{2}{S}\left(2-\frac{r}{R}\right)$$

Equality holds if and only if the triangle is equilateral.

Note.

The inequality strengthen the proposed problem by Elneniz Hesenov in RMM 12/2020.

Refference:

ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro

ABOUT AN INEQUALITY BY ERTAN YILDIRIM-XIV

By Marin Chirciu-Romania

1) In ΔABC the following relationship holds:

$$6r \le \sum_{cyc} \frac{a^2}{r_b + r_c} \le \frac{a^3 + b^3 + c^3}{4F}$$

Proposed by Ertan Yildirim-Izmir-Turkey

Solution. Lemma. 2) In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{a^2}{r_b + r_c} = 2(2R - r)$$

Proof. Using $r_a = \frac{F}{s-a}$, we get:

$$\sum_{cyc} \frac{a^2}{r_b + r_c} = \sum_{cyc} \frac{a^2}{\frac{F}{s-b} + \frac{F}{s-c}} = \frac{1}{F} \sum_{cyc} \frac{a^2(s-b)(s-c)}{a} = \frac{1}{sr} \sum_{cyc} a(s-b)(s-c) =$$

$= \frac{1}{sr} \cdot 2sr(2R - r) = 2(2R - r)$, which follows from $\sum a(s-b)(s-c) = 2sr(2R - r)$

For RHS, using Lemma and $\sum a^3 = 2s(s^2 - 3r^2 - 6Rr)$ inequality becomes as:

$$2(2R - r) \le \frac{2s(s^2 - 3r^2 - 6Rr)}{4sr} \Leftrightarrow 4r(2R - r) \le s^2 - 3r^2 - 6Rr \Leftrightarrow s^2 \ge 14Rr - r^2$$

Which follows from $s^2 \ge 16Rr - 5r^2$ (Gerretsen) and $R \ge 2r$ (Euler).

Equality holds if and only if triangle is equilateral. For LHS, using Lemma, we get:

$$2(2R - r) \ge 6r \Leftrightarrow R \ge 2r \text{ (Euler)}.$$

3) In ΔABC the following relationship holds:

$$6r \le \sum_{cyc} \frac{a^2}{h_b + h_c} \le 2(2R - r)$$

Proposed by Marin Chirciu-Romania

Solution. **Lemma. 4) In ΔABC the following relationship holds:**

$$\sum_{cyc} \frac{a^2}{h_b + h_c} = \frac{4R(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr}$$

Proof. Using $h_a = \frac{2F}{a}$, we get:

$$\sum_{cyc}\frac{a^2}{h_b+h_c}=\sum_{cyc}\frac{a^2}{\frac{2F}{b}+\frac{2F}{c}}=\frac{1}{2F}\sum_{cyc}\frac{a^2bc}{b+c}=\frac{abc}{2F}\sum_{cyc}\frac{a}{b+c}=\frac{4RF}{2F}\cdot\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}=$$

$=\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}$, which follows from $\sum\frac{a}{b+c}=\frac{2(s^2-r^2-Rr)}{s^2+r^2+2Rr}$. Let's get back to the main problem.

For RHS, using Lemma and $\sum a^3=2s(s^2-3r^2-6Rr)$

$$\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}\le 2(2R-r)\Leftrightarrow 2R(s^2-r^2-Rr)\le(2R-r)(s^2+r^2+2Rr)$$

$\Leftrightarrow s^2\le 6R^2+2Rr-r^2$, which follows from $s^2\le 4R^2+4Rr+3r^2$ (Gerretsen) and $R\ge 2r$ (Euler). Remains to prove that: $4R^2+4Rr+3r^2\le 6R^2+2Rr-r^2\Leftrightarrow$

$R^2-Rr-2r^2\ge 0\Leftrightarrow(R-2r)(R+r)\ge 0$. Equality holds if and only if triangle is equilateral. For LHS. Using Lemma, inequality becomes as:

$$\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}\ge 6r\Leftrightarrow 2R(s^2-r^2-Rr)\ge 3r(s^2+r^2+2Rr)\Leftrightarrow$$

$s^2(22R-3r)\ge r(2R^2+8Rr+3r^2)$, which follows from $s^2\ge 16Rr-5r^2$ (Gerretsen) and $R\ge 2r$ (Euler). Remains to prove that:

$$(16Rr-5r^2)(2R-3r)\ge r(2R^2+8Rr+3r^2)\Leftrightarrow$$

$$(16R-5r)(2R-3r)\ge 2R^2+8Rr+3r^2\Leftrightarrow$$

$$32R^2-48Rr-10Rr+15r^2\ge 2R^2+8Rr+3r^2\Leftrightarrow$$

$$30R^2-6Rr+12r^2\ge 0\Leftrightarrow 5R^2-11Rr+2r^2\ge 0\Leftrightarrow(R-2r)(5R-r)\ge 0$$

5) In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^2}{h_b+h_c}\le\sum_{cyc}\frac{a^2}{r_b+r_c}$$

Proposed by Marin Chirciu-Romania

Solution. Using Lemma's we have:

$$\sum_{cyc}\frac{a^2}{h_b+h_c}=\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr};\ \sum_{cyc}\frac{a^2}{r_b+r_c}=2(2R-r)$$

Inequality, becomes as: $\frac{4R(s^2-r^2-Rr)}{s^2+r^2+2Rr}\le 2(2R-r)\Leftrightarrow$

$2R(s^2-r^2-Rr)\le(2R-r)(s^2+r^2+2Rr)\Leftrightarrow s^2\le 6R^2+2Rr-r^2$, which follows from $s^2\le 4R^2+4Rr+3r^2$ (Gerretsen) and $R\ge 2r$ (Euler). Remains to prove that:

$$4R^2+4Rr+3r^2\le 6R^2+2Rr-r^2\Leftrightarrow R^2-Rr-2r^2\ge 0\Leftrightarrow(R-2r)(R+r)\ge 0$$

Equality holds if and only if triangle is equilateral.

6) In ΔABC the following relationship holds:

$$6r \le \sum_{cyc} \frac{a^2}{h_b + h_c} \le \sum_{cyc} \frac{a^2}{r_b + r_c} \le \frac{a^3 + b^3 + c^3}{4F}$$

Solution. See up these inequalities. Equality holds if and only if triangle is equilateral.

REFERENCE:

ROMANIAN MATHEMATICAL MAGAZINE-INETRACTIVE JOURNAL-www.ssmrmh.ro

ABOUT AN INEQUALITY BY GEORGE APOSTOLOPOULOS-V

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{4r}{R^2} \le \frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} \le \frac{R}{2r^2}$$

Proposed by George Apostolopoulos – Greece

Solution

We prove the following lemma:

Lemma.

2) In ΔABC the following relationship holds:

$$\frac{h_a}{r_b r_c} + \frac{h_b}{r_c r_a} + \frac{h_c}{r_a r_b} = \frac{1}{2R}\left[1 + \left(\frac{4R + r}{s}\right)^2\right]$$

Proof.

Using the formulas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$\sum \frac{h_a}{r_b r_c} = \sum \frac{\frac{2S}{a}}{\frac{S}{s-b} \cdot \frac{S}{s-c}} = \frac{2}{S} \sum \frac{(s-b)(s-c)}{a} = \frac{2}{rs} \cdot \frac{r[s^2 + (4R+r)^2]}{4Rs} = \frac{1}{2R}\left[1 + \left(\frac{4R+r}{s}\right)^2\right]$, which follows from

$$\sum \frac{(s-b)(s-c)}{a} = \frac{r[s^2 + (4R + r)^2]}{4Rs}$$

Let's get back to the main problem

LHS inequality. Using the Lemma the inequality can be written:

$$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \ge \frac{4r}{R^2} \Leftrightarrow s^2(R-8r)+R(4R+r)^2 \ge 0$$

We distinguish the following cases:

Case 1). If $(R-8r) \ge 0$, the inequality is obvious.

Case 2). If $(R-8r) < 0$, the inequality can be rewritten $R(4R+r)^2 \ge s^2(8r-R)$, which follows from Blundon – Gerretsen's inequality $s^2 \le \frac{R(4R+r)^2}{2(2R-r)}$

It remains to prove that $R(4R+r)^2 \ge \frac{R(4R+r)^2}{2(2R-r)}(8r-R) \Leftrightarrow R \ge 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

RHS inequality. Using the Lemma the inequality can be written:

$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right] \ge \frac{R}{2r^2} \Leftrightarrow s^2(R^2-r^2) \ge r^2(4R+r)^2$, which follows from Gerretsen's inequality $s^2 \ge 16Rr-5r^2 \ge \frac{r(4R+r)^2}{R+r}$.

It remains to prove that $\frac{r(4R+r)^2}{R+r}(R^2-r^2) \ge r^2(4R+r)^2 \Leftrightarrow R \ge 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

Remark. Let's interchange h_a with r_a.

3) In ΔABC the following relationship holds:

$$\frac{1}{r} \le \frac{r_a}{h_b h_c}+\frac{r_b}{h_c h_a}+\frac{r_c}{h_a h_b} \le \frac{R}{2r^2}$$

Proposed by Marin Chirciu – Romania

Solution We prove the following lemma:

Lemma.

4) In ΔABC the following relationship holds:

$$\frac{r_a}{h_b h_c}+\frac{r_b}{h_c h_a}+\frac{r_c}{h_a h_b} = \frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

Proof.

Using the formulas $h_a = \frac{2S}{a}$ and $r_a = \frac{S}{s-a}$ we obtain:

$$\sum \frac{r_a}{h_b h_c} = \sum \frac{\frac{S}{s-a}}{\frac{2S}{b}\cdot\frac{2S}{c}} = \frac{1}{4S}\sum \frac{bc}{s-a} = \frac{1}{4rs}\cdot\frac{s^2+(4R+r)^2}{s} = \frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]$$

which follows from $\sum\frac{bc}{s-a}=\frac{s^2+(4R+r)^2}{s}$

Let's get back to the main problem. LHS inequality.

Using the Lemma the inequality can be written:

$$\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\ge\frac{1}{r}\Leftrightarrow(4R+r)^2\ge 3s^2\ \text{(Doucet's inequality)}$$

Equality holds if and only if the triangle is equilateral. RHS inequality

Using the Lemma the inequality can be written:

$\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\le\frac{R}{2r^2}\Leftrightarrow s^2(2R-r)\ge r(4R+r)^2$, which follows from Gerretsen's inequality $s^2\ge 16Rr-5r^2\ge\frac{r(4R+r)^2}{R+r}$.

It remains to prove that $\frac{r(4R+r)^2}{R+r}(2R-r)\ge r(4R+r)^2\Leftrightarrow R\ge 2r$ (Euler's inequality)

Equality holds if and only if the triangle is equilateral.

Remark.

Between the sums $\frac{h_a}{r_br_c}+\frac{h_b}{r_cr_a}+\frac{h_c}{r_ar_b}$ and $\frac{r_a}{h_bh_c}+\frac{r_b}{h_ch_a}+\frac{r_c}{h_ah_b}$ the following relationship holds:

5) In ΔABC the following relationship holds:

$$\frac{h_a}{r_br_c}+\frac{h_b}{r_cr_a}+\frac{h_c}{r_ar_b}\le\frac{r_a}{h_bh_c}+\frac{r_b}{h_ch_a}+\frac{r_c}{h_ah_b}$$

Proposed by Marin Chirciu – Romania

Using the following Lemmas, the inequality holds:

$$\frac{1}{2R}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\le\frac{1}{4r}\left[1+\left(\frac{4R+r}{s}\right)^2\right]\Leftrightarrow R\ge 2r\ \text{(Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark. The following sequence of inequalities can be written:

6) In ΔABC the following relationship holds:

$$\frac{4r}{R^2}\le\frac{h_a}{r_br_c}+\frac{h_b}{r_cr_a}+\frac{h_c}{r_ar_b}\le\frac{r_a}{h_bh_c}+\frac{r_b}{h_ch_a}+\frac{r_c}{h_ah_b}\le\frac{R}{2r^2}$$

Solution See inequalities 1) and 3) Equality holds if and only if the triangle is equilateral.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY MARIAN URSĂRESCU-VII

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\frac{R}{r}+\frac{r}{R}+6\sum\frac{r_a^2}{bc}\geq 16$$

Proposed by Marian Ursărescu – Romania

Solution We prove the following lemma:

Lemma.

2) In ΔABC the following relationship holds:

$$\sum\frac{r_a^2}{bc}=\frac{2R(4R+r)-s^2}{2Rr}$$

Proof.

Using $r_a=\frac{S}{s-a}$ we obtain $\sum\frac{r_a^2}{bc}=\sum\frac{\frac{S^2}{(s-a)^2}}{bc}=\frac{S^2}{abc}\sum\frac{a}{(s-a)^2}=\frac{4R(4R+r)-s^2}{r^2s}$, which follows from $\sum\frac{a}{(s-a)^2}=\frac{4R(4R+r)-2s^2}{r^2s}$. Let's get back to the main problem.

Using Lemma the inequality from enunciation can be written:

$\frac{R}{r}+\frac{r}{R}+6\cdot\frac{2R(4R+r)-s^2}{2Rr}\geq 16\Leftrightarrow 3s^2\leq 25R^2-10Rr+r^2$ which follows from Gerreten's inequality: $s^2\leq 4R^2+4Rr+3r^2$. It remains to prove that:

$$3(4R^2+4Rr+3r^2)\leq 25R^2-10Rr+r^2\Leftrightarrow 13R^2-22Rr-8r^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(13R+4r)\geq 0,\text{ obviously from Euler's inequality } R\geq 2r.$$

Equality holds if and only if the triangle is equilateral.

Remark.We can develop the inequality:

3) In ΔABC the following inequality holds:

$$\frac{R}{r}+\frac{r}{R}+n\sum\frac{r_a^2}{bc}\geq\frac{5}{2}+\frac{9n}{4},\text{ where } n\geq 0.$$

Proposed by Marin Chirciu – Romania

Solution

We prove that: $\frac{R}{r}+\frac{r}{R}\geq\frac{5}{2}$, (1) and $\sum\frac{r_a^2}{bc}\geq\frac{9}{4}$ (2)

Indeed $\frac{R}{r}+\frac{r}{R}\geq\frac{5}{2}\Leftrightarrow 2R^2-5Rr+2r^2\geq 0\Leftrightarrow (R-2r)(2R+r)\geq 0$, obviously from Euler's inequality $R\geq 2r$.

For the inequality $\sum\frac{r_a^2}{bc}\geq\frac{9}{4}$ we use the following Lemma and we write the inequality:

$\frac{2R(4R+r)-s^2}{2Rr}\geq\frac{9}{4}\Leftrightarrow 2s^2\leq 16R^2+4Rr-9r^2$, which follows from Gerretsen's inequality

$s^2\leq 4R^2+4Rr+3r^2$. It remains to prove that:

$$2(4R^2+4Rr+3r^2)\leq 16R^2+4Rr-9r^2\Leftrightarrow 8R^2-13Rr-6r^2\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(8R+3r)\geq 0,\text{ which follows from Euler's inequality } R\geq 2r.$$

From (1), (2) and the condition from hypothesis $n\geq 0$ it follows the conclusion:

$$\frac{R}{r}+\frac{r}{R}+n\sum\frac{r_a^2}{bc}\geq\frac{5}{2}+\frac{9n}{4}$$

Equality holds if and only if the triangle is equilateral.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY MARTIN LUKAREVSKI-I

By Marin Chirciu-Romania

Let a,b,c be the sides of a triangle ABC, m_a,m_b,m_c the corresponding medians and R,r its circumradius and inradius respectively. Prove that:

$$\sum\frac{a^2}{m_b^2+m_c^2}\geq\frac{4r}{R}$$

Proposed by Martin Lukarevski – Macedonia

Solution

Using the median formula $m_a^2=\frac{2b^2+2c^2-a^2}{4}$ we have:

$$LHS=\sum\frac{a^2}{m_b^2+m_c^2}=\sum\frac{a^2}{\frac{2a^2+2c^2-b^2}{4}+\frac{2a^2+2b^2-c^2}{4}}=$$

$$=\sum\frac{4a^2}{4a^2+b^2+c^2}=4\sum\frac{a^4}{a^2(4a^2+b^2+c^2)}\overset{(1)}{\geq}4\cdot\frac{(\sum a^2)^2}{\sum a^2(4a^2+b^2+c^2)}=$$

$= 4 \cdot \frac{\sum a^4+2\sum b^2c^2}{4\sum a^4+2\sum b^2c^2} = 2 \cdot \frac{\sum a^4+2\sum b^2c^2}{2\sum a^4+\sum b^2c^2} \overset{(2)}{\geq} \frac{4r}{R} =$ RHS, where (1) follows from Berström's inequality, and (2) is equivalent to

$$\frac{\sum a^4 + 2\sum b^2c^2}{2\sum a^4 + \sum b^2c^2} \geq \frac{2r}{R} \Leftrightarrow (R-2r)\sum a^4 + (2R-2r)\sum b^2c^2 \geq 0$$

We distinguish the following cases:

Case 1). If inequality is obvious.

Case 2). If $(R-4r) < 0$ inequality rewrites itself:

$(2R-2r)\sum b^2c^2 \geq (4r-R)\sum a^4$, true from the identities known in triangle:

$$\sum a^4 = 2[s^4 - s^2(8Rr+6r^2) + r^2(4R+r)^2] \text{ and}$$

$$\sum b^2c^2 = s^4 + s^2(2r^2 - 8Rr) + r^2(4Rr+r)^2$$

We showed that:

$$(2R-2r)[s^4 + s^2(2r^2-8Rr) + r^2(4R+r)^2] \geq$$

$$\geq (4r-R) \cdot 2[s^4 - s^2(8Rr+6r^2) + r^2(4R+r)^2]$$

$$\Leftrightarrow (R-r)[s^4 + s^2(2r^2-8Rr) + r^2(4R+R)^2] \geq$$

$$\geq (4r-R)[s^4 - s^2(8Rr+6r^2) + r^2(4R+r)^2]$$

$$\Leftrightarrow s^2[(2R-5r)s^2 + r(-16R^2+36Rr+22r^2)] + r^2(4R+r)^2(2R-5r) \geq 0$$

True from Gerretsen's inequality $s^2 \geq 16Rr - 5r^2 \geq \frac{r(4R+r)^2}{R+r}$

It remain to prove that:

$$\frac{r(4R+r)^2}{R+r}[(2R-5r)(16Rr-5r^2) + r(-16R^2+36Rr+22r^2)] +$$

$$+r^2(4R+r)^2(2R-5r) \geq 0$$

$$\Leftrightarrow \frac{1}{R+r}[(2R-5r)(16R-5r) + (-16R^2+36Rr+22r^2)] + (2R-5r) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow [(2R-5r)(16R-5r) + (-16R^2+36Rr+22r^2)] + (R+r)(2R-5r) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 18R^2 - 57Rr + 42r^2 \geq 0 \Leftrightarrow 6R^2 - 19Rr + 14r^2 \geq 0 \Leftrightarrow (R-2r)(6R-7r) \geq 0$$

True from Euler's inequality $R \geq 2r$. Equality holds if and only if the triangle is equilateral.

Remark.

Let's find an inequality having an opposite sense:

JP.267. In ΔABC the following relationship holds:

$$\sum \frac{a^2}{m_b^2 + m_c^2} \leq 2$$

Proposed by Nguyen Viet Hung – Vietnam,

Romanian Mathematical Magazine , Number 18, Autumn 2020

Solution We prove the following lemma.

Lemma.

In ΔABC the following relationship holds:

$$m_a \geq \sqrt{s(s-a)}$$

Proof.

$$m_a \geq \sqrt{s(s-a)} \Leftrightarrow m_a^2 \geq s(s-a) \Leftrightarrow \frac{2b^2 + 2c^2 - a^2}{4} \geq \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \Leftrightarrow$$

$$\Leftrightarrow 2b^2 + 2c^2 - a^2 \geq (a+b+c)(b+c-a) \Leftrightarrow 2b^2 + 2c^2 - a^2 \geq (b+c)^2 - a^2 \Leftrightarrow$$

$\Leftrightarrow 2b^2 + 2c^2 \geq (b+c)^2 \Leftrightarrow (b-c)^2 \geq 0$, obviously with equality from $b = c$.

Let's get back to the main problem.

Using Lemma we obtain:

$$\sum \frac{a^2}{m_b^2 + m_c^2} \leq \sum \frac{a^2}{s(s-b) + (s-c)} = \sum \frac{a^2}{s(s-b+s-c)} = \sum \frac{a^2}{sa} = \sum \frac{a}{s} = 2$$

Equality holds if and only if the triangle is equilateral.

Remark. We can write the double inequality:

In ΔABC the following relationship holds:

$$\frac{4r}{R} \leq \sum \frac{a^2}{m_b^2 + m_c^2} \leq 2$$

Solution See inequalities 1) and 3) from above.

Equality holds if and only if the triangle is equilateral.

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro

ABOUT AN INEQUALITY BY SEYRAN IBRAHIMOV-I

By Marin Chirciu-Romania

1) Prove that in any ΔABC the following relationship holds:

$$\left(\sqrt{ab}+\sqrt{bc}+\sqrt{ca}\right)R^2 \geq 24\sqrt{3}(R-r)r^2$$

Proposed by Seyran Ibrahimov – Azerbaijan

Solution: We prove the following lemma:

2) In ΔABC the following relationship holds:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 6r\sqrt{3}$$

Proof. From means inequality we have:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \geq 3\sqrt[3]{\sqrt{(abc)^2}} = 3\sqrt[6]{(abc)^2} \overset{(1)}{\geq} \sqrt[6]{(12r^2)^3} = 3\sqrt{12r^2} = 6r\sqrt{3}$$

where (1) is equivalent with $(abc)^2 \geq (12r^2)^3 \Leftrightarrow (4Rrs)^2 \geq (12r^2)^3 \Leftrightarrow$

$\Leftrightarrow 16R^2r^2s^2 \geq 144 \cdot 12r^6 \Leftrightarrow R^2s^2 \geq 27r^2 \cdot 4r^2$, which follows from $R^2 \geq 4r^2$ (Euler) and $s^2 \geq 27r^2$ (Mitrinovic). Equality holds if and only if the triangle is equilateral.

Let's get back to the main problem.

Using Lemma it suffices to prove that:

$6r\sqrt{3}R^2 \geq 24\sqrt{3}(R-r)r^2 \Leftrightarrow R^2 \geq 4r(R-r) \Leftrightarrow (R-2r)^2 \geq 0$, obviously, with equality if $R = 2r$, namely for equilateral triangle.

Remark. Let's find an inequality having an opposite sense for $\sqrt{ab}+\sqrt{bc}+\sqrt{ca}$

3) In ΔABC the following relationship holds:

$$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \leq 2s$$

Solution: From means inequality we have

$\sqrt{ab}+\sqrt{bc}+\sqrt{ca} \leq \frac{a+b}{2}+\frac{b+c}{2}+\frac{c+a}{2} = a+b+c = 2s$. Equality holds if and only if the triangle is equilateral.

Remark. We can write the double inequality:

4) In ΔABC the following inequality holds: $6r\sqrt{3} \leq \sqrt{ab}+\sqrt{bc}+\sqrt{ca} \leq 2s$

Solution: See inequalities 2) and 3). Equality holds if and only if the triangle is equilateral.

Reference: Romanian Mathematical Magazine-www.ssmrmh.ro

PROBLEMS FOR JUNIORS

J.3161 Let ABC be a triangle and its side lengths be a, b, c. Let the area of the triangle with side lengths $\sqrt{a(b+c-a)}, \sqrt{b(c+a-b)}, \sqrt{c(a+b-c)}$ be F^* and the area of the triangle ABC be F. Prove that:

$$\frac{F}{F^*} \geq \sqrt{6\left(\frac{r}{R}\right) - \frac{R}{r}}$$

Proposed by Mehmet Șahin-Turkiye

J.3162 Let ABC be a triangle and its side lengths be a, b, c. Prove that:

$$18r^2 \leq \sqrt{ab(s-a)(s-b)} + \sqrt{bc(s-b)(s-c)} + \sqrt{ca(s-c)(s-a)} \leq 2r(r_a + r_b + r_c)$$

Proposed by Mehmet Șahin-Turkiye

J.3163 Let n ≥ 2 , $x_1, x_2, \dots, x_n$ are positive real numbers. Prove that:

$$\sqrt{\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2} < \sum_{i=1}^{n}\frac{1}{x_i} - \frac{1}{\sum_{i=1}^{n} x_i}$$

Proposed by Mehmet Șahin-Turkiye

J.3164 Prove that:

$$\frac{1}{\sin^2\left(\frac{3\pi}{7}\right)} - \frac{1}{\sin^2\left(\frac{\pi}{7}\right)} = -\frac{16}{\sqrt{7}}\sin\left(\frac{\pi}{7}\right) + \frac{8}{\sqrt{7}}\sin\left(\frac{2\pi}{7}\right) - 4$$

Proposed by Vasile Mircea Popa-Romania

J.3165 Let M be an interior point in ΔABC with the area F and $F_a =$ area $MBC, F_b =$ area MCA, $F_c =$ area MAB. Prove that:

$$\frac{a^4}{F_b} + \frac{b^4}{F_c} + \frac{c^4}{F_a} \geq 48F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

J.3166 If $m \geq 0, M$ is an interior point in ΔABC the area F and $F_a =$ area $MBC, F_b =$ area MCA,

$F_c=$ area MAB then:

$$\frac{(ab)^{m+1}}{F_c^m}+\frac{(bc)^{m+1}}{F_a^m}+\frac{(ca)^{m+1}}{F_b^m}\geq 4^{m+1}\left(\sqrt{3}\right)^{m+1}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3167 Let be $m\geq 0$ and ABC a triangle with the area F and the semiperimeter s, then:

$$s^m\cdot(a^{m+2}+b^{m+2}+c^{m+2})\geq 2^{m+2}\cdot\left(\sqrt{3}\right)^{m+1}\cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3168 If $m\geq 0$ and ABC is a triangle with the area F then the following inequality holds:

$$a^m+b^m+c^m+a^{3m+4}+b^{3m+4}+3^{3m+4}\geq 2^{2m+3}\left(\sqrt{3}\right)^{1-m}\cdot F^{m+1}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3169 Let be $m\geq 0, M$ an interior point in triangle ABC with the area F and d_a,d_b,d_c the distances of point M to the sides BC,CA respectively AB. Prove that:

$$\frac{x^{m+1}\cdot a^{m+2}}{(y+z)^{m+1}\cdot d_a^m}+\frac{y^{m+1}\cdot b^{m+2}}{(z+x)^{m+1}\cdot d_b^m}+\frac{z^{m+1}\cdot c^{m+2}}{(x+y)^{m+1}\cdot d_c^m}\geq 2\left(\sqrt{3}\right)^{m+1}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3170 Let be M an interior point in ΔABC with the area F and d_a,d_b,d_c the distances of point M to the sides BC,CA,AB then:

$$\frac{a^3}{d_a}+\frac{b^3}{d_b}+\frac{c^3}{d_c}\geq 24F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3171 If $x,y,z>0$ and M is an interior point in ΔABC with the area F and d_a,d_b,d_c are the distances of point M to the sides BC,CA,AB, then:

$$\frac{x^2a^3}{d_a}+\frac{y^2b^3}{d_b}+\frac{z^2c^3}{d_c}\geq 8F\cdot\left(\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3172 Let be $m\geq 0, M$ an interior point in ΔABC with the area F and $F_a=$ area MBC,

$F_b=$ area MCA, $F_c=$ area MAB. Prove that:

$$\frac{x^{m+1}\cdot a^{2m+2}}{(y+z)^{m+1}\cdot F_b^m}+\frac{y^{m+1}\cdot b^{2m+2}}{(z+x)^{m+1}F_c^m}+\frac{z^{m+1}\cdot c^{2m+2}}{(x+y)^{m+1}\cdot F_a^m}\geq 2^{m+1}\left(\sqrt{3}\right)^{m+1}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3173 Let be $x, y, z > 0$ and M an interior point in ΔABC with the area F and $F_a =$ area MBC, $F_b =$ area MCA, $F_c =$ area MAB, then:

$$\frac{x^2a^4}{F_b}+\frac{y^2b^4}{F_c}+\frac{z^2c^4}{F_a}\ge 16F\cdot\left(\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3174 If $x, y, z > 0$ and ABC is a triangle of area F, then:

$$\sqrt{\frac{x^2a^4}{(y+z)^2}+\frac{y^2b^4}{(z+x)^2}}+\sqrt{\frac{y^2b^4}{(z+x)^2}+\frac{z^2c^4}{(x+y)^2}}+\sqrt{\frac{z^2c^4}{(x+y)^2}+\frac{x^2a^4}{(y+z)^2}}\ge 2\cdot\sqrt{6}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3175 If $m \ge 0$, then in any ΔABC with the area F the following inequality holds:

$$(a^m+b^m+c^m)(a^{3m+4}+b^{3m+4}+c^{3m+4})\ge 16^{m+1}\left(\sqrt{3}\right)^{1-m}\cdot F^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3176. Let be $x, y, z > 0$ and M an interior point in ΔABC with the area F and $F_a =$ area MBC, $F_b =$ area $MCA, F_c =$ area MAB. Prove that:

$$\frac{x^2\cdot a^4}{(y+z)^2F_b}+\frac{y^2\cdot b^4}{(z+x)^2F_c}+\frac{z^2\cdot c^4}{(x+y)^2F_a}\ge 12\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3177 Let M be an interior point in ΔABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CA respectively AB then:

$$(a^6+b^6+c^6)\left(\frac{1}{d_a^2}+\frac{1}{d_b^2}+\frac{1}{d_c^2}\right)\ge 576F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

J.3178 Let be $m \ge 0, M$ an interior point in ΔABC with the area F and $F_a =$ area MBC, $F_b =$ area $MCA, F_c =$ area MAB, then: $\frac{(ab)^{2m+2}}{F_c^m}+\frac{(bc)^{2m+2}}{F_a^m}+\frac{(ca)^{2m+2}}{F_b^m}\ge 16^{m+1}\cdot F^{m+2}$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

J.3179 In any ΔABC with the area F the following relationship holds:

$$(a^6+b^6+c^6)\left(\frac{1}{h_a^2}+\frac{1}{h_b^2}+\frac{1}{h_c^2}\right)\ge 64\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3180 If M is an interior point in ΔABC with the area F and $F_a=$ area $MBC, F_b=$ area MCA,

$F_c=$ area MAB then:

$$\frac{a^4b^4}{F_c}+\frac{b^4c^4}{F_a}+\frac{c^4a^4}{F_b}\ge 256\cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

J.3181 Let be $x,y>0$, then in any ΔABC the following inequality holds:

$$\frac{a^4}{x(r_ar_b+yr_br_c)}+\frac{b^4}{xr_br_c+yr_cr_a}+\frac{c^4}{xr_cr_a+yr_ar_b}\ge\frac{48}{x+y}r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3182 Let $MABC$ be a pyramid having the apex M and the base the triangle ABC with the area F. If d_a,d_b,d_c are the distances of the point M to the sides BC,CA respectively AB and S is the lateral area of the pyramid then:

$$\frac{a^4b^3}{d_b}+\frac{b^4c^3}{d_c}+\frac{c^4a^3}{d_a}\ge 128\cdot\frac{F^3}{S}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3183 If M is an interior point in ΔABC with the area F and d_a,d_b,d_c the distances of point M to the sides BC,CA, respectively AB, then:

$$\frac{a^2b}{d_b}+\frac{b^2c}{d_c}+\frac{c^2a}{d_a}\ge 24F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

J.3184 In any ΔABC the following inequality holds:

$$\frac{a^4}{r_br_c}+\frac{b^4}{r_cr_a}+\frac{c^4}{r_ar_b}\ge 48r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

J.3185 If $m\ge 0$ and ABC is a triangle with the area F, then:

$$\frac{a^{m+1}b^m}{h_b}+\frac{b^{m+1}c^m}{h_c}+\frac{c^{m+1}a^m}{h_a}\ge 2^{2m+1}\cdot\left(\sqrt{3}\right)^{1-m}\cdot F^m$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3186 Let be $m\ge 0, x,y,z>0$ and ABC a triangle and M an interior point in the triangle. If d_a,d_b,d_c are the distances of point M to the sides BC,CA respectively AB then:

$$\frac{x^{m+1}\cdot a^{m+2}}{(y+z)^{m+1}\cdot d_a^m}+\frac{y^{m+1}\cdot b^{m+2}}{(z+x)^{m+1}\cdot d_b^m}+\frac{z^{m+1}\cdot c^{m+2}}{(x+y)^{m+1}\cdot d_c^m}\ge 2\left(\sqrt{3}\right)^{m+1}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3187 Let be $x, y \geq 0, M$ an interior point in ΔABC with the area F and $F_a =$ area MBC,

$F_b =$ area $MCA, F_c =$ area MAB, then:

$$\frac{a^2b^2}{xF_a + yF_b} + \frac{b^2c^2}{xF_b + yF_c} + \frac{c^2a^2}{xF_c + yF_a} \geq \frac{48}{x+y} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

J.3188 Let be $t \geq 0$ and M an interior point in the convexe polygon $A_1A_2 \dots A_n, n \geq 3$ with the area F and the length sides $a_k = A_kA_{k+1}, k = \overline{1,n}, A_{n+1} = A_1$ and d_k the distance of point M to the sides $A_kA_{k+1}, k = \overline{1,n}$, then:

$$\frac{a_1^{t+2}}{d_1^t} + \frac{a_2^{t+2}}{d_2^t} + \dots + \frac{a_n^{t+2}}{d_n^t} \geq 2^{t+2} \cdot F \cdot \tan^{t+1}\left(\frac{\pi}{n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3189 If $x, y > 0$ then in any triangle ABC the following inequality holds:

$$\frac{a^2b^2}{xr_ar_b + yr_br_c} + \frac{b^2c^2}{xr_br_c + yr_cr_a} + \frac{c^2a^2}{xr_cr_a + yr_ar_b} \geq \frac{48}{x+y} \cdot r^2$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

J.3190 If $x, y > 0, M$ is an interior point in triangle ABC with the area F and $F_a =$ area MBC,

$F_b =$ area $MCA, F_c =$ area MAB, then:

$$\frac{a^4}{xF_b + yF_c} + \frac{b^4}{xF_c + yF_a} + \frac{c^4}{xF_a + yF_b} \geq \frac{48}{x+y} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

J.3191 Let be ABC a triangle with the area F and F a point in space such that

$M \notin [AB] \cup [BC] \cup [CA]$ and let be $F_a =$ area $MBC, F_b =$ area $MCA, F_c =$ area MAB and

$S = F_a + F_b + F_c$. Prove that:

$$\frac{a^4}{F_b} + \frac{b^4}{F_c} + \frac{c^4}{F_a} \geq \frac{48}{S} \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3192 Let be $t \geq 0, x, y > 0$ and M an interior point in the convexe polygon $A_1A_2 \dots A_n, n \geq 3$ with the area F with the sides having the lengths $a_k = A_kA_{k+1}, k = \overline{1,n}, A_{n+1} = A_1$ and $F_k =$ area $MA_kA_{k+1}, \forall k = \overline{1,n}$, then:

$$\frac{a_1^{2t+2}}{(xF_1 + yF_2)^t} + \frac{a_2^{2t+2}}{(xF_2 + yF_3)^t} + \dots + \frac{a_{n-1}^{2t+2}}{(xF_{n-2} + yF_n)^t} + \frac{a_n^{2t+2}}{(xF_n + yF_1)^t} \geq \frac{4^{t+1}}{(x+y)^t} \cdot F \cdot \tan^{t+1}\left(\frac{\pi}{n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3193 Let be $t \geq 0$ and M an interior point in the convexe polygon $A_1A_2 \dots A_n, n \geq 3$ with the area F with the sides having the lengths $a_k = A_kA_{k+1}, k = \overline{1,n}, A_{n+1} = A_1$ and

$F_k =$ area $MA_kA_{k+1}, \forall k = \overline{1,n}$, then:

$$\frac{a_1^{2t+2}}{F_1^t} + \frac{a_2^{2t+2}}{F_2^t} + \dots + \frac{a_n^{2t+2}}{F_n^t} \geq 4^{t+1} \cdot F \cdot \tan^{t+1}\left(\frac{\pi}{n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3194 If $t \geq 0$ then in any convexe polygon $A_1A_2 \dots A_n, n \geq 3$ with the area F and the sides with the length $a_k = A_kA_{k+1}, A_{n+1} = A_1$ the following inequality holds:

$$a_1^{2t+2} + a_2^{2t+2} + \dots + a_n^{2t+2} \geq \frac{4^{t+1} \cdot F^{t+1}}{n^t} \cdot \tan^{t+1}\frac{\pi}{n}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3195 In any ΔABC the following inequality holds:

$$\frac{a^2b^2}{r_br_c} + \frac{b^2c^2}{r_cr_a} + \frac{c^2a^2}{r_ar_b} \geq 48r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

J.3196 In any ΔABC the following inequality holds:

$$\frac{AH}{a \cdot r_a} + \frac{BH}{b \cdot r_b} + \frac{CH}{c \cdot r_c} \leq \frac{s}{3r^2}$$

Proposed by Sarkhan Adgozalev-Georgia

J.3197 In any ΔABC the following inequality holds:

$$\frac{AI}{bc \cdot r_a} + \frac{BI}{ca \cdot r_b} + \frac{CI}{ab \cdot r_c} \leq \frac{3R}{4r}$$

Proposed by Sarkhan Adgozalev-Georgia

J.3198 If x, y, z are positive real numbers then in ΔABC the following relationship holds:

$$\frac{x}{y+z}{\omega_a}^2 + \frac{y}{z+x}{\omega_b}^2 + \frac{z}{x+y}{\omega_c}^2 \geq 27r^2 - \frac{s^2}{2}$$

Equality holds if and only if $x = y = z$ and ΔABC is equilateral.

Proposed by Mehmet Șahin-Turkiye

J.3199 $In\ \Delta ABC,\ I - incenter,\ O_A,\ O_B,\ O_C - circumcenters\ of\ \Delta BIC,\ \Delta AIC,\ \Delta AIB.$

$$\frac{s}{2r^2} \geq \frac{b+c}{AO_A \cdot AI} + \frac{c+a}{BO_B \cdot BI} + \frac{a+b}{CO_C \cdot CI} \geq \frac{3\sqrt{3}}{R}$$

Proposed by Sarkhan Adgozalev-Georgia

J.3200 In acute ΔABC the following relationship holds:

$$\sum \frac{a^2}{b^2+c^2-a^2} \geq \left(\frac{2p}{3R}\right)^2$$

Proposed by Marin Chirciu – Romania

J.3201 In ΔABC the following relationship holds:

$$\sum \sqrt{a(b+c-a)} \geq 6r\sqrt{3}$$

Proposed by Marin Chirciu – Romania

J.3202 In ΔABC the following relationship holds:

$$\sum a\sqrt{\frac{a}{b+c-a}} \geq 6r\sqrt{3}$$

Proposed by Marin Chirciu – Romania

J.3203 In acute ΔABC the following relationship holds:

$$\sum \sqrt{a(b^2+c^2-a^2)} \geq 6r\sqrt{2p}$$

Proposed by Marin Chirciu – Romania

J.3204 In acute ΔABC the following relationship holds:

$$\sum \left(\frac{a}{b^2+c^2-a^2}\right)^n \sqrt{a(b^2+c^2-a^2)} \geq \left(\frac{2p}{9R^2}\right)^n 3R\sqrt{2p}, n \in \mathbb{N}^*$$

Proposed by Marin Chirciu – Romania

J.3205 In $\Delta ABC, (h_a, h_b, h_c)$ can represent the lengths of a the sides of a triangle. Then:

$$\sum \left(\frac{h_a}{h_b+h_c-h_a}\right)^n \sqrt{h_a(h_b+h_c-h_a)} \geq 9r, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

J.3206 In $\Delta ABC, (r_a, r_b, r_c)$ can represent the lengths of the sides in a triangle. Then:

$$\sum \left(\frac{r_a}{r_b+r_c-r_a}\right)^n \sqrt{r_a(r_b+r_c-r_a)} \geq 9r, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

J.3207 In ΔABC the following relationship holds:

$$\sum \left(\frac{m_a}{m_b+m_c-m_a}\right)^n \sqrt{m_a(m_b+m_c-m_a)} \geq 9r, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania

J.3208 In ΔABC the following relationship holds:

$$\sum\sqrt{\frac{\cos\frac{A}{2}}{\cos\frac{B}{2}+\cos\frac{C}{2}-\cos\frac{A}{2}}}\geq 3$$

Proposed by Marin Chirciu – Romania

J.3209 In ΔABC the following relationship holds:

$$\sum\left(\frac{1}{r_b}+\frac{\lambda}{r_c}\right)^2\geq\frac{2(\lambda+1)^2}{3Rr},\lambda\geq 0$$

Proposed by Marin Chirciu – Romania

J.3210 In ΔABC the following relationship holds:

$$\sum\left(\frac{1}{m_b}+\frac{\lambda}{m_c}\right)^2\geq\frac{4(\lambda+1)^2}{3R^2},\lambda\geq 0$$

Proposed by Marin Chirciu – Romania

J.3211 Prove that the number $4^{99}+6^{100}+3^{200}$ is a perfect square.

Proposed by Mădălina Panduru – Romania

J.3212 In ΔABC the following relationship holds:

$$\sqrt{6}\leq\sum\sqrt{\tan^2\frac{B}{2}+\tan^2\frac{C}{2}}\leq\sqrt{6\left(\frac{R}{r}-1\right)}$$

Proposed by Marin Chirciu – Romania

J.3213 Let be $n\in\mathbb{N}^*$. Prove that:

a. $1\cdot 2+2\cdot 3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$ b. $n!\leq(n+1)^{\frac{n-1}{2}}\left(\frac{n+2}{3}\right)^{\frac{n}{2}}$. In which case do we have inequality ?

Proposed by Dorina Goiceanu, Camelia Dană –Romania

J.3214 In ΔABC the following relationship holds:

$$3\sqrt{6}\leq\sum\sqrt{\cot^2\frac{B}{2}+\cot^2\frac{C}{2}}\leq\left(\frac{2R}{r}-1\right)\sqrt{6}$$

Proposed by Marin Chirciu – Romania

J.3215 If $x, y, z > 0$ and $\lambda \geq 8$ then:

$$\sqrt{\frac{x^3}{y^3 + \lambda xyz}} + \sqrt{\frac{y^3}{z^3 + \lambda xyz}} + \sqrt{\frac{z^3}{x^3 + \lambda xyz}} \geq \frac{3}{\sqrt{\lambda + 1}}$$

Proposed by Marin Chirciu – Romania

J.3216 In ΔABC the following relationship holds:

$$6\sqrt{2} \leq \sum \sqrt{\csc^2 \frac{B}{2} + \csc^2 \frac{C}{2}} \leq 3\frac{R}{r}\sqrt{\frac{R}{r}}$$

Proposed by Marin Chirciu – Romania

J.3217 Prove that the numbers $10017, 100117, 1001117, \dots, 100\underbrace{11\dots1}_{k\ times}7, \dots k \in \mathbb{N}^*$ can be divided by 53.

Proposed by Dan Grigorie –Romania

J.3218 In ΔABC the following relationship holds:

$$\left(1 + \frac{r}{R}\right)\sqrt{2} \leq \sum \sqrt{\sin^2 \frac{B}{2} + \sin^2 \frac{C}{2}} \leq \sqrt{6 \cdot \left(1 - \frac{r}{2R}\right)}$$

Proposed by Marin Chirciu – Romania

J.3219 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{m_a^4}{m_b^2 + m_c(m_a + m_b)}} + \sqrt[3]{\frac{m_b^4}{m_c^2 + m_a(m_b + m_c)}} + \sqrt[3]{\frac{m_c^4}{m_a^2 + m_b(m_c + m_a)}} \geq 3\sqrt[3]{3r^2}$$

Proposed by Marin Chirciu – Romania

J.3220 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{r_a^4}{r_b^2 + r_c(r_a + r_b)}} + \sqrt[3]{\frac{r_b^4}{r_c^2 + r_a(r_b + r_c)}} + \sqrt[3]{\frac{r_c^4}{r_a^2 + r_b(r_c + r_a)}} \geq 3\sqrt[3]{3r^2}$$

Proposed by Marin Chirciu – Romania

J.3221 Find the prime numbers p such that $p^2 + 2000$ to be square of a natural number.

Proposed by Dan Grigorie –Romania

J.3222 In ΔABC the following relationship holds:

$$\sum\left(\frac{1}{s_b}+\frac{\lambda}{s_c}\right)^2 \geq \frac{4(\lambda+1)^2}{3R^2}, \lambda \geq 0$$

Proposed by Marin Chirciu – Romania

J.3223 Find all natural numbers n such that $\sqrt{n+1}+\sqrt{4n-3}$ to be natural number

Proposed by Gabriela Militaru Cismaru-Romania

J.3224 Compare the numbers $A = 2^{\sqrt{\log_2 2024}}$ and $B = 2024^{\sqrt{\log_{2024} 2}}$

Proposed by Ionuț Ivănescu, Mihaela Mirea –Romania

J.3225 Let be $a, b, c \in \mathbb{Z}$. The number $A = (7a + b - 2c + 1) \cdot (3a - 5b + 4c + 10)$ is odd or even.

Proposed by Horia Mușat –Romania

J.3226 Find $n \in \mathbb{N}^*$ such that $A = n^{11} + n + 1$ to be prime number.

Proposed by Mihaela Stăncele, Monica Stanca –Romania

J.3227 Let be $n \in \mathbb{N}$ and $x_1, x_2, \dots, x_n$ real numbers such that $x_1 + x_2 + \cdots + x_n = \frac{n-1}{2}$. Find the smallest value of the expression $E_{(x_1,x_2,\dots,x_n)} = x_1 + x_2^2 + \cdots + x_n^2$

Proposed by Lucian Tuțescu, Neculai Stanciu –Romania

J.3228 Find the real numbers x and y such that:

$$(xy+9)^2 + (y^2+xy)^2 = 18y^2$$

Proposed by Dan Grigorie, Lavinia Trincu –Romania

J.3229 Find the real numbers x and y such that:

$$(xy+4)^2 + (xy-y^2)^2 = 8y^2$$

Proposed by Luiza Cremeneanu, Constantina Prunaru –Romania

J.3230 a. Solve in the set $(1,\infty)$ the equation $\log_{x+1} x + \log_{x+2}(x+1) = \log_3 2 + \log_4 3$

b. Solve in the set $(0,1)$ the equation $\log_{x+1} x + \log_{x+2}(x+1) = \log_{\frac{3}{2}} \frac{1}{2} + \log_{\frac{5}{2}} \frac{3}{2}$

Proposed by Dan Lucian Grigorie, Mihai Călugăru –Romania

J.3231 Find $n \in \mathbb{N}$ such that $A = \frac{1+2^n+3^n}{1+2^{n-1}+3^{n-1}}$ to be natural number.

Proposed by Loredana Surcel, Cosmin Alexandru-Romania

J.3232 Find the functions $f: \mathbb{R} \to \mathbb{R}$ continuous in $x_0 = 0$ which verify the relationship:

$$f(x)f(3x)f(9x) = 3^{13x}, \forall x \in \mathbb{R}$$

Proposed by Cezar Ozun, Felicia Ozun-Romania

J.3233 Let be $x, y > 0$ such that $x^2 + xy + y^2 = 3$. Prove that

$$\sqrt{3x^2 + 1} + \sqrt{3xy + 2} + \sqrt{3y^2 + 3} \leq 3\sqrt{5}$$

Proposed by Felicia Ozun-Romania

J.3234 a. Find the general term of the sequence $(x_n)_{n\geq 1}$ defined through $5x_{n+1} = x_n + 10$,

$\forall n \in \mathbb{N}$and $x_1 = 1$.

b. Find $\lim\limits_{n\to\infty} x_n$

Proposed by Cezar Ozun-Romania

J.3225 Prove that if $x_1, x_2, \dots, x_n \in \mathbb{R}_+^*, n, k \in \mathbb{N}, n \geq 1$ such that $x_1 \cdot x_2 \dots x_n \geq 1$, then

$$x_1^{3k} + x_2^{3k} + \dots + x_n^{3k} \leq x_1^{4k} + x_2^{4k} + \dots + x_n^{4k}$$

Proposed by Ovidiu Gabriel Dinu-Romania

J.3226 Let $\varphi(m)$ be the Euler exponent of the natural number m, that is, the number of prime numbers with m and less than m.Show that there exists an infinity of natural numbers n such that $\varphi(m) = \frac{2\cdot n}{5}$

Proposed by Simona Chiriță, Gilena Dobrică –Romania

J.3227 Solve the equation in the set of real numbers:

$$\sqrt[3]{x^3 + 1} + \sqrt{x - 1} + \sqrt{x - 2} > 5 - x$$

Proposed by Eugenia Turcu, Carmen Terheci –Romania

J.3228 Solve in $\mathbb{R}$ the equation $\log_7(6^x + 1) = \log_6(7^x - 1)$

Proposed by Cezar Ozun, Gabriel Tica-Romania

J.3229 Find $x \in \mathbb{Z}$ such that $x^2 + 8x + 3$ to be the square of an integer.

Proposed by Gigi Zaharia-Romania

J.3230 In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{s_a^4}{s_b^2 + s_c(s_a + s_b)}} + \sqrt[3]{\frac{s_b^4}{s_c^2 + s_a(s_b + s_c)}} + \sqrt[3]{\frac{s_c^4}{s_a^2 + s_b(s_c + s_a)}} \geq 3\sqrt[3]{3r^2}$$

Proposed by Marin Chirciu – Romania

J.3231 In ΔABC the following relationship holds:

$$\sum \sqrt[3]{\frac{\cot^4\frac{A}{2}}{\cot^2\frac{B}{2}+\cot\frac{C}{2}\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)}} \ge \sqrt[3]{3}$$

Proposed by Marin Chirciu – Romania

J.3232 In ΔABC the following relationship holds:

$$\sum \sqrt[3]{\frac{\sin^4\frac{A}{2}}{\sin^2\frac{B}{2}+\sin\frac{C}{2}\left(\sin\frac{A}{2}+\sin\frac{B}{2}\right)}} \ge \left(1+\frac{r}{R}\right)^{\frac{2}{3}}$$

Proposed by Marin Chirciu – Romania

J.3232 In ΔABC the following relationship holds:

$$\sum \sqrt[3]{\frac{\cos^4\frac{A}{2}}{\cos^2\frac{B}{2}+\cos\frac{C}{2}\left(\cos\frac{A}{2}+\cos\frac{B}{2}\right)}} \ge 3\left(\frac{1}{4}+\frac{r}{2R}\right)^{\frac{2}{3}}$$

Proposed by Marin Chirciu – Romania

J.3233 In ΔABC the following relationship holds:

$$\sqrt{\frac{m_a^3}{m_a+\lambda m_b}}+\sqrt{\frac{m_b^3}{m_b+\lambda m_c}}+\sqrt{\frac{m_c^3}{m_c+\lambda m_a}} \ge \frac{9r}{\sqrt{\lambda+1}},\ \lambda \ge 0$$

Proposed by Marin Chirciu – Romania

J.3234 In ΔABC the following relationship holds:

$$\sqrt{\frac{w_a^3}{w_a+\lambda w_b}}+\sqrt{\frac{w_b^3}{w_b+\lambda w_c}}+\sqrt{\frac{w_{c63}}{w_c+\lambda w_a}} \ge \frac{9r}{\sqrt{\lambda+1}}, \lambda \ge 0$$

Proposed by Marin Chirciu – Romania

J.3235 In ΔABC the following relationship holds:

$$\sqrt{\frac{s_a^3}{s_a+\lambda s_b}}+\sqrt{\frac{s_b^3}{s_b+\lambda s_c}}+\sqrt{\frac{s_c^3}{s_c+\lambda s_a}} \ge \frac{9r}{\sqrt{\lambda+1}},\ \lambda \ge 0$$

Proposed by Marin Chirciu – Romania

J.3236 Prove that for any point M on the incircle in ΔABC the following relationship holds:

$$\frac{MA^2}{r_a}+\frac{MB^2}{r_b}+\frac{MC^2}{r_c} \ge \frac{MA^2}{h_a}+\frac{MB^2}{h_b}+\frac{MC^2}{h_c}$$

Proposed by Marin Chirciu – Romania

J.3237 In ΔABC the following relationship holds:

$$\sqrt{\frac{\cos^3\frac{A}{2}}{\cos\frac{A}{2}+\lambda\cos\frac{B}{2}}}+\sqrt{\frac{\cos^3\frac{B}{2}}{\cos\frac{B}{2}+\lambda\cos\frac{C}{2}}}+\sqrt{\frac{\cos^3\frac{C}{2}}{\cos\frac{C}{2}+\lambda\cos\frac{A}{2}}}\geq\frac{3\sqrt{3}}{\sqrt{\lambda+1}}\left(\frac{1}{4}+\frac{r}{2R}\right),\lambda\geq 0$$

Proposed by Marin Chirciu – Romania

J.3238 In ΔABC the following relationship holds:

$$\sqrt{\frac{\sec^3\frac{A}{2}}{\sec\frac{A}{2}+\lambda\sec\frac{B}{2}}}+\sqrt{\frac{\sec^3\frac{B}{2}}{\sec\frac{B}{2}+\lambda\sec\frac{C}{2}}}+\sqrt{\frac{\sec^3\frac{C}{2}}{\sec\frac{C}{2}+\lambda\sec\frac{A}{2}}}\geq\frac{2\sqrt{3}}{\sqrt{\lambda+1}},\lambda\geq 0$$

Proposed by Marin Chirciu – Romania

J.3239 In ΔABC the following relationship holds:

$$\frac{2p}{R}\leq\frac{a}{AI}+\frac{b}{BI}+\frac{c}{CI}\leq\frac{3R}{r}\sqrt{1-\frac{r}{2R}}$$

Proposed by Marin Chirciu – Romania

J.3240 In ΔABC the following relationship holds:

$$\frac{4p}{R}\leq\frac{b+c}{AI}+\frac{c+a}{BI}+\frac{a+b}{CI}\leq\frac{2p}{r}$$

Proposed by Marin Chirciu – Romania

J.3241 In ΔABC the following relationship holds:

$$\frac{2p}{R}\leq\frac{b+c-a}{AI}+\frac{c+a-b}{BI}+\frac{a+b-c}{CI}\leq\frac{p}{r}$$

Proposed by Marin Chirciu – Romania

J.3242 In ΔABC the following relationship holds:

$$\sqrt{\frac{\cot^3\frac{A}{2}}{\cot\frac{A}{2}+\lambda\cot\frac{B}{2}}}+\sqrt{\frac{\cot^3\frac{B}{2}}{\cot\frac{B}{2}+\lambda\cot\frac{C}{2}}}+\sqrt{\frac{\cot^3\frac{C}{2}}{\cot\frac{C}{2}+\lambda\cot\frac{A}{2}}}\geq\frac{3\sqrt{3}}{\sqrt{\lambda+1}},\ \lambda\geq 0$$

Proposed by Marin Chirciu – Romania

J.3243 In ΔABC the following relationship holds:

$$\sum\tan^2\frac{A}{2}\tan^2\frac{B}{2}\left(\sqrt[3]{\frac{p}{r}}\tan\frac{A}{2}+\sqrt[3]{\frac{p}{r}}\tan\frac{B}{2}-2\right)\geq 0$$

Proposed by Marin Chirciu – Romania

J.3244 In ΔABC the following relationship holds:

$$\sum (m_a \sin A)^2 \leq \left(\frac{9R}{4}\right)^2$$

Proposed by Marin Chirciu – Romania

J.3245 In ΔABC the following relationship holds:

$$\sum \cot^2\frac{A}{2}\cot^2\frac{B}{2}\left(\sqrt[3]{\frac{r}{p}}\cot\frac{A}{2}+\sqrt[3]{\frac{r}{p}}\cot\frac{B}{2}-2\right) \geq 0$$

Proposed by Marin Chirciu – Romania

J.3246 In ΔABC the following relationship holds:

$$\sum (s_a \sin A)^2 \leq \left(\frac{9R}{4}\right)^2$$

Proposed by Marin Chirciu – Romania

J.3247 If $a_1, a_2, \dots, a_n > 0$ such that $a_1 + a_2 + \dots + a_n = 1 + \frac{1}{2^n}$ then:

$$\frac{a_1}{2a_1+1}+\frac{a_2}{4a_2+1}+\dots+\frac{a_n}{2^n a_n+1} \leq \frac{1}{2}$$

Proposed by Marin Chirciu – Romania

J.3248 If $a_k > 0\ \ \forall k = \overline{1,n}$, then prove that:

$$\frac{\prod_{k=1}^{n}(a_k^n + n - 1)}{\left(\sum_{k=1}^{n} a_k\right)^n} \geq 1$$

Proposed by Neculai Stanciu – Romania

J.3249 If $x, y, z \in \mathbb{R}$, then prove that:

$$8(x^2+y^2+z^2) - 4(x+y+z) + 3 \geq 4(xy+yz+zx)$$

Proposed by Neculai Stanciu – Romania

J.3250 If $x_k > 0 (k = 1,2,\dots,n)$, then prove that:

$$\sum \frac{x_1^2 + x_1x_2 + x_2^2}{(x_1+x_2)(x_1^2+x_2^2)} \leq \frac{3}{4}\sum_{k=1}^{n}\frac{1}{x_k}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3251 If $a_k > 0$ $(k = 1,2,\dots,n)$ then prove that:

$$16\sum\frac{a_1a_2(a_1+a_2)}{(3a_1+a_2)(a_1+3a_2)}+\sum\frac{a_1^2+a_2^2}{a_1+a_2}\le 3\sum_{k=1}^{n}a_k$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3252 If $ABCDEFGHIJKLM$ is a regular 13 – gon, then prove that: $AE(AC-AB)=AF(AD-AC)$

Proposed by Neculai Stanciu – Romania

J.3253 Prove that any triangle ABC with usual notations is true the following:

$$\sum a^3(b^2+c^2)\ge(12Rr)^2s$$

Proposed by Neculai Stanciu – Romania

J.3254 Prove that the sum:

$$(n^2+2n+1)^3+(n^2+8n+16)^3+(9n^2+42n+49)^3+(9n^2+48n+64)^3$$

is divisible by $2n^2+10n+13$ for all $n\in\mathbb{N}$.

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3255 If $a,b,c>0$, then prove that:

$$\sum\frac{6a^3}{\sqrt{2a(a+b)^3}+\sqrt{2b^2(a^2+b^2)}}-\sum a\ge 0$$

Proposed by Neculai Stanciu – Romania

J.3256 If $ABCDEFGIJK$ is a regular 11 – gon, then prove that:

$$AE^2-AD^2=AB\cdot AE\cdot\left(\frac{AD}{AB}-\frac{AE}{AC}\right)$$

Proposed by Neculai Stanciu – Romania

J.3257 Let ABC be a triangle, the altitude h_a interscts the circumcircle in point E, the bisector of angle $\angle EAC$ intersects the circumcircle in point D. The tangent line to circle in point D intersects the lines AE and BC in points M and N.

1) Determine all triangles ABC for which $BHCE$ is rhombus.

2) Determine all triangle ABC for which $ABMN$ is cyclic.

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3258 Let ABC be a triangle, and J denote the midpoint of GH. Prove that the center of Euler circle lies on the incircle if and only if $IJ=\frac{\sqrt{2}}{3}IH$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3259 If $a_k > 0$ $(k = 1,2, \dots, n)$, then prove that:

$$\left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} a_k^{n-1}\right) - \sum_{k=1}^{n} a_k^n - n(n-1)\prod_{k=1}^{n} a_k \geq 0$$

Proposed by Neculai Stanciu – Romania

J.3260 If $a, b, c > 0$, then prove that:

$$6\left(\sum a\right)\left(\sum ab\right) - \left(\sum a\right)\left(\sum a^2\right) - \left(\sum a\right)^3 \leq 18abc$$

Proposed by Neculai Stanciu – Romania

J.3261 If $a_k > 0(k = 1,2, \dots, n)$ and $s = \sum_{k=1}^{n} a_k$, then:

$$\left(\frac{n-1}{n}\right)^2 \sum_{k=1}^{n} a_k \cdot \sum_{k=1}^{n} \frac{a_k}{(s - a_k)^2} \geq 1$$

Proposed by Neculai Stanciu – Romania

J.3262 If $x_k > 0$ $(k = 1,2, \dots, n)$, then:

$$\frac{1}{n}\sum_{cyclic} \sqrt{1 + \frac{6}{x_1 + x_2 + x_3} + \frac{3}{x_1x_2 + x_2x_3 + x_3x_1}} - \frac{n}{\sum_{k=1}^{n} x_k} \geq 1$$

Proposed by Neculai Stanciu – Romania

J.3263 If $x_k > 0(k = 1,2, \dots, n)$, then prove that:

$$\sum_{cyclic} \sqrt{\frac{1}{3} \cdot (x_1^4 + x_1^2x_2^2 + x_2^4)} \geq \sum_{k=1}^{n} x_k^2 \geq \sum_{cyclic} x_1 \sqrt{\frac{1}{3} \cdot (2x_1^2 + x_2x_3)}$$

Proposed by Neculai Stanciu – Romania

J.3264 If $a_k > 0(k = 1,2, \dots, n)$ and $s = \sum_{k=1}^{n} a_k$, then:

$$\sum_{k=1}^{n} a_k^3 \geq \frac{1}{n(n_1)}\left(\sum_{k=1}^{n} a_k \sqrt{s - a_k}\right)^2$$

Proposed by Neculai Stanciu – Romania

J.3265 If $a_k > 0$ $(k = 1,2, \dots, n)$, then prove that:

$$\sum_{cyclic} \left(\frac{a_1}{\sqrt{a_2^2 + (n^2 - 2)a_2a_3 + a_3^2}} + \frac{a_2}{\sqrt{a_3^2 + (n^2 - 2)a_3a_1 + a_1^2}} + \frac{a_3}{\sqrt{a_1^2 + (n^2 - 2)a_1a_2 + a_2^2}} \right) \geq 3$$

Proposed by Neculai Stanciu – Romania

J.3266 If $a, b, c > 0$, then prove that:

$$\sum \frac{a}{\sqrt{a^2 + 3ab + 3b^2 + 2bc}} \geq 1$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

J.3267 If $a, b, c > 0$ such that $a^n + b^n + c^n = 3, n \in \mathbb{N}, n \in \mathbb{N}$ and $\lambda \geq 0$ then:

$$\frac{1}{\lambda + (a+b)^n} + \frac{1}{\lambda + (b+c)^n} + \frac{1}{\lambda + (c+a)^n} \geq \frac{3}{\lambda + 2^n}$$

Proposed by Marin Chirciu – Romania

J.3268 If $a, b, c > 0$ such that $abc = 1$ and $n \in \mathbb{N}, \lambda \geq 2$ then:

$$\frac{1}{a^n + \lambda} + \frac{1}{b^n + \lambda} + \frac{1}{c^n + \lambda} \leq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

J.3269 Let $-2 \leq \lambda \leq 2$ fixed. If $a, b, c > 0, abc = a + b + c$ then find the maximum of expression:

$$A = \lambda(ab + bc + ca) - a^2 - b^2 - c^2$$

Proposed by Marin Chirciu – Romania

J.3270 If a, b, c are the dimensions a cuboid with the diagonal d and $n \in \mathbb{N}^*$ then:

$$d \leq \sqrt[2n]{\frac{3^{n-1}a^{2n+1}}{b} + \frac{3^{n-1}\, b^{2n+1}}{c} + \frac{3^{n-1}c^{2n+1}}{a}}$$

Proposed by Marin Chirciu – Romania

J.3271 If $a, b, c \geq 0, abc = 1$ and $\lambda \geq 0$ then:

$$\frac{a^4}{a + \lambda bc} + \frac{b^4}{b + \lambda ca} + \frac{c^4}{c + \lambda ab} \geq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

J.3272 If $a, b, c > 0$ such that $a^2 + b^2 + c^2 = 3$ and $n \geq 0, \lambda \geq 0$ then:

$$n(a+b)(b+c)(c+a) + \lambda abc \leq 8n + \lambda$$

Proposed by Marin Chirciu – Romania

J.3273 If $a, b, c \geq 0, abc = 1$ and $\lambda \geq 0, n \in \mathbb{N}, n \geq 3$ then:

$$\frac{a^n}{a + \lambda bc} + \frac{b^n}{b + \lambda ca} + \frac{c^n}{c + \lambda ab} \geq \frac{3}{\lambda + 1}$$

Proposed by Marin Chirciu – Romania

J.3274 In ΔABC the following relationship holds:

$$\frac{18r^2}{R} \le \frac{w_a^2}{m_a} + \frac{w_b^2}{m_b} + \frac{w_c^2}{m_c} \le 4R + r$$

Proposed by Marin Chirciu – Romania

J.3275 If $a, b, c > 2$ such that $\frac{1}{a-1} + \frac{1}{b-1} + \frac{1}{c-1} = 1$, then find the minimum of expression

$P = (a+\lambda)(b+\lambda)(c+\lambda)$, where $\lambda \ge 0$.

Proposed by Marin Chirciu – Romania

J.3276 In ΔABC the following relationship holds:

$$3 \le \sum \frac{s_a + s_b}{s_b + s_c} \le \frac{7R}{4r} - \frac{1}{2}$$

Proposed by Marin Chirciu – Romania

J.3277 In ΔABC the following relationship holds:

$$\sum s_a^2 s_b \sum s_a s_b^2 \ge \frac{1}{3}\left(\frac{2rs^2}{R}\right)^3$$

Proposed by Marin Chirciu – Romania

J.3278 In ΔABC the following relationship holds:

$$\sum \frac{s_a^4 + s_b^2 s_c^2}{s_b^2 + s_c^2} \ge \frac{2rs^2}{R}$$

Proposed by Marin Chirciu – Romania

J.3279 In ΔABC the following relationship holds:

$$\sum \frac{w_a^4 + w_b^2 w_c^2}{w_b^2 + w_c^2} \ge 3\sqrt{3}S$$

Proposed by Marin Chirciu – Romania

J.3280 In ΔABC the following relationship holds:

$$\sum \frac{s_a^4 + s_b^2 s_c^2}{s_b^2 + s_c^2} \ge \frac{2rs^2}{R}$$

Proposed by Marin Chirciu – Romania

J.3281 In ΔABC the following relationship holds:

$$\sum \frac{s_a^2}{s_a^4 s_b^2 + s_c^2} \le \frac{9R^2}{8F^2}$$

Proposed by Marin Chirciu – Romania

J.3282 In ΔABC the following relationship holds:

$$\frac{18r^2}{R} \le \frac{s_a^2}{m_a} + \frac{s_b^2}{m_b} + \frac{s_c^2}{m_c} \le 4R + r$$

Proposed by Marin Chirciu – Romania

J.3283 In ΔABC the following relationship holds:

$$\frac{a^2 + \lambda b^2}{b^2 + c^2} + \frac{b^2 + \lambda c^2}{c^2 + a^2} + \frac{c^2 + \lambda a^2}{a^2 + b^2} \le \frac{3}{2}(\lambda + 1)\left(\frac{R}{2r}\right)^3, \lambda \ge \frac{1}{2}$$

Proposed by Marin Chirciu – Romania

J.3284 In ΔABC the following relationship holds:

$$\left(\frac{3}{2p}\right)^2 \le \frac{1}{(s_a + s_b)^2} + \frac{1}{(s_b + s_c)^2} + \frac{1}{(s_c + s_a)^2} \le \left(\frac{3R}{4S}\right)^2$$

Proposed by Marin Chirciu – Romania

J.3285 In ΔABC the following relationship holds:

$$\left(\frac{3r}{2R}\right)^2 \le \frac{1}{\left(\sec\frac{A}{2} + \sec\frac{B}{2}\right)^2} + \frac{1}{\left(\sec\frac{B}{2} + \sec\frac{C}{2}\right)^2} + \frac{1}{\left(\sec\frac{C}{2} + \sec\frac{A}{2}\right)^2} \le \frac{1}{4}\left(2 + \frac{r}{2R}\right)$$

Proposed by Marin Chirciu – Romania

J.3286 In ΔABC the following relationship holds:

$$\sum (h_a \sin A)^2 \le \frac{3}{4}p^2$$

Proposed by Marin Chirciu – Romania

J.3287 In ΔABC the following relationship holds:

$$\sum (r_a \sin A)^2 \ge \frac{3S^2}{R^2}$$

Proposed by Marin Chirciu – Romania

J.3288 In ΔABC the following relationship holds:

$$p^2 + 9r^2 \le (AI + BI + CI)^2 \le p^2 + r^2 + 4Rr$$

Proposed by Marin Chirciu – Romania

J.3289 In ΔABC the following relationship holds:

$$\frac{m_a^2+\lambda m_a h_a+h_a^2}{m_a+h_a}\geq\frac{\lambda+2}{2}h_a,\ \ -2\leq\lambda\leq 2$$

Proposed by Marin Chirciu – Romania

J.3290 In ΔABC the following relationship holds:

$$\frac{w_a^2+\lambda w_a h_a+h_a^2}{w_a+h_a}\geq\frac{\lambda+2}{2}h_a, -2\leq\lambda\leq 2$$

Proposed by Marin Chirciu – Romania

J.3291 In ΔABC the following relationship holds:

$$\frac{m_a^2+\lambda m_a w_a+w_a^2}{m_a+w_a}\geq\frac{\lambda+2}{2}w_a, \text{ where } -2\leq\lambda\leq 2$$

Proposed by Marin Chirciu – Romania

J.3292 In ΔABC the following relationship holds:

$$\frac{s_a^2+\lambda s_a h_a+h_a^2}{s_a+h_a}\geq\frac{\lambda+2}{2}h_a,\ \ -2\leq\lambda\leq 2$$

Proposed by Marin Chirciu – Romania

J.3293 In $\Delta ABC, I$ – incentre, R_a, R_b, R_c – circumradii in $\Delta BIC, \Delta CIA, \Delta AIB$, then:

$$\sqrt{6}\leq\sqrt{\frac{R_a}{r_a}}+\sqrt{\frac{R_b}{r_b}}+\sqrt{\frac{R_c}{r_c}}\leq\sqrt{\frac{3R}{r}}$$

Proposed by Marin Chirciu – Romania

J.3294 In ΔABC the following relationship holds:

$$\frac{a^2}{h_b^n}+\frac{b^2}{h_c^n}+\frac{c^2}{h_a^n}\geq\frac{4}{(3r)^{n-2}},\ \ n\geq 2$$

Proposed by Marin Chirciu – Romania

J.3295 In ΔABC the following relationship holds:

$$\frac{a^2}{r_b^n}+\frac{b^2}{r_c^n}+\frac{c^2}{r_a^n}\geq\frac{4}{(3r)^{n-2}},\ \ n\geq 2$$

Proposed by Marin Chirciu – Romania

J.3296 In ΔABC the following relationship holds:

$$\frac{a^2}{m_b^n}+\frac{b^2}{m_c^n}+\frac{c^2}{m_a^n}\geq 36r^2\left(\frac{2}{3R}\right)^n, n\geq 2$$

Proposed by Marin Chirciu – Romania

J.3297 In ΔABC the following relationship holds:

$$9r \le \sum \frac{m_a^2 + m_a m_b + m_b m_c + m_c m_a}{2m_a + m_b + m_c} \le 4R + r$$

Proposed by Marin Chirciu – Romania

J.3298 In ΔABC the following relationship holds:

$$9r \le \sum \frac{w_a^2 + w_a w_b + w_b w_c + w_c w_a}{2w_a + w_b + w_c} \le 4R + r$$

Proposed by Marin Chirciu – Romania

J.3299 In ΔABC the following relationship holds:

$$\frac{27Rp}{2(4R+r)^2} \le \sum \frac{\tan^2\frac{A}{2} + 1}{2\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2}} \le \frac{4R+r}{p}$$

Proposed by Marin Chirciu – Romania

J.3300 In ΔABC the following relationship holds:

$$\frac{9p}{4R+r} \le \sum \frac{\cot^2\frac{A}{2} + 1 + \frac{4R}{r}}{2\cot\frac{A}{2} + \cot\frac{B}{2} + \cot\frac{C}{2}} \le \frac{p}{r}$$

Proposed by Marin Chirciu – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

PROBLEMS FOR SENIORS

S.3159 In any acute triangle ABC holds:

$$a^2\sqrt{\cos A}+b^2\sqrt{\cos B}+c^2\sqrt{\cos C}\le\frac{9\sqrt{2}}{2}R^2$$

Proposed by Vasile Mircea Popa-Romania

S.3160 Let be $m\ge 0, M$ an interior point in triangle ABC with the area F and $F_a=$ area MBC, $F_b=$ area $MCA, F_c=$ area MAB, then:

$$\frac{a^{2m+2}}{F_b^m}+\frac{b^{2m+2}}{F_c^m}+\frac{c^{2m+2}}{F_a}\ge 4^{m+1}\left(\sqrt{3}\right)^{m+1}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

S.3161 Let be $x,y>0$ and M an interior point in triangle ABC with the area F and d_a,d_b,d_c the distances of point M to the sides BC,CA respectively AB Prove that:

$$\frac{a^4b^3}{xd_b+yh_b}+\frac{b^4c^3}{xd_c+yh_c}+\frac{c^4a^3}{xd_a+yh_a}\ge\frac{128}{x+3y}\cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

S.3162 Let be $x,y>0, x\ge y$ and M an interior point in ΔABC with the area F and $F_a=$ area MBC, and F_b,F_c tha analogs. Prove that:

$$\frac{a^2b^2}{xF-yF_c}+\frac{b^2c^2}{xF-yF_a}+\frac{c^2a^2}{xF-yF_b}\ge\frac{48}{3x-y}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.3163 In any ΔABC with the area F the following inequality holds:

$$\sqrt{a^4+b^4}+\sqrt{b^4+c^4}+\sqrt{c^4+a^4}\ge 4\sqrt{6}F$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.3164 In any ΔABC with the area F the following inequality holds:

$$\sqrt{a^8+b^8}+\sqrt{b^8+c^8}+\sqrt{c^8+a^8}\geq 16\cdot\sqrt{2}\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

S.3165 Let be $m\geq 0$ and M an interior point in ΔABC with the area F and F_a =area MBC,

F_b = area MCA, F_c = area MAB. Prove that:

$$\frac{x^{m+1}a^{2m+2}}{(y+z)^{m+1}F_b^m}+\frac{y^{m+1}b^{2m+2}}{(z+x)^{m+1}F_c^m}+\frac{z^{m+1}c^{2m+2}}{(x+y)^{m+1}F_a^m}\geq 2^{m+1}\left(\sqrt{3}\right)^{m+1}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

S.3166 Let be $x,y,z>0, m\geq 0$ and ABC a triangle with the area F then:

$$\frac{x^{m+1}\cdot a^{2m}}{(y+z)^{m+1}h_a^2}+\frac{y^{m+1}\cdot b^{2m}}{(z+x)^{m+1}h_b^2}+\frac{z^{m+1}\cdot c^{2m}}{(x+y)^{m+1}\cdot h_c^2}\geq 2^{m-1}\left(\sqrt{3}\right)^{1-m}\cdot F^{m-1}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

S.3167 Let be M the interior bisector's leg of the angle $\widehat{BAC}$ of triangle ABC with the area F and d_b, d_c the distances of point M to the sides AC respectively AB. Prove that:

$$\frac{a^2\cdot b}{d_b}+\frac{b^2\cdot c}{d_c}+\frac{c^2\cdot a}{h_a}\geq 12\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

S.3168 In any ΔABC the following inequality holds:

$$\frac{a}{h_a}+\frac{b}{h_b}+\frac{c}{h_c}\geq 2\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3169 With the usual notations in ΔABC and $x,y\geq 0, x+y=4$ the following inequality holds:

$$\left(\frac{1}{h_a^x}+\frac{1}{h_b^y}+\frac{1}{h_c^x}\right)\cdot\left(\frac{1}{h_a^y}+\frac{1}{h_b^x}+\frac{1}{h_c^y}\right)\geq 12$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3170 If $x>0$ then in any triangle ABC with the area F the following inequality holds:

$$3x^2+a^4+b^4+c^4\geq 8x\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3171 In any triangle ABC with the semiperimer s and area F the following inequality holds:

$$2s + ab^2 + bc^2 + ca^2 \geq 8\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3172 Let M be an interior point in ΔABC with the area F and $F_a =$ area $MBC, F_b =$ area MCA,

$F_c =$ area MAB, then:

$$\frac{(ab + bc + 2F_a)^2}{ca + F_b} + \frac{(bc + ca + 2F_b)^2}{ab + F_c} + \frac{(ca + ab + 2F_c)^2}{bc + F_a} \geq 4\left(4\sqrt{3} + 1\right)F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3173 If $x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{xa}{y+z} + \frac{xa^3}{y+z} + 2 \cdot \frac{yb^2}{z+x} + \frac{zc^4}{x+y} + \frac{z}{x+y} \geq 4\sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3174 In any ΔABC with the area F the following inequality holds:

$$\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} \geq \frac{\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3175 In any tetrahedron having the sum of the squares of the averages m and the total area F the following inequality holds:

$$m \geq 2\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3176 In any ΔABC with the area F the following inequality holds:

$$a^2b + b^2c + c^2a \geq 2F \cdot \sqrt{\frac{bc}{\sin^2\frac{C}{2}} + \frac{ca}{\sin^2\frac{A}{2}} + \frac{ab}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

S.3177 Let be $m \geq 0, x, y, z > 0$ and M an interior point in triangle ABC with the area F and d_a, d_b, d_c the distances of point M to the sides BC, CA respectively AB. Prove that:

$$\frac{x^{m+1}\cdot a^{m+2}}{(h_a+d_a)^m}+\frac{y^{m+1}\cdot b^{m+2}}{(h_b+d_b)^m}+\frac{z^{m+1}\cdot c^{m+2}}{(h_c+d_c)^m}\geq\frac{F}{2^{2m-1}}\cdot\left(\sum_{cyc}\frac{xy}{\sin^2\frac{C}{2}}\right)^{\frac{m+1}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze-Romania

S.3178 Let be $m\geq 0, x, y, z>0$ and ABC a triangle with the area F, then:

$$\frac{x^{m+1}\cdot a^{m+2}}{h_a^m}+\frac{y^{m+1}\cdot b^{m+2}}{h_b^m}+\frac{z^{m+1}\cdot c^{m+2}}{h_c^m}\geq\frac{2F}{3^m}\left(\sum_{cyc}\frac{xy}{\sin^2\frac{C}{2}}\right)^{\frac{m+1}{2}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu-Romania

S.3179 If $x, y, z>0$ and ABC is a triangle with the area F, and the other notations are the usual ones, then:

$$\frac{xa}{h_a}+\frac{yb}{h_b}+\frac{zc}{h_c}\geq\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.3180 Let be $s, t\geq 0, s+t=4, x, y, z>0$ and ABC is a triangle with the area F, then:

$$((xa^2)^s+(yb^2)^t+(zc^2)^s)((xa^2)^t+(yb^2)^s+(zc^2)^t)\geq\frac{16}{81}\left(\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}\right)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.3181 If ABC is a triangle with the area F and the other notations are the usual ones, and

$x, y, z>0$, then:

$$\frac{x}{h_a^2}+\frac{y}{h_b^2}+\frac{z}{h_c^2}\geq\frac{1}{2}\cdot\sqrt{\frac{xy}{\sin^2\frac{C}{2}}+\frac{yz}{\sin^2\frac{A}{2}}+\frac{zx}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

S.3182 If $t, u, v, x, y, z\geq 0$ and $t+x, u+y, v+z=4$ then in ΔABC with the area F the following inequality holds: $a^t+b^u+c^v+a^x+b^y+c^z\geq 8\sqrt{3}\cdot F$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3183 In any ΔABC with the semiperimeter s and the area F the following inequality holds:

$$2s+a^3b^4+b^3c^4+c^3a^4\geq 32\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3184 In any triangle ABC with the semiperimeter s and the area F the following inequality holds:

$$2s + a^3 + b^3 + c^3 \geq 8 \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3185 In any ΔABC with the area F the following inequality holds:

$$a^4 + b^4 + c^4 + 3 \geq 8 \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3186 In any triangle ABC with the semiperimeter s and the area F the following inequality holds:

$$2s + a^7 + b^7 + c^7 \geq 32 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3187 In any ΔABC with the area F the following inequality holds:

$$a^2b^2 + b^2c^2 + c^2a^2 + 3 \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3188 If in triangle ABC c_a, c_b, c_c three concurrent cevians, then:

$$\sqrt{\frac{c_a(c_b + c_c)}{h_a^2 + h_b h_c}} + \sqrt{\frac{c_b(c_c + c_a)}{h_b^2 + h_c h_a}} + \sqrt{\frac{c_c(c_a + c_b)}{h_c^2 + h_a h_b}} \geq \frac{6r}{R}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3189 If $x, y \geq 0, x + y = 3$ and ABC is a triangle with the area F and semiperimeter s, then:

$$2s + a^x b^y + b^x c^y + c^x a^y \geq 8 \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3190 In any ΔABC with the area F and the semiperimeter s the following inequality holds:

$$2s + a^3 + b^3 + c^3 \geq 8\sqrt{3}F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3191 In any ΔABC with the area F and the semiperimeter s the following inequality holds:

$$2s + ab^2 + bc^2 + ca^2 \geq 8 \cdot \sqrt{3} \cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

S.3192 Let ABC be an acute triangle and I be the center of the incircle . X,Y,Z are outside the triangle ABC and $X \in CI$, $Y \in AI$, $Z \in BI$, $m(\widehat{XAC}) = m(\widehat{YBA}) = m(\widehat{BCZ}) = 90^\circ$

Prove that:

a)$\text{Area}(XYZ) = \frac{r}{2}\left[\frac{b(s-b)}{s-c} + \frac{c(s-c)}{s-a} + \frac{a(s-a)}{s-b}\right]$ b)$\text{Area}(XYZ) \geq \text{Area}(ABC)$

where r and s are the inradius and semiperimeter of the triangle ABC respectively.

Proposed by Mehmet Șahin-Turkiye

S.3193 If x, y, z are positive real numbers then in ΔABC the following relationship holds:

$$\frac{x+y}{z}\left(\frac{a}{b+c}\right)^2 + \frac{y+z}{x}\left(\frac{b}{c+a}\right)^2 + \frac{z+x}{y}\left(\frac{c}{a+b}\right)^2 \geq \frac{27(s^2+r^2-2Rr)}{16s^2}$$

Equality holds if and only if $x = y = z$ and ΔABC is equilateral.

Proposed by Mehmet Șahin-Turkiye

S.3194 In acute ΔABC the following relationship holds:

$$\rho_H \leq R - \frac{3r}{2}$$

where $\rho_H, R\ and\ r$ are inradius of orthic triangle,circumradius of ΔABC and inradius of ΔABC.

Proposed by Mehmet Șahin-Turkiye

S.3195 The points D, E, F are taken on the sides [BC], [CA], [AB] of the triangle ABC respectively,such that |BD| = [CE] = [AF]. Show that the smallest value of the area of the ΔDEF is

$$\frac{4(a+b+c)abc - (ab+bc+ca)^2}{32Rs}$$

Proposed by Mehmet Șahin-Turkiye

S.3196 Let ABC be a triangle and $[AD]$ be its interior bisector . Let the of the inscribed circles of triangles ABD and ACD be r_1 and r_2 ,respectively. Prove that

$$r_1 < r_2 \Leftrightarrow c < b$$

Proposed by Mehmet Șahin-Turkiye

S.3197 If x, y, z are positive real numbers then in ΔABC the following relationship holds:

$$\frac{x}{y+z}{m_a}^2 + \frac{y}{z+x}{m_b}^2 + \frac{z}{x+y}{m_c}^2 \geq \frac{16Fs - 27R^3}{8R}$$

Equality holds if and only if $x = y = z$ and ΔABC is equilateral.

Proposed by Mehmet Șahin-Turkiye

S.3198 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$3\sum_{cyc}(h_b+h_c)(p_a-w_a)\geq 4s(h_a+h_b+h_c-9r).$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.3199 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$p_a+p_b+p_c\geq w_a+w_b+w_c+\frac{16}{15}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

S.3200 In $\Delta ABC, M\in Int(ABC)$ the following relationship holds:

$$b\cdot AM+c\cdot BM+a\cdot CM\geq\sqrt{a^2b^2+b^2c^2+c^2a^2}$$

Proposed by Bogdan Fuștei-Romania

S.3201 In $\Delta ABC, I$ —incenter, the following relationship holds:

$$\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}\leq\frac{s\sqrt{3}-AI-BI-CI}{r}$$

Proposed by Bogdan Fuștei-Romania

S.3202 In $\Delta ABC, x, y, z>0$ the following relationship holds:

$$x^2+y^2+z^2\geq\sqrt{2}\cdot\sum_{cyc}\frac{yz\cdot\cos\frac{A}{2}}{\sqrt{1+\sin\frac{A}{2}}}$$

Proposed by Bogdan Fuștei-Romania

S.3203 In ΔABC the following relationship holds:

$$\sqrt{2}\sum_{cyc}\frac{\sqrt{1+\sin\frac{A}{2}}}{\cos\frac{A}{2}}\geq 3(2-\sqrt{3})+\frac{2R}{s}\left(\sum_{cyc}\cos\frac{A}{2}\right)^2$$

Proposed by Bogdan Fuștei-Romania

S.3204 In $\Delta ABC, x, y, z>0$ the following relationship holds:

$$\sum_{cyc}\sqrt{\frac{x}{y+z}}\cdot\cos\frac{A}{2}\leq\sqrt{\frac{(x+y+z)^3}{(x+y)(y+z)(z+x)}}$$

Proposed by Bogdan Fuștei-Romania

S.3205 In non-obtuse ΔABC the following relationship holds:

$$m_a + m_b + m_c \geq 2R + 5r$$

Proposed by Bogdan Fuștei-Romania

S.3206 In $\Delta ABC, I$ —incenter, the following relationship holds:

$$\sum_{cyc} \frac{1}{\sin\frac{A}{4}} \geq \frac{AI + BI + CI + s}{r} + 3\left(\sqrt{2} + \sqrt{6} - \sqrt{3} - 2\right)$$

Proposed by Bogdan Fuștei-Romania

S.3207 In $\Delta ABC, M \in Int(ABC)$ the following relationship holds:

$$AM \cdot m_a + BM \cdot m_b + CM \cdot m_c \geq \frac{R^2}{2(2R^2 + r^2)}(m_a + m_b + m_c)^2$$

Proposed by Bogdan Fuștei-Romania

S.3208 Solve in $\mathbb{R}$ the following system:

$$\begin{cases} 11(x^2 + y^2) + 4xy + 9yz + zx = (3x + 2y + z)^2 \\ 11(y^2 + z^2) + 4yz + 9zx + xy = (3y + 2z + x)^2 \\ 11(z^2 + x^2) + 4z + 9xy + yz = (3z + 2x + y)^2 \end{cases}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.3209 Prove that:

$$\left(\prod_{cyclic} \frac{x_1^{2k+2} + x_2^{2k+2}}{x_1^{2k} + x_2^{2k}}\right) \Big/ \left(\prod_{i=1}^{n} x_i^2\right) \geq 1$$

where $x_i > 0 \ \left(i = \overline{1, n}\right)$ and $k \in \mathbb{N}$

Proposed by Neculai Stanciu – Romania

S.3210 Solve in $(0, \infty)$ the following system:

$$\begin{cases} \frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} = y + z + t + \frac{4(x-y)^2}{x+y+z} \\ \frac{y^2}{z} + \frac{z^2}{t} + \frac{t^2}{y} = z + t + x + \frac{4(y-z)^2}{y+z+t} \\ \frac{z^2}{t} + \frac{t^2}{x} + \frac{x^2}{z} = t + x + y + \frac{4(z-t)^2}{z+t+x} \\ \frac{t^2}{x} + \frac{x^2}{y} + \frac{y^2}{t} = x + y + z + \frac{4(t-x)^2}{t+x+y} \end{cases}$$

Proposed by Neculai Stanciu, Mihaly Bencze – Romania

S.3211 If $x_k > 0 (k = 1,2, \dots, 2023)$ and $\sum_{k=1}^{2023} x_k \geq 2023$, then prove that:

$$\sum_{k=1}^{2023} x_k^{2024} \geq 2023$$

Proposed by Neculai Stanciu – Romania

S.3212 Prove that:

$$((a^3 - 2a)^2 + (2a^2 - 1)^2) \cdot ((a^5 - 2a^3 + 2a)^2 + (2a^4 - 2a^2 + 1)^2) \cdot$$
$$\cdot ((a^7 - 2a^5 + 2a^3 - 2a)^2 + (2a^6 - 2a^4 + 2a^2 - 1)^2):$$
$$:\left((a^6 + 1)(a^{10} + 1)(a^{14} + 1)\right) = 1 \ \forall a \in \mathbb{C}$$

Proposed by Neculai Stanciu – Romania

S.3213 If a, b, c are positive real numbers such that $a + b + c = 3$, then prove that:

$$\frac{32}{3}\left(\frac{1}{a(b+c)^5} + \frac{1}{b(c+a)^5} + \frac{1}{c(a+b)^5}\right) \geq 1$$

Proposed by Neculai Stanciu – Romania

S.3214 Prove that:

$$\left(\sum \frac{x^{n+2} + y^{n+2}}{x^n + y^n}\right) \geq 6 \prod \sqrt[3]{\frac{x^2 y^2}{x^2 + y^2}}, \forall x, y, z > 0$$

Proposed by Neculai Stanciu – Romania

S.3215 If $a, b, c, d \in \mathbb{C}$, then compute:

$$\left(\sum((a+b)^3 + (a+c)^3 + (a+d)^3 + (a-b)^3 + (a-c)^3 + (a-d)^3)\right):\left(\left(\sum a\right)\left(\sum a^2\right)\right)$$

Proposed by Neculai Stanciu – Romania

S.3216 Prove that:

$$\left(((a^3 - 2a)^2 + (2a^2 - 1)^2)((a^6 - 2a^2)^2 + (2a^4 - 1)^2)((a^9 - 2a^3)^2 + (2a^6 - 1)^2)\right):$$
$$:\left((a^6 + 1)(a^{12} + 1)(a^{18} + 1)\right) = 1 \ \forall a \in \mathbb{C}$$

Proposed by Neculai Stanciu – Romania

S.3217 Prove that for all $n \geq 2$ exist $a_1, a_2, \dots, a_n \in \mathbb{N}$ such that:

$$\sum_{k=1}^{n} \frac{1}{a_k} = \frac{3}{2}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.3218 Prove that the following equations:

(a) $x_1^5+x_2^5+x_3^5+x_4^5+x_5^5=0$ and (b) $x_1^7+x_2^7+x_3^7+x_4^7+x_5^7+x_6^7+x_7^7+x_8^7=0$

has an infinitely solutions in $\mathbb{Z}$

Proposed by Neculai Stanciu – Romania

S.3219 If $a,b,c>0$, then prove that:

$$\sum\frac{a}{b}-\frac{\sum a^2}{\sum ab}\geq 2$$

Proposed by Neculai Stanciu – Romania

S.3220 If $x_i>0\ (i=1,2,\dots,n)$ and $k\in\mathbb{N}$, then prove that:

$$2\sum_{i=1}^{n}x_i^{2k+2}\geq\sum_{cyclic}x_1x_2\left(x_1^{2k}+x_2^{2k}\right)\geq\sum_{cyclic}x_1^2x_2^2\left(x_1^{2k-2}+x_2^{2k-2}\right)\geq\cdots\geq$$

$$\geq\sum_{cyclic}x_1^kx_2^k(x_1^2+x_2^2)\geq 2\sum_{cyclic}x_1^{k+1}x_2^{k+1}$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.3221 If $x_i>0\ (i=1,2,\dots,n)$ and $k\in\mathbb{N}^*$, then prove that:

$$2\sum_{i=1}^{n}x_i^{2k+1}\geq\sum_{cyclic}x_1x_2\left(x_1^{2k-1}+x_2^{2k-1}\right)\geq\sum_{cyclic}x_1^2x_2^2\left(x_1^{2k-3}+x_2^{2k-3}\right)\geq\cdots\geq$$

$$\geq\sum_{cyclic}x_1^kx_2^k(x_1+x_2)$$

Proposed by Neculai Stanciu – Romania

S.3222 If a,b,c are three positive real numbers, then prove that:

$$\sum_{cyclic}\frac{1}{2(2a+b+c)}\geq\sum_{cyclic}\frac{1}{3(a+b)+2c}$$

Proposed by Neculai Stanciu – Romania

S.3223 If $a,b,c>0$, then prove that :

$$3\sqrt{6}\left(\sum a^2-\sum ab\right)\geq\left(\sum|a-b|\right)^2$$

Proposed by Mihaly Bencze, Neculai Stanciu – Romania

S.3224 Solve for positive real numbers the following system:

$$\begin{cases}\frac{x^3}{(1+y)(1+z)}=\frac{6y-x-z-2}{8}\\ \frac{y^3}{(1+z)(1+x)}=\frac{6z-y-x-2}{8}\\ \frac{z^3}{(1+x)(1+y)}=\frac{6x-z-y-2}{8}\end{cases}$$

Proposed by Neculai Stanciu – Romania

S.3225 Prove that in all triangle ABC holds the inequality:

$$7\sum a^4+64s^2Rr\geq 5(s^2+r^2+4Rr)^2$$

Proposed by Neculai Stanciu – Romania

S.3226 If $a,b,c>0$, then prove that:

$$\sum\frac{2a^2+c(b-c)}{(b+c)(a+b+c)}\geq 1$$

Proposed by Neculai Stanciu – Romania

S.3227 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{1}{(a+b)(m_a+m_b-m_c)^2}\geq\frac{2\sqrt{3}}{9R^3}$$

Proposed by Daniel Sitaru – Romania

S.3228 $f(x)=x^{2n}\sinh\left(\frac{1}{x}\right), x>0, n\in\mathbb{N}^*, f^{(n)}$ – "n" – th derivative

Find:

$$\Omega=\lim_{n\to\infty}f^{(n)}(\pi)$$

Proposed by Daniel Sitaru – Romania

S.3229

$$X\in M_2(\mathbb{R}), X=\begin{pmatrix}a & b\\ c & d\end{pmatrix}, X^5=\begin{pmatrix}3 & 2\\ 4 & 5\end{pmatrix}$$

Find: $\Omega=a-b+c-d$

Proposed by Daniel Sitaru – Romania

S.3230 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a^2m_b^2}{(a+b)a^2b^2}\geq\frac{9\sqrt{3}}{32R}$$

Proposed by Daniel Sitaru – Romania

S.3231 In ΔABC the following relationship holds:

$$\sum_{cyc} \tan^2\frac{A}{2} \geq \frac{26}{27} + 27\left(\sum_{cyc} \cot^2\frac{A}{2}\right)^{-3}$$

Proposed by Daniel Sitaru – Romania

S.3232 Find:

$$\Omega = \lim_{n\to\infty} \frac{1}{n} \cdot \sum_{k=2}^{n} \sqrt{\frac{(k-1)k}{2k-1+2\sqrt{(k-1)k}}} \cdot \left(\sum_{k=1}^{n} \sqrt{k}\right)^{-1}$$

Proposed by Daniel Sitaru – Romania

S.3233 In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{r_a r_b a^2 b^2}{(r_a + r_b) m_a^2 m_b^2} \geq 8r$$

Proposed by Daniel Sitaru – Romania

S.3224 If $a, b, c \geq 0$, and $a + b + c = \frac{3}{2}$ then:

$$\frac{a+b}{(b+c)^8} + \frac{b+c}{(c+a)^8} + \frac{c+a}{(a+b)^8} \geq 4(a^2 + b^2 + c^2)$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

S.3225 Let $a, b, c > 0$. Prove that:

$$a + b + c + 2\sqrt[3]{abc} + \frac{4}{\sqrt[3]{abc}} \geq \sqrt{a^2+8} + \sqrt{b^2+8} + \sqrt{c^2+8}$$

Proposed by Phan Ngoc Chau-Vietnam

S.3226 If $\lambda, \mu > 0$ then in ΔABC the following relationship holds:

$$4\sqrt{3}(\lambda + \mu) \cdot \frac{r}{R} \leq \sum_{cyc} \frac{\lambda a + \mu b}{r_c} \leq \frac{3R(\lambda+\mu)}{F} \cdot \sqrt{9R^2 - s^2}$$

Proposed by Mehmet Șahin – Turkiye

S.3227 If $a, b, c \geq 0, (n > 0)$ and $a + b + c = \frac{3}{2}$ then:

$$\frac{a+b}{\sqrt[n]{b+c}} + \frac{b+c}{\sqrt[n]{c+a}} + \frac{c+a}{\sqrt[n]{a+b}} + \frac{ab+bc+ca}{n} \geq 3 \cdot \left(1 + \frac{1}{4n}\right)$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

S.3228 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a^4}{(b+c)b^2c^2}\geq\frac{9\sqrt{3}}{32R}$$

Proposed by Daniel Sitaru – Romania

S.3229 In $\Delta ABC, AA' = w_a, BB' = h_b, CC' = m_c, AA' \cap BB' \cap CC' \neq \emptyset$. Prove that:

$$\frac{\sin B+\sin C}{\cos C}+\tan B=\frac{\sin^2 B\cdot\sin C}{\cos B\cdot\cos^2 C}$$

Proposed by Daniel Sitaru – Romania

S.3230 In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{a^4}{(b+c)m_b^2m_c^2}\geq\frac{8\sqrt{3}}{9R}$$

Proposed by Daniel Sitaru – Romania

S.3231 In ΔABC the following relationship holds:

$$(2a+b)(2c+b)+(2b+c)(2a+c)+(2c+a)(2c+b)\leq 81R^2$$

Proposed by Daniel Sitaru – Romania

S.3232 In ΔABC the following relationship holds:

$$\frac{a}{\sin\frac{A}{3}}+\frac{b}{\sin\frac{B}{3}}+\frac{c}{\sin\frac{C}{3}}<18R$$

Proposed by Daniel Sitaru – Romania

S.3233 If $a,b,c,x,y,z>0$ then:

$$\left(\frac{a^4}{x^2}+2\right)\cdot\left(\frac{b^4}{y^2}+2\right)\cdot\left(\frac{c^4}{z^2}+2\right)\geq\frac{3(a+b+c)^4}{(x+y+z)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.3234 If $a,x\in\mathbb{R}$ and $b,b+x>0$. Find x such that:

$$\left(\frac{a}{a+x}\right)^2-\frac{1}{2}\left(\frac{a}{a+x}-\frac{x}{b}\right)^2+\left(\frac{b}{b+x}\right)^2\leq\left(\frac{b}{b+x}+\frac{x}{b}\right)\left(\frac{a}{a+x}-\frac{x}{b}+\frac{1}{2}\right)$$

Proposed by Sidi Abdallah Lemrabott-Mauritania

S.3235 Find all value of $\alpha,\beta\in\mathbb{R}$ such that $|a-b|+\alpha(a+b)\geq\sqrt{\beta(a^2+b^2)}, \forall a,b\geq 0$

Proposed by Nguyen Van Canh-Vietnam

S.3236 Let $a, b, c \geq 0: \sqrt{ab} + \sqrt{bc} + \sqrt{ca} = 3$. Prove that:

$$\frac{1}{a+b+4} + \frac{1}{b+c+4} + \frac{1}{c+a+4} \leq \frac{1}{2}$$

Proposed by Phan Ngoc Chau-Vietnam

S.3237 Let $x \in \mathbb{R}$. Prove that: $2022^{x^2} \geq 1 + x + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6}$

Proposed by Nguyen Van Canh-Vietnam

S.3238 If I – incenter in ΔABC then holds:

$$\frac{1}{r}\sum_{cyc} AI \geq \sum_{cyc}\sqrt{\frac{h_a}{r_a}} \cdot \sqrt{1 + \sum_{cyc}\frac{n_a}{h_a}}$$

Proposed by Bogdan Fuștei – Romania

S.3239 In ΔABC the following relationship holds:

$$2\sum_{cyc}\sqrt{a} \leq 2\sum_{cyc}\frac{a}{\sqrt{b}+\sqrt{c}-\sqrt{a}} \leq \frac{abc}{8r^2(4R+r)^2} \cdot \left(\sum_{cyc}\sqrt{a}\right)^3$$

Proposed by Daniel Sitaru – Romania

S.3240 In any ΔABC the following relationship holds:

$$\sum_{cyc}\frac{m_a}{m_b\left(5(m_a^2+m_b^2)+m_a(6m_b+11m_c)\right)} \geq \frac{4}{81R^2}$$

Proposed by Zaza Mzhavanadze – Georgia

S.3241 Let be $t \geq 0; n \in \mathbb{N}, n \geq 2, a, b, x_k \in \mathbb{R}_+^* = (0, \infty), \forall k = \overline{1,n}$, then:

$$a^2 \cdot \sum_{k=1}^{n} x_k^{t+1} + b^2 \cdot \sum_{k=1}^{n}\frac{1}{x_k^{t+1}} \geq 2ab \cdot n$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.3242 In any ΔABC and $\forall n, m \in \mathbb{N}$, the following relationship holds:

$$\frac{r_a^n(r_a^2+r_br_c)}{(r_b+r_c)^m} + \frac{r_b^n\left(r_b^2+r_cr_a\right)}{(r_c+r_a)^m} + \frac{r_c^n(r_c^2+r_ar_b)}{(r_a+r_b)^m} \geq \frac{3^{n-m+3} \cdot r^{n-m+2}}{2^{m-1}}$$

Proposed by Zaza Mzhavanadze – Georgia

S.3243 Compare (without computer): $2023^{2022^{\pi}}$ and $2022^{2023\sqrt{\pi}}$

Proposed by Nguyen Van Canh-Vietnam

S.3244 In ΔABC the following relationship holds:

$$\sum_{cyc} h_a\,(r_b + r_c - 2m_a) \geq 0$$

Proposed by Bogdan Fuștei – Romania

S.3245 If $x, y, z > 0$ then:

$$\left(\frac{x^4}{y^4} + 2\right)\cdot\left(\frac{y^4}{z^4} + 2\right)\cdot\left(\frac{z^4}{x^4} + 2\right)\cdot\left(\frac{81\cdot x^2y^2z^2}{(x^3+y^3+z^3)^2} + 2\right) \geq 36$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru – Romania

S.3246 If $a, b, c > 0, a + b + c = 1$ then:

$$\sum_{cyc}(a+b)\cdot\left(a^a b^b\right)^{\frac{1}{a+b}} \leq a^2 + b^2 + c^2 + a^2 b^b c^c$$

Proposed by Daniel Sitaru – Romania

S.3247 Let $\lambda \geq 0$ fixed. Solve for real numbers equation

$$\sqrt{2x-\lambda} + \sqrt{x+\lambda} = \sqrt{5x+2\lambda}$$

Proposed by Marin Chirciu – Romania

S.3248 Let $a \geq 0, \lambda \geq 2$ fixed. If $x > 0$ find the maximum of

$$E = \frac{x^3 + x}{x^4 + ax^3 + (\lambda^2+2)x^2 + ax + 1}$$

Proposed by Marin Chirciu – Romania

S.3249 If $a, b, c > 0$ such that $a + b + c = \lambda$ and $\lambda > 0$ then find $\max P$

$$P = \frac{ab}{\sqrt{ab+\lambda c}} + \frac{bc}{\sqrt{bc+\lambda a}} + \frac{ca}{\sqrt{ca+\lambda b}}$$

Proposed by Marin Chirciu – Romania

S.3250 If $x > 0, a > 0, b > 0$ solve

$$\sqrt{2ax+a^2} + \sqrt{2bx+b^2} = \left(x + \frac{ab}{2}\right)\left(\frac{1}{\sqrt{a+1}} + \frac{1}{\sqrt{b+1}}\right) + \frac{b}{\sqrt{a+1}} + \frac{a}{\sqrt{b+1}}$$

Proposed by Marin Chirciu – Romania

S.3251 If $x, y, z > 0$ and $n \geq 0$ then:

$$n\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\ge\frac{3(n-2)}{2}+\sum\frac{nx+y+z}{y+z}$$

Proposed by Marin Chirciu – Romania

S.3252 If $x,y,z>0$ and $n\ge 0,\lambda\ge 0$ then:

$$n\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)+3\ge\frac{3n\lambda}{\lambda+1}+\sum\frac{nx+y+\lambda z}{y+\lambda z}$$

Proposed by Marin Chirciu – Romania

S.3253 If $x_1,x_2,\dots,x_n\in\mathbb{R}$ such that $x_1^2+x_2^2+\cdots x_n^2=1$ find:

$$\max\left(x_1+2\sqrt{2}x_2+\cdots+n\sqrt{n}x_n\right)$$

Proposed by Marin Chirciu – Romania

S.3254 If $a,b,c>0$ such that $a+b+c=3$ and $\lambda\ge\frac{-3}{2}$ then:

$$\sum\frac{a^3}{a^2+b^2}+\lambda\ge\frac{2\lambda+3}{18}(ab+bc+ca)^2$$

Proposed by Marin Chirciu – Romania

S.3255 If $x_1,x_2,\dots,x_n>0$ find the minimum of

$$E=\sqrt[n]{x_1x_2\dots x_n}+\frac{1}{x_1}+\frac{1}{2x_2}+\frac{1}{4x_3}+\cdots+\frac{1}{2^{n-1}x_n}$$

Proposed by Marin Chirciu – Romania

S.3256 If $x_1,x_2,\dots,x_n>0$ such that $x_1+x_2+\cdots+x_n=1$ find minimum of

$$P=\sqrt{x_1^2+\frac{1}{x_1^2}}+\sqrt{x_2^2+\frac{1}{x_2^2}}+\cdots+\sqrt{x_n^2+\frac{1}{x_n^2}}$$

Proposed by Marin Chirciu – Romania

S.3257 If $a_1,a_2,\dots,a_n\ge 0$ and $n\in\mathbb{N},n\ge 2$ then:

$$a_1a_2\dots a_n+\sqrt[n]{(1+a_1^n)(1+a_2^n)\dots(1+a_n^n)}\ge 1$$

Proposed by Marin Chirciu – Romania

S.3258 If $a,b,c>1$ or $a,b,c\in(0,1),a,b,c$ – sides ΔABC and $n\in\mathbb{N}$, then:

$$\sum\frac{(\log_b^c\ )^n}{b+c-a}\ge\frac{9}{a+b+c}$$

Proposed by Marin Chirciu – Romania

S.3259 If $a, b, c > 0$ such that $\frac{a}{a+\lambda}+\frac{b}{b+\lambda}+\frac{c}{c+\lambda}=1$ and $\lambda \geq 0$ then find the maximum of abc.

Proposed by Marin Chirciu – Romania

S.3260 If $a, b, c \geq 1$ and $\lambda \geq 0$ then:

$$\sum \frac{3bc+2a+1}{b+\lambda c} \geq \frac{18}{\lambda+1}$$

Proposed by Marin Chirciu – Romania

S.3261 Let I_a, I_b, I_c be the excenters in ΔABC. Prove that:

$$8p\frac{r}{R} \leq \frac{(I_bI_c)^2}{b+c}+\frac{(I_cI_a)^2}{c+a}+\frac{(I_aI_b)^2}{a+b} \leq p\left(\frac{R}{r}\right)^2$$

Proposed by Marin Chirciu – Romania

S.3262 Let $a > 1$ fixed. Solve for real numbers:

$$(a^x+a)\left(\frac{2a}{a-1}-x\right)=\frac{2a(a+1)}{a-1}$$

Proposed by Marin Chirciu – Romania

S.3263 If $x, y, z > 0$ such that $x+y+z=1$ and $6n \geq k > 0$ then:

$$n\sum \frac{y+z}{x}+3k\sum yz \geq 6n+k$$

Proposed by Marin Chirciu – Romania

S.3264 If $a, b, c > 1$ or $a, b, c \in (0,1)$, a, b, c – sides ΔABC and $n \in \mathbb{N}$, then:

$$\sum \left(\frac{\log_b^c}{b+c-a}\right)^n \geq 3\left(\frac{3}{a+b+c}\right)^n$$

Proposed by Marin Chirciu – Romania

S.3265 Let $\lambda \geq 0$ fixed. If $a, b, c > 0$ such that $\sqrt{ab}+\sqrt{bc}+\sqrt{ca}=1$, find the minimum value of the expression:

$$E=\frac{a^2}{a+\lambda b}+\frac{b^2}{b+\lambda c}+\frac{c^2}{c+\lambda a}$$

Proposed by Marin Chirciu – Romania

S.3266 If $x, y, z \geq 1$ and $a > 1$ then

$$\sum x \log_a(a^y + a^z) \leq 2(xy + yz + zx)$$

Proposed by Marin Chirciu – Romania

S.3267 If $a_1, a_2, \dots, a_n \geq 0$ and $n \in \mathbb{N}, n \geq 2$ then:

$$a_1 a_2 \dots a_n + \sqrt[n]{(1 + a_1^n)(1 + a_2^n) \dots (1 + a_n^n)} \geq 1$$

Proposed by Marin Chirciu – Romania

S.3268 In ΔABC the following relationship holds:

$$\sum \frac{ah_a}{h_b + h_c} \leq p \leq \sum \frac{ar_a}{r_b + r_c}$$

Proposed by Marin Chirciu – Romania

S.3269 In ΔABC the following relationship holds:

$$\sum \frac{\sin^4\frac{B}{2} + \lambda \sin^4\frac{C}{2}}{1 + n\sin^2\frac{A}{2}} \geq \frac{3(\lambda + 1)}{4(n + 4)}\left(\frac{3}{2} - \frac{r}{R}\right), \text{ where } \lambda \geq 0 \text{ and } n \leq 4.$$

Proposed by Marin Chirciu – Romania

S.3270 If $a, b, c, t, v, x, y > 0$ then:

$$\frac{a^2t + u}{xb + yc} + \frac{b^2t + u}{xc + ya} + \frac{c^2t + u}{xa + yb} \geq \frac{6tu}{x + y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3271 In any triangle with the area F the following inequality holds:

$$(a^2 + 2)(a^2b^2 + 2) \cdot (b^2c^4 + 2) \geq 144 \cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3272 If $x, y, z > 0$ then in ΔABC with the area F the following inequality holds:

$$\frac{x \cdot a^3}{(y + z)^2 \cdot h_a} + \frac{y \cdot b^3}{(z + x)^2 \cdot h_b} + \frac{z \cdot c^3}{(x + y)^2 \cdot h_c} \geq \frac{6F}{x + y + z}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3273 If $m \geq 0$ and ABC is any triangle then:

$$\frac{a \cdot b^{m+1}}{(a + 2b)^m} + \frac{b \cdot c^{m+1}}{(b + 2c)^m} + \frac{c \cdot a^{m+1}}{(c + 2a)^m} \geq 4 \cdot 3^{2-m} \cdot r^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3274 If $m \geq 0$ and a is the hypothesis lenght of a right triangle ABC, then:

$$(a^{2m+2}+b^{2m+2}+c^{2m+2})\cdot\left(\frac{1}{a^{2m+2}}+\frac{1}{b^{2m+2}}+\frac{1}{c^{2m+2}}\right)\geq\frac{10^{m+1}}{3^{2m}}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3275 In any ΔABC having the area F the following inequality holds:

$$(m_a^4+1)\cdot(m_b^4+1)\cdot(m_c^4+1)\geq\frac{81}{4}\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3276 Le be $m \geq 0, x, y, z > 0$, then in ΔABC with the area F the following inequality holds:

$$\frac{x^{m+1}\cdot a^{m+2}}{(y+z)^{m+1}\cdot h_a^m}+\frac{y^{m+1}\cdot b^{m+2}}{(z+x)^{m+1}\cdot h_b^m}+\frac{z^{m+1}\cdot c^{m+2}}{(x+y)^{m+1}\cdot h_c^m}\geq 2(\sqrt{3})^{1-m}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3277 Let ABC be a triangle with the area F, then:

$$\frac{a^7}{xr+y\cdot h_a}+\frac{b^7}{xr+y\cdot h_b}+\frac{c^7}{xr+y\cdot h_c}\geq\frac{64}{x+3y}\cdot F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3278 If $x, y, z > 0$, then:

$$(x^2y^2+2)\cdot(y^2z^2+2)\cdot(z^2x^2+2)\cdot\left(\frac{1}{(x+y)^4}+1\right)\cdot\left(\frac{1}{(y+z)^4}+1\right)\cdot\left(\frac{1}{(z+x)^4}+1\right)\geq\frac{729}{64}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3279 In any ΔABC with the area F the following inequality holds:

$$(a^2b^2+1)(b^2c^2+1)+(b^2c^2+1)(c^2a^2+1)+(c^2a^2+1)(a^2b^2+1)\geq 64\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3280 Let be $m \geq 0$ and ABC a triangle with the area F, then:

$$\frac{a\cdot b^{m+1}}{(r+h_a)^m}+\frac{b\cdot c^{m+1}}{(r+h_b)^m}+\frac{c\cdot a^{m+1}}{(r+h_c)^m}\geq 2^{2-m}\cdot(\sqrt{3})^{m+1}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3281 In any ΔABC with the area F and semiperimeter s the following inequality holds:

$$r_a\cdot a^4+r_b\cdot b^4+r_c\cdot c^4\geq\frac{16}{4R+r}\cdot s^2\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3282 If $x, y, z > 0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{xa^3}{(y+z)h_a}+\frac{yb^3}{(z+x)h_b}+\frac{zc^3}{(x+y)h_c}\geq 4F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3283 Let be M an interior point in ΔABC with the area F and d_a, d_b, d_c the distances M to the sides BC, CA respectively AB, then:

$$\frac{a^5\cdot b^4}{h_a\cdot d_b^2}+\frac{b^5\cdot c^4}{h_c\cdot d_c^2}+\frac{c^5\cdot a^4}{h_c\cdot d_a^2}\geq 512F^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3284 Let be $m\in\mathbb{N}, m\geq 2$ and $a_k>1, \forall k=\overline{1,m}$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n]{a_1}+\sqrt[n]{a_2}+\cdots+\sqrt[n]{a_m}-m\right)n$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3285 In any ΔABC with the area F the following inequality holds

$$\frac{1}{h_a^2}\csc B\csc C+\frac{1}{h_b^2}\csc C\cdot\csc A+\frac{1}{h_c^2}\csc A\csc B\geq\frac{12r}{p}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3286 In any ΔABC with the area F the following inequality holds:

$$\frac{\cot A}{a^2}+\frac{\cot B}{b^2}+\frac{\cot C}{c^2}\geq\frac{3}{4F}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3287 Let be $n\in\mathbb{N}, n\geq 3, a_k\in\mathbb{R}_+^*=(0,\infty), \forall k=\overline{1,n}, s_n=\sum_{k=1}^n a_k$ and $t,u,v,w\geq 0$ such that $t+u=v+w=2$ and $x,y>0$ with $xs_n>y\max\limits_{1\leq k\leq n}a_k$. Prove that:

$$\sum_{k=1}^n\frac{a_k^t}{(xs_n-ya_k)^v}+\sum_{k=1}^n\frac{a_k^u}{(xs_n-ya_k)^w}\geq\frac{2n}{nx-y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.3288 Let be $n\in\mathbb{N}, n\geq 3, a_k\in\mathbb{R}_+^*=(0,\infty), \forall k=\overline{1,n}$ and $t,u\geq 0, t+u=2, x,y>0$,

$xs_n>y\cdot\max\limits_{1\leq k\leq n}a_k$ where $s_n=\sum_{k=1}^n a_k$, then:

$$\sum_{k=1}^n\frac{a_k^t}{(xs_n-ya_k)^u}+\sum_{k=1}^n\frac{ak^u}{(xs_n-ya_k)^t}\geq\frac{2n}{nx-y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.3289 If $a,b,c>0, t,u\geq 0, x,y>0$ in $t+u=2$, then:

$$\frac{a^t}{(bx+cy)^u}+\frac{b^t}{(cx+ay)^u}+\frac{c^t}{(ax+by)^u}+\frac{a^u}{(bx+cy)^t}+\frac{b^u}{(cx+ay)^t}+\frac{c^u}{(ax+by)^t}\geq\frac{6}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.3290 In any ΔABC with the area F the following inequality holds:

$$\left(\frac{1}{h_ah_b}+2\right)\left(\frac{1}{h_bh_c}+2\right)\left(\frac{1}{h_ch_a}+2\right)\geq\frac{9\sqrt{3}}{F}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3291 If $a,b,c,x,y,z>0$ then:

$$\frac{a^2+b^2+c^2}{x+y}+\frac{a}{b(bx+cy)}+\frac{b}{ca(cx+ay)}+\frac{c}{ab(ax+by)}\geq\frac{6}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze – Romania

S.3292 If $a,b,c>0, t,u\geq 0, x,y>0$ with $t+u=4$ then

$$\frac{a^t}{(bx+cy)^u}+\frac{b^t}{(cx+ay)^u}+\frac{c^t}{(ax+by)^u}+\frac{a^u}{(bx+cy)^t}+\frac{b^4}{(cx+ay)^t}+\frac{c^u}{(ax+by)^t}\geq\frac{6}{(x+y)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.3293 If $a,b,c,x,y>0$ and $abc\leq 1$ then:

$$\frac{1}{a^3(xb+yc)}+\frac{1}{b^3(xc+ya)}+\frac{1}{c^3(xa+yb)}\geq\frac{3}{x+y}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți – Romania

S.3294 Let be $m\geq 0, a,b,c,d>0$ and $a+b+c>d$, then:

$$\frac{(a+d)^{m+1}}{(b+d)^m}+\frac{(b+d)^{m+1}}{(c+d)^m}+\frac{(c+d)^{m+1}}{(a+d)^m}\geq 4d$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.3295 If $a,b,c,x,y>0$, then:

$$\frac{a}{bx+cy+tb}+\frac{b}{cx+ay+tc}+\frac{c}{ax+by+ta}\geq\frac{3}{x+y+t}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.3296 If $m\geq 0$ and $a,b,c>0$ with $abc\leq 1$ then:

$$\frac{1}{a^{2m+1}(b+c)^m}+\frac{1}{b^{2m+1}(c+a)^m}+\frac{1}{c^{2m+1}(a+b)^m}\geq\frac{3}{2^m}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți– Romania

S.3297 If $m, n, x, y, z > 0$ then:

$$\frac{mx+ny}{nx+my+(m+n)z}+\frac{my+nz}{(m+n)x+ny+mz}+\frac{mz+nx}{mx+(m+n)y+nz}\geq\frac{3}{2}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu– Romania

S.3298 If $a, b, c > 0$, then:

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq 2$$

Proposed by D.M. Bătinețu – Giurgiu– Romania

S.3299 In any ΔABC with the area F and the semiperimeter s the following inequality holds:

$$(a^3+b^3+c^3)s\geq 2^4\cdot F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru– Romania

S.3300 In any ΔABC with the area F the following inequality holds:

$$a^3+b^3+c^3\geq 2F\cdot\sqrt{\frac{ab}{\sin^2\frac{C}{2}}+\frac{bc}{\sin^2\frac{A}{2}}+\frac{ca}{\sin^2\frac{B}{2}}}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze– Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

UNDERGRADUATE PROBLEMS

U.3154

$$lim_{n\to\infty}\left(\prod_{k=1}^{n}\left(\int_0^{\frac{\pi}{4052}}\sqrt[\frac{n}{k}]{\frac{\sin(x)+\sin(3x)+\sin(5x)+\cdots+\sin(4051x)}{\cos(x)+\cos(3x)+\cos(5x)+\cdots+\cos(405x)}}\,dx\right)\right)^{\frac{1}{2n}}=\sqrt{\frac{\pi}{\alpha}},\ \ \alpha=?$$

Proposed by Amin Hajiyev-Azerbaijan

U.3155 *Prove that*:

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\sin(2x)\sin(2y)\sin(2z)}{1+\sin^2(x)+\sin^2(x)\sin^2(y)+\sin^2(x)\sin^2(y)\sin^2(z)}dxdydz=$$

$$=Li_3\left(-\frac{1}{2}\right)+Li_3\left(\frac{2}{3}\right)-Li_3\left(\frac{3}{4}\right)-Li_3\left(\frac{1}{3}\right)+2Li_2\left(-\frac{1}{2}\right)\ln(2)+\frac{13}{8}\zeta(3)$$
$$+\frac{1}{6}\ln^3\left(\frac{3}{2}\right)+\frac{1}{3}\ln^3(2)+\frac{1}{2}\ln^2\left(\frac{4}{3}\right)\ln(2)-\frac{1}{2}\ln^2\left(\frac{3}{2}\right)\ln(3)$$
$$-\frac{\pi^2}{4}\ln(2)+\frac{\pi^2}{6}\ln(3)+\frac{1}{3}\ln^3(3)-\frac{1}{2}\ln(2)\ln^2(3)$$

Proposed by Amin Hajiyev-Azerbaijan

U.3156 Find:

$$\Omega=\int_0^{\infty}\frac{x\ \ln^2(x)}{\left(x^2+1\right)^2}dx$$

Proposed by Vasile Mircea Popa-Romania

U.3157 *Prove*:

$$\int_0^{\infty} \frac{tanh^2(x)}{x\,cosh(2x)}dx = 2ln\left(\frac{A^6\pi 2^{5/6}}{\Gamma^2\left(\frac{1}{4}\right)\sqrt{e}}\right), A\ is\ Glaisher - Kinkelin\ constant$$

Proposed by Shobhit Jain − India

U.3158 Find:

$$\Omega = \int_{-1}^{1} \frac{1}{\sqrt[5]{(1-x)^4(1+x)}}dx$$

Proposed by Vasile Mircea Popa-Romania

U.3159 Find:

$$\Omega = \lim_{n\to\infty}\left[\frac{1}{n}\int_0^{\frac{\pi}{4}} \ln\left(1+e^{n\cos x}\right)dx\right]$$

Proposed by Vasile Mircea Popa-Romania

U.3160 Find:

$$\Omega = \int_0^{\infty} \frac{x\arctan(x)}{\left(x^2+1\right)\left(x+1\right)^2}dx$$

Proposed by Vasile Mircea Popa-Romania

U.3161 If p_a, p_b, p_c − Spieker cevians in ΔABC then:

$$p_a + p_b + p_c \le h_a + h_b + h_c + \frac{16}{5}[max\{a,b,c\} - min\{a,b,c\}].$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3162

If p_a, p_b, p_c − Spieker cevians in ΔABC then:

$$p_a + p_b + p_c \ge h_a + h_b + h_c + \frac{4}{3}\cdot\frac{a^2+b^2+c^2-ab-bc-ca}{s}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3163 If $p_a, p_b, p_c -$ Spieker cevians, $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$\frac{p_a - h_a}{n_a + h_a} + \frac{p_b - h_b}{n_b + h_b} + \frac{p_c - h_c}{n_c + h_c} \geq \frac{4r}{3s} \cdot \left(\frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c} - 3\right)$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3164 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$\frac{p_a}{h_a} + \frac{p_b}{h_b} + \frac{p_c}{h_c} \geq 3 + \frac{2}{3} \cdot \frac{a^2 + b^2 + c^2 - ab - bc - ca}{F}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3165 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$p_a + p_b + p_c + 12r \geq \frac{7}{3}(h_a + h_b + h_c)$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3166 If we have the integral, then find α

$$\int_0^\infty \left(\frac{\sinh\left(\sqrt{\frac{2\pi}{3}}x\right)}{\cosh\left(\sqrt{\frac{3\pi}{2}}x\right)} + \frac{\sinh\left(\sqrt{\frac{4\pi}{5}}x\right)}{\cosh\left(\sqrt{\frac{5\pi}{4}}x\right)} \right) dx = \log(\alpha)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3167 Find:

$$\int_{-\infty}^{\infty} \frac{\tanh^2(2\pi x) + 1}{\cosh(2\pi x) - \frac{1}{\sqrt{2}}} \frac{\sinh(2\pi x)}{\sinh\left(\frac{\pi x}{2}\right)} dx = 3\sqrt{5 + \frac{7}{\sqrt{2}}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3168 Prove the summation:

$$\sum_{n=0}^{\infty} \frac{(-1)^n\, n\, (4n+3)\, (2n+1)}{(2n+3)(4n+1)} \frac{(4n+1)!!}{(4n+2)!!} = -\frac{641}{40\sqrt{2\left(6409\sqrt{2} + 9041\right)}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3169 If we have the integrals

$$\int_0^\infty \left(e^{\frac{2x}{3}}-\frac{2}{3}\right)^2 \frac{\tanh\left(\frac{x}{2}\right)+\tanh(2x)}{x} e^{-4x}dx=\frac{2}{9}\log(P)$$

$$\int_0^\infty \left(e^{\frac{2x}{3}}+\frac{2}{3}\right)^2 \frac{\tanh\left(\frac{x}{2}\right)+\tanh(2x)}{x} e^{-4x}dx=\frac{2}{9}\log(Q)$$

then show that: $\sqrt[12]{P\,Q}=\frac{\sqrt[3]{2}\Gamma\left(\frac{1}{3}\right)^9}{5\times 3^{\frac{2}{3}}\sqrt{5}\pi^{\frac{9}{2}}}$

Proposed by Srinivasa Raghava-AIRMC-India

U.3170 Prove the integral

$$\int_0^1\left(\sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}+\frac{(4n-1)!!}{(4n)!!}+\frac{(6n-1)!!}{(6n)!!}\right)x^n\right)dx=\frac{118}{15}-\frac{2\sqrt[4]{3}\sqrt{2}}{5}-\frac{2\sqrt{2}}{3}-\frac{4\sqrt{2}}{5\sqrt[4]{3}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3171 Prove the product

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{(2n)^4-(2n)^2}\right)\left(1-\frac{1}{(2n)^2-2n}\right)=\frac{2}{\sqrt{\pi}}\frac{\sin\left(\frac{\pi\sqrt{\phi}}{2}\right)\sinh\left(\frac{\pi}{2}\left(\frac{1}{\phi}\right)\right)}{\Gamma\left(\frac{1}{2\phi^2}\right)\Gamma\left(\frac{\phi^2}{2}\right)},\ \phi \text{ is the Golden ratio}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3172 If we have the sum

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^{\frac{n(n+1)}{2}}}{(2n+1)^2}\left(1+\frac{1}{2n+1}-\frac{1}{3n+2}+\frac{1}{4n+3}\right)=\pi\alpha$$

then find the value of the expression

$$\sqrt[3]{\alpha^8-24\alpha^7+232\alpha^6-1152\alpha^5+3068\alpha^4-3984\alpha^3+1328\alpha^2+1536\alpha-35}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3173 If we have the integral

$$\alpha=\int_{-\infty}^{\infty}\frac{\tanh(\pi x)+1}{\cosh(2\pi x)+\frac{1}{2}}\frac{\sinh\left(\frac{2\pi x}{3}\right)}{\sinh\left(\frac{\pi x}{2}\right)}dx,\ \text{then evaluate the below expression}$$

$$\sqrt[3]{243\alpha^6-14256\alpha^4-31104\alpha^3+18432\alpha}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3174 Prove that:

$$\frac{1}{\sqrt{\pi}}\int_0^\infty \frac{\cosh\left(\frac{x}{12}\sqrt{3\pi}\right)}{1+2\cosh\left(2x\sqrt{\frac{\pi}{3}}\right)}dx=\frac{1+\sqrt{2}-\sqrt{3}}{4}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3175 If we have the integral

$$\alpha=\left(\frac{3}{\pi}\right)^4\int_0^\infty \frac{x^3+x+1}{x^4+x^2+1}\frac{\log^3(x)}{\sqrt[3]{x}}dx$$

then prove that $7\alpha^2-2552\alpha-36491=0$

Proposed by Srinivasa Raghava-AIRMC-India

U.3176 Prove that:

$$\int_0^\infty \left(\frac{\sqrt{\tanh(\pi x)}}{\sqrt{\sinh(\pi x)}+\sqrt{\cosh(\pi x)}}\right)e^{-\pi x}dx=\frac{2}{\pi}-\frac{4\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)^2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3177 Prove the integral:

$$\int_0^\infty \left(\sqrt{\tanh(\pi x)}+1\right)\left(\sqrt{\coth(\pi x)+1}\right)e^{-\pi x}dx=\frac{2}{\pi}+\frac{4\sqrt{2}\Gamma\left(\frac{5}{4}\right)^2}{\pi^{\frac{3}{2}}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3178 Prove the summation:

$$\sum_{n=1}^\infty \frac{F_{2n}}{2^{6n-1}(4n-3)}\binom{4n}{2n}=\frac{1}{3}\sqrt{\frac{1}{5}\left(3+\sqrt{5}-\sqrt{2\sqrt{5}+5}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3179 Prove the integral:

$$\int_0^\infty \frac{\sinh\left(\frac{\pi x}{2}\right)+\sinh(2\pi x)}{(e^{2\pi x}+1)\left(e^{\frac{\pi x}{2}}+1\right)}\frac{e^{-\pi x}}{x}dx=\frac{1}{2}\log\left(\frac{3\times 2^{\frac{13}{4}}(\sqrt{2}-1)\pi^{\frac{3}{2}}}{5\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{5}{8}\right)^2}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3180 Prove the Integral:

$$\int_0^\infty \frac{\tanh^2(\pi x)}{1+\cosh\left(\frac{\pi x}{2}\right)}\sqrt{\frac{\tanh\left(\frac{\pi x}{2}\right)}{1+\cosh(\pi x)}}dx=\frac{\sqrt{2}}{\pi}+\frac{1}{2}-\frac{\log\left(1+\sqrt{2}\right)}{\pi}-\frac{4\sqrt{\pi}}{\Gamma\left(\frac{1}{4}\right)^2}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3181 Prove the integral relation:

$$\int_0^\infty \left(\frac{\sinh(2\pi x)-\sinh\left(\frac{\pi x}{2}\right)}{\left(e^{2\pi x}+1\right)\left(e^{\frac{\pi x}{2}}+1\right)}\right)\frac{e^{-\frac{\pi x}{2}}}{x}dx=\log\left(\frac{\sqrt{15}\sqrt[4]{\pi}\Gamma(\frac{1}{8}}{2^{\frac{17}{8}}\Gamma\left(\frac{1}{4}\right)^{\frac{3}{2}}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3182 Prove that:

$$\int_{-\infty}^\infty \frac{\cosh\left(\sqrt{\frac{2\pi}{3}}x\right)}{\cosh\left(\sqrt{\frac{3\pi}{2}}x+\frac{3}{4}\right)}dx=(1-e)\sqrt{\frac{2\pi}{3e}}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3183 If we define α through this integral relation:

$$\int_{-\infty}^\infty \frac{\tanh^2(2\pi x)+1}{\cosh(2\pi x)-\frac{1}{\sqrt{2}}}\frac{\cosh(2\pi x)}{\cosh\left(\frac{\pi x}{2}\right)}dx=\alpha\int_{-\infty}^\infty \frac{\tanh^2(2\pi x)+1}{\cosh(2\pi x)+\frac{1}{\sqrt{2}}}\frac{\cosh(2\pi x)}{\cosh\left(\frac{\pi x}{2}\right)}dx$$

then prove that: $\alpha^4+4\alpha^3-42\alpha^2+36\alpha+17=0$

Proposed by Srinivasa Raghava-AIRMC-India

U.3184 Prove that:

$$\int_0^\infty \left({}_2F_1\left(-\frac{1}{2},-\frac{1}{3};\frac{1}{6};e^{-x}\right)+{}_2F_1\left(-\frac{1}{2},\frac{1}{3},\frac{1}{6};e^{-x}\right)+{}_2F_1\left(\frac{1}{2},-\frac{1}{3};\frac{1}{6};e^{-x}\right)\right)e^{-x}dx=\frac{5}{2}+\frac{\Gamma\left(\frac{1}{3}\right)^5}{2^{\frac{8}{3}}\pi^2\Gamma\left(-\frac{1}{3}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3185 Prove the summation:

$$\sum_n^\infty \frac{\binom{2n}{n}}{n(2n-1)(4n-3)2^{2n}}=\frac{2}{9}\left(-5+\log(8)+\frac{\Gamma\left(\frac{1}{4}\right)^2}{\sqrt{2\pi}}\right)$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3185 If we define the function $\psi(q)$

$\psi(q)=\int_{-\infty}^{\infty}\frac{\tanh(4\pi x)-q}{\cosh^2(2\pi x)-q}dx$, then prove the relation

$$\frac{\psi(q)}{q}=2q(q-1)\frac{\partial^2\psi(q)}{\partial q^2}+(2q+1)\frac{\partial\psi(q)}{\partial q}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3186 Prove the integral relation:

$$\int_0^\infty\sqrt{\frac{\tanh\left(x\sqrt{\frac{\pi}{8}}\right)}{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}}dx=\frac{2-\sqrt{2}}{2^{\frac{7}{4}}\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{8}\right)^2}{\Gamma\left(\frac{1}{4}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3187 Prove that:

$$\int_{-\infty}^{\infty}\frac{\tanh\left(\frac{\pi x}{3}\right)+\tanh(3\pi x)}{e^{2\pi x}-1}\frac{\cosh\left(\frac{\pi x}{2}\right)}{\cosh(2\pi x)}dx=\frac{\sqrt{2}-3}{\sqrt{3}}+\sqrt{4+\sqrt{2}-\frac{\sqrt{3}}{2}(2+\sqrt{2})}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3188 If $\Omega_1=\sum_{n=1}^{\infty}\zeta(n+1)\left(n!-e\left\lfloor\frac{n!}{e}\right\rfloor\right)$; $\Omega_2=\int_0^1\int_0^1\frac{1-xy\,e}{(1-xy)(1+\ln(x)+\ln(y))}dx\,dy$

Prove that: $\Omega_1-\Omega_2=e$, $\lfloor.\rfloor$ – floor function.

Proposed by Amin Hajiyev-Azerbaijan

U.3189 Find:

$$\Omega=\int_1^{\sqrt{3}}\left(\frac{\tan^{-1}x}{x-\tan^{-1}x}\right)^2dx$$

Proposed by Daniel Sitaru – Romania

U.3190 If

$$\Omega_1=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}x\ln\left(\frac{1}{\cos(x)}+\frac{1}{\sin(x)}\right)dx;\ \Omega_2=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}y\ln\left(\frac{1}{\cos(y)}-\frac{1}{\sin(y)}\right)dy$$

then show that:

$$\Omega_1+\Omega_2=\frac{21}{64}\zeta(3)+\frac{9\pi^2}{32}\ln(2)$$

Proposed by Ankush Kumar Parcha-India

U.3191 Find:

$$\Omega = \lim_{n\to\infty} \frac{n+2}{2^n} \sum_{k=0}^{n} \frac{(-1)^k \cdot 2^{n-k}}{k+1} \cdot \binom{n}{k}$$

Proposed by Daniel Sitaru – Romania

U.3192 Prove the summation:

$$\sum_{n=0}^{\infty} \left(\frac{(2n-1)!!}{(2n)!!} + \frac{(4n-1)!!}{(4n)!!} \right) \frac{1}{4n+1} = \frac{\pi}{4} + \frac{\log(17+12\sqrt{2})}{8} + \frac{\sqrt{\pi}\Gamma\left(\frac{5}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$$

Proposed by Srinivasa Raghava-AIRMC-India

U.3193 If $0 < a \leq b$ then:

$$e^a + e^{-a} + 2\int_a^b \frac{x^2}{\ln\left(x+\sqrt{x^2+1}\right)} dx \leq e^b + e^{-b}$$

Proposed by Daniel Sitaru – Romania

U.3194 Prove the below closed form:

$$\iint_{\left[0,\frac{\pi}{2}\right]^2} \ln(\sin^2(x) + \cos^2(y))\, dx\, dy = \pi G - \frac{\pi^2}{2} \ln 2$$

Where, G is the Catalan's constant

Proposed by Ankush Kumar Parcha-India

U.3195 If

$$\Omega := \int_0^{\infty} \ln(1+x^2) \frac{\ln(x)}{x^2} dx + \frac{1}{4}\int_0^{\infty} \ln\left(1+\frac{y^2}{4}\right) \frac{\ln(y)}{y^2} dxy + \frac{1}{9}\int_0^{\infty} \ln\left(1+\frac{z^2}{9}\right) \frac{\ln(z)}{z^2} dz + \cdots$$

Then show that: $\Omega = \pi[\zeta(3) - \zeta'(3)]$

Proposed by Ankush Kumar Parcha-India

U.3196 Find a closed form of:

$$\Omega = \int_0^{\infty} \frac{\left((1-4x^2)\sin(\pi x) - 4\tanh\left(\frac{\pi}{2}\right) x \cos(\pi x)\right) \operatorname{csch}(\pi x)}{\cos(2\pi x) - \cosh(\pi)} dx$$

Proposed by Fethi Toubal-Algeria

U.3197 Find:

$$\Omega=\int_0^1\int_0^1\int_0^1\frac{dx\,dy\,dz}{\sqrt{xyz}(1+4x)(1+3x+2y+z)}$$

Proposed by Hikmat Mammadov-Azerbaijan

U.3198 Prove the below closed form:

$$\sum_{n,m\in\mathbb{N}^2}\frac{1}{e^{(m+n)\pi}+e^{(m-n)\pi}}=\frac{1-G\sqrt{2}}{4(1-e^{\pi})}$$

where, G is Gauss's constant.

Proposed by Ankush Kumar Parcha-India

U.3199 Prove the below closed form

$$\int\int_{[0,1]^2}\tan^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)dx\,dy=-\frac{\pi\sqrt{3}}{12}+\frac{3}{4}\ln 3-\frac{i\pi}{2}$$

Proposed by Ankush Kumar Parcha-India

U.3200 Prove the below closed form:

$$\int\int_{[0,1]^2}\sqrt{\frac{x}{y}+\frac{y}{x}}\,dx\,dy=\frac{\sqrt{2}}{3}(2+\varpi)$$

where, ϖ is lemniscate constant.

Proposed by Ankush Kumar Parcha-India

U.3201 Prove the below closed form:

$$\int_{\mathbb{R}}e^{x}e^{-e^{x}}e^{-e^{-e^{x}}}e^{-e^{x}}e^{x}dx=-1+e^{-1}+\gamma+\delta e^{-1}$$

where δ is Euler – Gmopertz constant, γ is Euler – Mascheroni constant, e is Euler's number.

Proposed by Ankush Kumar Parcha-India

U.3202 Prove the below closed form:

$$\int\int\int_{[0,1]^3}\sum_{x,y,z}\frac{\ln(x+y)}{\sqrt{x+y}}dx\,dy\,dz=8\sqrt{2}\ln 2-64\left(\frac{\sqrt{2}-1}{3}\right)$$

Proposed by Ankush Kumar Parcha-India

U.3203 If

$$\Omega := \int\int_{[0,1]^2} \tanh^{-1}\left(\frac{x}{y}+\frac{y}{x}\right)\frac{dx\,dy}{(x+y)}$$

Then, show that:

$$\Omega = 2\mathcal{R}\left\{Li_2\left(\frac{1+i\sqrt{3}}{4}\right) - Li_2\left(\frac{\sqrt{3}+i}{2\sqrt{3}}\right) + Li_2\left(\frac{\sqrt{3}+i}{\sqrt{3}}\right)\right\} - \frac{\pi^2}{18} - \ln 2\ln 3 + \ln^2(2) - i\pi\ln 2$$

where, $\mathcal{R}(z)$ is the real part of complex number, $Li_2(z)$ is the dilogarithm of Spence's function

Proposed by Ankush Kumar Parcha-India

U.3204 If

$$\Omega(x,y,z) = \sqrt{\frac{\ln(xyz)}{xyz}}$$

Then, show that

$$\int\int\int_{[0,1]^3}\left(\Omega(x,y,z) + \frac{1}{\Omega(x,y,z)}\right)dx\,dy\,dz = i\left(\frac{135+\sqrt{3}}{9}\right)\sqrt{\frac{\pi}{2}}$$

Proposed by Ankush Kumar Parcha-India

U.3205 Prove the below closed form:

$$\int\int_{[0,1]^2}\frac{dx\,dy}{(x+y)^2+(x+y)+1} = -\frac{7\pi}{12\sqrt{3}} + \ln 3 - \frac{\ln 7}{2} + \frac{5}{2\sqrt{3}}\sin^{-1}\left(\frac{11}{14}\right)$$

Proposed by Ankush Kumar Parcha-India

U.3206 Prove the below closed form:

$$\int\int_{[0,1]^2}\frac{x+xy+y}{x^2+(x+y)^2+y^2}dx\,dy = \frac{\pi}{12\sqrt{3}} + \frac{3}{4}\ln 3$$

Proposed by Ankush Kumar Parcha-India

U.3207 Prove the below closed form:

$$\int\int\int_{[0,1]^2}\frac{dx\,dy\,dz}{\sqrt{x+xy+xyz}} = 8\left[\sqrt{3}-\sqrt{2}+\ln\left(\frac{\sqrt{6}+\sqrt{3}-\sqrt{2}-1}{2}\right)\right]$$

Proposed by Ankush Kumar Parcha-India

U.3208 Prove the below closed form:

$$\int\int_{[0,1]^2}\ln\left(\frac{x\sqrt{x}}{y\sqrt{y}}+\frac{y\sqrt{y}}{x\sqrt{x}}\right)dx\,dy = \frac{\pi}{\sqrt{3}} + 2\ln 2 - \frac{3}{2}$$

Proposed by Ankush Kumar Parcha-India

U.3209 Prove the below closed form:

$$\Omega = \int\int_{[0,1]^2} \ln\left(\frac{\sinh(x)}{\cosh(y)} + \frac{\cosh(x)}{\sinh(y)}\right) dx\, dy =$$
$$= \frac{1}{2} Li_3\left(-\frac{1}{e^2}\right) - \frac{1}{4} Li_3\left(-\frac{1}{e^4}\right) - \frac{1}{4} Li_2\left(\frac{1}{e^4}\right) + \frac{3\zeta(3)}{16} + \frac{\pi^2}{24} + \ln 2$$

Proposed by Ankush Kumar Parcha-India

U.3210 Prove the below closed form:

$$\int\int_{[0,1]^2} \sinh^{-1}(x+y)\, dx\, dy = \frac{7}{4}\ln\left(2+\sqrt{5}\right) - \frac{\ln\left(1+\sqrt{2}\right)}{2} + \frac{3}{2}\left(\sqrt{2}-\sqrt{5}\right)$$

Proposed by Ankush Kumar Parcha-India

U.3211 Prove the below closed form:

$$\int_0^1 \frac{\ln(x)}{1+2x+3x^2} dx = \frac{1}{\sqrt{2}} \Im\left\{Li_2\left(\frac{-1-i\sqrt{2}}{3}\right)\right\} + \frac{\ln 3}{4\sqrt{2}} \cos^{-1}\left(\frac{1}{3}\right) - \frac{\pi}{4\sqrt{2}} \ln 3$$

Proposed by Ankush Kumar Parcha-India

U.3212 Prove the below closed form:

$$\int\int_{[0,1]^2} \frac{\ln(xy)\ln(x+y)}{x+y} dx\, dy = -\frac{\zeta(3)}{4} + \frac{\pi^2}{6} - 2\ln^2(2) + 8\ln 2$$

Proposed by Ankush Kumar Parcha-India

U.3212 Prove the below closed form:

$$\int\int\int_{[0,1]^3} \sum_{x,y,z} \frac{\sqrt{x}+\sqrt{y}}{\sqrt{y}+\sqrt{z}} dx\, dy\, dz = \frac{41}{18} - \frac{16}{9}\ln 2$$

Proposed by Ankush Kumar Parcha-India

U.3213 Prove the below closed form:

$$\int\int_{[0,1]^2} (x+y)\tan^{-1}(x+y)\, dx\, dy = \frac{7}{3}\tan^{-1}(2) - \frac{\pi}{3} + \frac{1}{6}\ln\left(\frac{4}{5}\right) - \frac{2}{3}$$

Proposed by Ankush Kumar Parcha-India

U.3214 If $p_a -$ Spieker cevian in ΔABC then:

$$p_a \le h_a + \frac{16R}{9}\left(\frac{b-c}{a}\right)^2$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3215 If $p_a -$ Spieker cevian in ΔABC then:

$$p_a \le h_a + \frac{64}{27}(R - 2r)$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3215 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$p_a + p_b + p_c \le h_a + h_b + h_c + \frac{25}{9}(R - 2r)$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3216 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$\frac{p_a + p_b + p_c}{h_a + h_b + h_c} \le \frac{3R}{10r} + \frac{2}{5}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3217 If $p_a -$ Spieker cevian, $n_a -$ Nagel cevian in ΔABC then:

$$\frac{s(b-c)^2}{a(2s+a)} \le n_a - p_a \le \frac{s|b-c|}{2s+a}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3218 If $p_a -$ Spieker cevian in ΔABC then:

$$\frac{2s(b-c)^2}{4s^2 - a^2} \le p_a - w_a \le \frac{2sa|b-c|}{4s^2 - a^2}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3219 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$(b+c)p_a + (c+a)p_b + (a+b)p_c \ge \frac{4\sqrt{3}}{3}s^2$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3220 If $p_a, p_b, p_c -$ Spieker cevians in ΔABC then:

$$bcp_a+cap_b+abp_c\geq\frac{4\sqrt{3}}{9}s^3$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3221 If p_a- Spieker cevian in ΔABC then:

$$\frac{(b-c)^2}{2(2s+a)}\leq p_a-m_a\leq\frac{a|b-c|}{2(2s+a)}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3222 If p_a,p_b,p_c- Spieker cevians in ΔABC then:

$$p_a+p_b+p_c\geq m_a+m_b+m_c+\frac{s(R-2r)}{10R}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3223 If p_a,p_b,p_c- Spieker cevians in ΔABC then:

$$p_a+p_b+p_c\geq\frac{9}{2}\sqrt{2Rr}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3224 If p_a,p_b,p_c- Spieker cevians in acute ΔABC then:

$$p_a+p_b+p_c\geq\frac{23R}{10}+\frac{22r}{5}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3225 If p_a,p_b,p_c- Spieker cevians in ΔABC then:

$$(p_a+p_b+p_c)^2\geq 4s^2-\frac{9r(64R-53r)}{25}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3226 Let be ΔABC with the area F and the other usual notations, and $x,y\geq 0$ with $x+y=4$ then:

$$\left(\sqrt{a^{2x}+b^{2y}}+\sqrt{b^{2x}+c^{2y}}+\sqrt{c^{2x}+a^{2y}}\right)\cdot\left(\sqrt{a^{2y}+b^{2x}}+\sqrt{b^{2y}+c^{2x}}+\sqrt{c^{2y}+a^{2x}}\right)\geq 96F^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3227 In any triangle ABC, with the usual notations the following inequality holds:

$$\frac{a^4b^3}{h_b}+\frac{b^4c^3}{h_c}+\frac{c^4a^3}{h_a}\geq\frac{128}{3}\cdot S^3$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3228 In any ΔABC with the area F the following inequality holds:

$$3+a^2b^2+b^2c^2+c^2a^2\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3229 If $x,y,z>0,t\geq 0$ then in ΔABC with the area F the following inequality holds:

$$\frac{y+z+2t}{x+t}\cdot r_a^2+\frac{z+x+2t}{y+t}\cdot r_b^2+\frac{x+y+2t}{z+t}\cdot r_c^2\geq 6\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3230 In any ΔABC with the area F and the semiperimeter s the following inequality holds:

$$2s+a^2b+b^2c+c^2a\geq 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3231 Let be X and Y be the interior point in ΔABC with the area F and x_a the distance from X to BC and y_a the distances from Y to BC and x_b,x_c and y_b,y_c the analogs of x_a respectively y_a. Prove that:

$$\frac{a^3}{x_a+y_a}+\frac{b^3}{x_b+y_b}+\frac{c^3}{x_c+y_a}\geq 12F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3232 If $x,y,z>0$ then in any ΔABC with the area F the following inequality holds:

$$\left(\frac{ax}{h_a}+\frac{by}{h_b}+\frac{cz}{h_c}\right)^2\geq 4(xy+yz+zx)$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3233 In any ΔABC with the area F the following inequality holds:

$$3s+a^4+b^4+c^4\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3234 In any ΔABC with the area F and semiperimeter s the following inequality holds:

$$2s+a^3+b^3+c^3\geq 8\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3235 Let be $x,y,z>0$ and ABC a triangle, then:

$$\frac{xa}{(y+z)h_a}+\frac{yb}{(z+x)h_b}+\frac{zc}{(x+y)h_c}\geq\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3236 In any convexe polygon $A_1A_2 \dots A_n, n \geq 3$ with the area F and the sides with the length

$a_k = A_kA_{k+1}, k = \overline{1,n}, A_{n+1} = A_1$ the following inequality holds:

$$a_1^4 + a_2^4 + \dots + a_n^4 \geq \frac{16F^2}{n} \cdot \tan^2 \frac{\pi}{n}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3237 If M is an interior point in ΔABC and d_a, d_b, d_c are the distances of point M to the sides BC, CA, AB, then:

$$\frac{a^2b}{d_b} + \frac{b^2c}{d_c} + \frac{c^2a}{d_a} \geq 24F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3238 If $m \geq 0$, then in any triangle:

$$(a^{3m+1} + b^{3m+1} + c^{3m+1})(a^{m+3} + b^{m+3} + c^{m+3}) \geq 16^{m+1} \cdot 3^{1-m} \cdot F^{2m+2}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți-Romania

U.3239 If the sequence $(a_n)_{n\geq1}$ is convergent and $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}^*$ and it exist

$\lim\limits_{n\to\infty} n(a_n - a) = b \in \mathbb{R}^*$ such that:

$$\lim_{n\to\infty} (a_{n+1}a_n - a^2) \cdot n = 2ab$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3240 If $f: \mathbb{R}_+^* \to \mathbb{R}_+^* = (0, \infty)$ is a continuous function such that $\lim\limits_{x\to\infty} \frac{f(x+1)}{x \cdot f(x)} = a > 0$. Find:

$$\lim_{x\to\infty} \left(\left(f(x+1)\right)^{\frac{1}{x+1}} - \left(f(x)\right)^{\frac{1}{x}} \right)$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3241 Let be $(i_n)_{n\geq1}, i_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ the sequence of Andrei G. Ioachimescu with

$i_n = i_a \in (-2, -1)$ where i_a is Ioachimescu constant and $H_n = \sum_{k=1}^{n} \frac{1}{k}, \forall n \in \mathbb{N}^*$. Find:

$$\lim_{n\to\infty} e^{H_n} \cdot (i_n - i_a)^2$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3242 Let be $s, t > 0$ and $(a_n)_{n\geq1}, (b_n)_{n\geq1}$ sequences of real strictly positive numbers such that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n n^{s+1}} = a > 0$, $\lim\limits_{n\to\infty} \frac{b_{n+1}}{b_n n^t} = b > 0$. Find:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}b_{n+1}}}{(n+1)^{s+t}}-\frac{\sqrt[n]{a_nb_n}}{n^{s+t}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3243 Let be $(i_n)_{n\geq 1}, i_n=-2\sqrt{n}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}, \forall n\in\mathbb{N}^*$ the sequence of "Andrei G. Ioachimescu" with $\lim_{n\to\infty} i_n=i_a\in(-2,-1)$ where i_a is Ioachimescu constant. Find:

$$\lim_{n\to\infty}(i_n-i_a)\cdot\sqrt{n}$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3244 Let be $(a_n)_{n\geq 1}, a_n\in\mathbb{R}_+^*=(0,\infty)$ such that $a_{n+1}=a_n+\frac{\sqrt[n]{n!}}{n}, \forall n\in\mathbb{N}^*$. Find:

$$\lim_{n\to\infty}\left(\frac{a_{n+1}}{\sqrt[n+1]{a_{n+1}}}-\frac{a_n}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3245 Let be $(i_n)_{n\geq 1}, i_n=-2\sqrt{n}+\sum_{k=1}^{n}\frac{1}{\sqrt{k}}$, the sequence of Ioachimescu constant

$i_a=\lim_{n\to\infty} i_n\in(-2,-1)$. Find:

$$\lim_{n\to\infty}(i_n-i_a)^2\cdot n$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3246 Let be $s,t\geq 0$ and $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{s+1}}=a>0, \lim_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^t}=b>0$. Find:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}\cdot b_{n+1}}}{(n+1)^{s+t}}-\frac{\sqrt[n]{a_n\cdot b_n}}{n^{s+t}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3247 Let be $(a_n)_{n\geq 0}, a_n>0, \forall n\in\mathbb{N}^*, a_0=a_1$ such that $a_{n+1}=\sqrt[3]{a_{n+1}^3+a_{n+1}^2}, \forall n\geq 1$ and

$(b_n)_{n\geq 1}\cdot b_n>0, \forall n\in\mathbb{N}^*$ with $\lim_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n}=b>0$. Find:

$$\lim_{n\to\infty}\frac{a_n}{\sqrt[n]{b_n}}$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3248 It is known that "Andrei G. Ioachimescu sequence" $(i_n)_{n\geq 1}$with $\lim_{n\to\infty} i_n=i_a=$(Ioachimescu constant). Find:

$$\lim_{n\to\infty}\sqrt{n}\cdot(i_{n+1}\cdot i_n-i_a^2)$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3249 Let be $(i_n)_{n\geq 1}, i_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ the sequence of the constant i_a of Ioachimescu, namely $\lim\limits_{n\to\infty} i_n = i_a$. Find:

$$\lim_{n\to\infty} (i_{n+1}^2 \cdot i_n - i_a^3) \cdot \sqrt[2n]{n!}$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3250 Let be $(\gamma_n)_{n\geq 1}, \gamma_n = -\ln n + \sum_{k=1}^{n} \frac{1}{k}$ the sequence of the constant γ of Euler – Mascheroni and $(a_n)_{n\geq 1}$ a sequence of real numbers such that $\lim\limits_{n\to\infty} a_n = a \in \mathbb{R}^*$ and

$\lim\limits_{n\to\infty} n\,(a_n - a) = b \in \mathbb{R}^*$. Find:

$$\lim_{n\to\infty} (a_n\gamma_n - a\gamma)\, n$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3251 If $(i_n)_{n\geq 1}$ is Ioachimescu sequence $i_n = -2\sqrt{n} + \sum_{k=1}^{n} \frac{1}{\sqrt{k}}$ with $\lim\limits_{n\to\infty} i_a$ = Ioachimescu constant, find:

$$\lim_{n\to\infty} (i_{n+2} i_{n+1} i_n - i_a^3) \cdot \sqrt{n}$$

Proposed by D.M. Bătinețu – Giurgiu-Romania

U.3252 If $(a_n)_{n\geq 1}$ is a sequence of real strictly positive numbers such that:

$$a_{n+1}(n+1)^{n+1} = a_n n^n,\ \forall n \in \mathbb{N}^*.\ \text{Find:}$$

$$\lim_{n\to\infty} \left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3253 Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that:

$$a_{n+1} = a_n + b_n,\ \forall n \in \mathbb{N}^* \text{ and } \lim_{n\to\infty} b_n = b > 0.\ \text{Find:}$$

$$\lim_{n\to\infty} \left(\frac{a_{n+1}}{\sqrt[n+1]{a_{n+1}}} - \frac{a_n}{\sqrt[n]{a_n}} \right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3254 If $s, t > 0$ and $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{n^s a_n} = a > 0$. Find:

$\lim\limits_{n\to\infty} \sqrt[n]{a_n} \cdot (\gamma_n - \gamma) e^{H_n}$ where $H_n = \sum_{k=1}^{n} \frac{1}{k}, \gamma_n = -\ln n + H_n$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3255 If $a, b > 1$ find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a}\cdot\sqrt[b]{b}-1\right)\cdot\sqrt[n]{n!}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3256 Let $s, t \geq 0$ and $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n n^s} = a > 0$, $\lim_{n\to\infty}\frac{b_{n+1}}{b_n n^t} = b > 0$. Find if $s + t = 1$, then:

$$\lim_{n\to\infty}\sqrt[n]{a_n}\cdot\left(\sqrt[n+1]{b_{n+1}}-\sqrt[n]{b_n}\right) = \frac{ab\cdot t}{e}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3257 If $(a_n)_{n\geq 1}, a_n \in \mathbb{R}_+^* = (0,\infty)$ and $a_{n+1} = a_n + n\sin\frac{\pi}{n}, \forall n \in \mathbb{N}^*$. Find:

$$\lim_{n\to\infty}\left(a_{n+1}\cdot\sqrt[n+1]{a_{n+1}}-a_n\cdot\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3258 Let be $s \geq 0$ and $(a_n)_{n\geq 1}, a_n \in \mathbb{R}_+^* = (0,\infty), \forall n \in \mathbb{N}^*$ with $\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^{s+1}} = a > 0$. Find:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^s}-\frac{\sqrt[n]{a_n}}{n^s}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3259 If $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ and $\lim_{n\to\infty}\frac{a_{n+1}}{n^2 a_n} = a$. Find:

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}}{\sqrt[n]{(2n-1)!!}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

U.3260 If $(a_n)_{n\geq 1}, a_n \in \mathbb{R}_+^* = (0,\infty)$ and $a_{n+1} = a_n + n\tan\frac{\pi}{n}, \forall n \in \mathbb{N}^*$. Find:

$$\lim_{n\to\infty}\left(a_{n+1}\cdot\sqrt[n+1]{a_{n+1}}-a_n\cdot\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3261 Let be $u, v > 0$ and the sequences $(a_n)_{n\geq 1}, (H_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*, H_n = \sum_{k=1}^n \frac{1}{k}$,

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^u} = a > 0$. Find:

$$\lim_{n\to\infty} e^{(u+v)H_n}\cdot\frac{1}{\sqrt[n]{a_n}}\cdot\sin\frac{\pi}{n^v}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

U.3262 Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$a_{n+1} = a_n + b_n, \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} b_n = 1$. Find:

$$\lim_{n\to\infty}\left(a_{n+1}\cdot\sqrt[n+1]{a_{n+1}} - a_n\cdot\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.3263 Let be $(a_n)_{n\geq 1}, a_n = n!$ and $(b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ sequences of real strictly positive numbers such that $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = b > 0$, $\lim\limits_{n\to\infty}\frac{c_{n+1}}{n\cdot c_n} = c > 0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}b_{n+1}}{c_{n+1}}} - \sqrt[n]{\frac{a_n\cdot b_n}{c_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3264 Let $t \in \mathbb{R}_+^* = (0,\infty)$ and the sequences $(a_n)_{n\geq 1}, a_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$ with

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n\cdot n^t} = a > 0$ and $(H_n)_{n\geq 1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$. Find:

$$\lim_{n\to\infty}\frac{e^{(t+1)H_{n+1}} - e^{(t+1)H_n}}{\sqrt[n]{a_n}}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

U.3265 Let be $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that:

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = a > 0$. Find: $\lim\limits_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)\cdot\sqrt[n]{n!}$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

U.3266 Let be $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that:

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n} = a > 0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)\cdot\sqrt[n]{(2n-1)!!}$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu -Romania

U.3267 If $a \in (1,\infty)$ then find:

$$\lim_{n\to\infty}\left(a^{\sin\frac{2\pi}{n}} - 1\right)\cdot\sqrt[n]{(n!)^2}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3268 Let be $(H_n)_{n\ge1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$ and $(a_n)_{n\ge1}$ a sequence of real strictly positive numbers such that $a_{n+1} = a_n \cdot e^{H_n}, \forall n \ge 1$. Find:

$$\lim_{n\to\infty}\left(\frac{n+1}{\sqrt[n+1]{a_{n+1}}} - \frac{n}{\sqrt[n]{a_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3269 If $(H_n)_{n\ge1}, H_n = \sum_{k=1}^{n}\frac{1}{k}$ and $(a_n)_{n\ge1}$ is a sequence of real strictly positive numbers such that $a_{n+1} = a_n \cdot e^{H_n}, \forall n \in \mathbb{N}^*$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}} - \sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3270 Let be $(a_n)_{n\ge1}, a_n = n!$ and $(b_n)_{n\ge1}, b_n > 0, \forall n \in \mathbb{N}^*$ with $\lim\limits_{n\to\infty}\frac{b_{n+1}}{b_n\cdot n^t} = b > 0$, where $t > 0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}^{t+1}}{b_{n+1}}} - \sqrt[n]{\frac{a_n^{t+1}}{b_n}}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3271 Find:

$$\lim_{n\to\infty}\frac{\left(\sqrt[n+1]{\big((n+1)!\big)^2} - \sqrt[n]{(n!)^2}\right)}{n}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3272 Let be $t \ge 0$ and $(a_n)_{n\ge1}, (b_n)_{n\ge1}, a_n = n!, b_n = (2n-1)!!$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}\cdot b_{n+1}^t} - \sqrt[n]{a_n\cdot b_n^t}\right)\cdot\frac{1}{n^t}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3273 Let be $t \in \mathbb{N}^*$ fixed. Find:

$$\lim_{n\to\infty}\frac{\left(\sqrt[n+1]{(n+1)!}\right)^{t+1} - \left(\sqrt[n]{n!}\right)^{t+1}}{\left(\sqrt[n]{(2n-1)!!}\right)^t}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3274 Find:

$$\lim_{n\to\infty}\frac{\left(\sqrt[n+1]{(n+1)!}\right)^2-\left(\sqrt[n]{n!}\right)^2}{\sqrt[n]{(2n-1)!!}}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3275 Let be $(a_n)_{n\geq 1}, a_n = n!$ and $(b_n)_{n\geq 1}, b_n > 0, \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty}\frac{n\cdot b_{n+1}}{b_n} = b > 0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^2\cdot b_{n+1}}-\sqrt[n]{a_n^2\cdot b_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3276 Let be $(a_n)_{n\geq 1}, a_n = \sqrt[n]{n!}, \forall n \in \mathbb{N}^*$ and $(b_n)_{n\geq 1}, (c_n)_{n\geq 1}$ with $b_n, c_n > 0, \forall n \in \mathbb{N}^*$ such that $\lim\limits_{n\to\infty}\frac{b_{n+1}}{n\cdot b_n} = b > 0$, $\lim\limits_{n\to\infty}\frac{n\cdot c_{n+1}}{c_n} = c > 0$. Prove that:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}\cdot b_{n+1}\cdot c_{n+1}}-\sqrt[n]{a_n\cdot b_n\cdot c_n}\right)=\frac{bc}{e}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3277 If $(a_n)_{n\geq 2}, a_n = \sqrt[n]{n!}, \forall n \geq 2$ and $(b_n)_{n\geq 1}, b_n > 0, \forall n \geq 1$ and $\lim\limits_{n\to\infty}\frac{b_{n+1}\cdot n}{b_n} = b > 0$. Find:

$$\lim_{n\to\infty}\left(a_{n+1}^2\sqrt[n+1]{b_{n+1}}-a_n^2\sqrt[n]{b_n}\right)=\frac{b}{e}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3278 Let be $a_n = \sqrt[n]{n!}, \forall n \geq 2$ and $(b_n)_{n\geq 1}, b_n > 0, \forall n \geq 1$ with $\lim\limits_{n\to\infty}\frac{n^2\cdot b_{n+1}}{b_n} = b > 0$. Find:

$$\lim_{n\to\infty}\sqrt[n]{b_n}\cdot\left(a_{n+1}^3-b_n^3\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3279 Let be $t > 0$ fixed. Find:

$$\lim_{n\to\infty}\frac{\sqrt[n+1]{\left((n+1)!\right)^{t+1}}-\sqrt[n]{(n!)^{t+1}}}{n^t}$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru-Romania

U.3280 Let be $(a_n)_{n\geq 1}, a_n > 0, \forall n \in \mathbb{N}^*$ such that $\lim\limits_{n\to\infty}\frac{n\,a_{n+1}}{a_n} = a > 0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\left((n+1)!\right)^2\cdot a_{n+1}}-\sqrt[n]{(n!)^2a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3281 Let be $s>0$ and $(a_n)_{n\geq1}, a_n>0, \forall n\in\mathbb{N}^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{n^{s+1}\cdot a_n}=a>0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)\frac{1}{n^s}$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3282 Let be $(H_n)_{n\geq1}, H_n=\sum_{k=1}^{n}\frac{1}{k}$. Find:

$$\lim_{n\to\infty}\left(-\ln\sqrt[n]{n!}+H_n\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Daniel Sitaru -Romania

U.3283 If $m\geq0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^{m+2}}{(b+c)^m}+\frac{b^{m+2}}{(c+a)^m}+\frac{c^{m+2}}{(a+b)^m}\geq 2^{2-m}\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3284 Let $s>0$ and $(a_n)_{n\geq1}, a_n>0, \forall n\in\mathbb{N}^*$ such that $\lim_{n\to\infty}\frac{a_{n+1}}{n^{s+1}a_n}=a>0$. Find:

$$\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{a_{n+1}}}{(n+1)^s}-\frac{\sqrt[n]{a_n}}{n^s}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Mihaly Bencze -Romania

U.3285 If $x,y,z>0$ then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^4}{x^2}+\frac{b^4}{y^2}+\frac{c^4}{z^2}+4(xy+yz+zx)\geq 16\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3286 If $a,b,c,x,y>0$ then:

$$a\cdot\sqrt{\frac{(x+y)a}{xb+yc}}+b\cdot\sqrt{\frac{(x+y)b}{xc+ya}}+c\cdot\sqrt{\frac{(x+y)c}{xa+yb}}\geq a+b+c$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3287 If $x,y>0$ and ABC is a triangle with the area F then:

$$(x^2+y^2)\cdot(a^2+b^2+c^2)\geq 8xy\cdot\sqrt{3}\cdot F+\sum_{cyc}(xa-yb)^2$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3288 Let be $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ sequences of real strictly positive numbers such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{n\cdot a_n}=a>0, \lim\limits_{n\to\infty}\frac{n\cdot b_{n+1}}{b_n}=b>0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}^2\cdot b_{n+1}}-\sqrt[n]{a_n^2 b_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3289 Let be $s>0, t\geq 0$ and $(a_n)_{n\geq 1}$ a sequence of real strictly positive numbers such that

$\lim\limits_{n\to\infty}\frac{a_{n+1}}{a_n}=a>0$. Find:

$$\lim_{n\to\infty}\left(\sqrt[n+1]{a_{n+1}}-\sqrt[n]{a_n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu -Romania

U.3290 In any triangle ABC with the area F, the following inequality holds:

$$\frac{a^2}{h_a(b+c)}+\frac{b^2}{h_b(c+a)}+\frac{c^2}{h_c(a+b)}\geq\sqrt{3}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3291 Let be $x, y>0$ and ABC a triangle, then:

$$\frac{a}{(bx+cy)^2}+\frac{b}{(cx+ay)^2}+\frac{c}{(ax+by)^2}\geq\frac{\sqrt{3}}{(x+y)^2\cdot R}$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3292 If $m\geq 0$, then in any triangle ABC with the area F the following inequality holds:

$$\frac{a^{m+2}}{b^m}+\frac{b^{m+2}}{c^m}+\frac{c^{m+2}}{a^m}\geq 4\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3293 In any triangle ABC with the area F the following inequality holds:

$$\left(\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\right)\cdot(a^4+2)\cdot(b^4+2)(c^4+2)\geq 27\cdot\sqrt{3}\cdot F$$

Proposed by D.M. Bătinețu – Giurgiu, Claudia Nănuți -Romania

U.3294 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$\frac{n_a-m_a}{h_a}+\frac{n_b-m_b}{h_b}+\frac{n_c-m_c}{h_c}\geq\frac{a^2+b^2+c^2-ab-bc-ca}{2F}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3295 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$n_a + n_b + n_c \geq m_a + m_b + m_c + \frac{{n_a}^2 + {n_b}^2 + {n_c}^2 - s^2}{2s}$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3296 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$n_a + n_b + n_c \geq 2s - 3(2\sqrt{3} - 3)r$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3297 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$(n_a + n_b + n_c)^2 \geq 9s^2 - 80Rr - 2r^2$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3298 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$n_a n_b + n_b n_c + n_c n_a \geq 3s^2 - 32Rr + 10r^2$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3299 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$n_a + n_b + n_c + \frac{2\sqrt{3} - 3}{3}.(h_a + h_b + h_c) \geq 2s$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

U.3300 If $n_a, n_b, n_c -$ Nagel cevians in ΔABC then:

$$3(n_a + n_b + n_c) + 9r \geq 4(m_a + m_b + m_c)$$

Proposed by Mohamed Amine Ben Ajiba-Morocco

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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PROBLEMS FOR JUNIORS

JP.586 If a,b,c are sides of acute ΔABC and

$$\frac{2}{a^2+c^2-b^2}=\frac{1}{b^2+c^2-a^2}+\frac{1}{c^2+a^2-b^2}$$

then: $\tan^2 B\geq \tan A\cdot\tan C$

Proposed by Daniel Sitaru – Romania

JP.587 If $x,y\geq 1$ then:

$$\ln(xy)\cdot(2\ln(xy)+1)\geq 4\left(\ln x\sqrt{\ln y}+\ln y\sqrt{\ln x}\right)$$

Proposed by Daniel Sitaru – Romania

JP.588 If $a,b,c,d>0, abcd=1$ then: $a^{b+c+d}b^{c+d+a}c^{d+a+b}d^{a+b+c}\leq 1$

Proposed by Marin Chirciu – Romania

JP.589 If $a,b,c>0$ and $1<\lambda\leq 2$ then:

$$\sum \frac{a(b+c)^2}{\lambda a+b+c} \le \frac{a^2+b^2+c^2}{\lambda-1}$$

Proposed by Marin Chirciu – Romania

JP.590 If $a, b, c > 0$ then:

$$\sum \frac{(b+c)^2}{2a^3+bc(b+c)} \le \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

Proposed by Marin Chirciu – Romania

JP.591 If $x, y, z > 0, x+y+z=1$ and $\lambda \ge 26$ then:

$$x^4+y^4+z^4+\lambda xyz \le \frac{\lambda+1}{27}$$

Proposed by Marin Chirciu – Romania

JP.592 Solve for real numbers:

$$\begin{cases} 2^x+3^y+5^z=10 \\ \left|\sqrt{x^2+y^2+z^2}-\sqrt{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}}\right| = \left|x-\frac{1}{x}\right|+\left|y-\frac{1}{y}\right|+\left|z-\frac{1}{z}\right| \end{cases}$$

Proposed by Daniel Sitaru – Romania

JP.593 Solve for real numbers:

$$\frac{x^2}{x^2+4\sqrt{x}+2}+\frac{2}{2+x\sqrt{x+2}}=\frac{4x}{5x+2}$$

Proposed by Daniel Sitaru – Romania

JP.594 Solve for real numbers: $\sin^{2022} x \cdot \cos^{2024} x = \frac{1}{2^{2022}}$

Proposed by Daniel Sitaru – Romania

JP.595 Find $x, y, z > 1$ such that:

$$\sum_{cyc} \frac{\log_2 x}{\log_2^6 x+\log_2^3 y+\log_2^3 z} = \frac{1}{27}\left(\sum_{cyc} \log_2 x\right)^3$$

Proposed by Daniel Sitaru – Romania

JP.596 In acute $\Delta ABC, AA', BB', CC'$ - are altitudes, $C' \in (AB), B' \in (AC)$,

$\{H\} = BB' \cap CC'$ and E, F are middle points of $[BH], [AC]$ respectively. Prove that:

$$4EF^2 \ge (EC'+EB')^2+(C'F+B'F)^2$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

JP.597 Let $ABCD$ be an convex quadrilateral, $\lambda \in \mathbb{R}$ and M, N be such that

$\overrightarrow{AN} = \lambda \cdot \overrightarrow{AB}, \overrightarrow{DN} = \lambda \cdot \overrightarrow{DC}, \overrightarrow{AD} = 3 \cdot \overrightarrow{BC}$. Find $\lambda \in \mathbb{R}$ such that $\overrightarrow{MN} = 7 \cdot \overrightarrow{BC}$

Proposed by Marian Ursărescu, Florică Anastase – Romania

JP.598 Let $n \geq 4$, and let $a_1, a_2, \dots, a_n$ be nonnegative real numbers such that

$a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_1a_2 + a_2a_3 + \cdots + a_na_1 = n$. Prove that:

$$\frac{1}{2a_1+5} + \frac{1}{2a_2+5} + \cdots + \frac{1}{2a_n+5} \geq \frac{n}{7}$$

Proposed by Vasile Cîrtoaje – Romania

JP.599 Prove that 3 is the largest positive value of the power k such that the inequality

$$\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq a_1^k + a_2^k + \cdots + a_n^k$$

holds for $n \geq 2$ and any positive real numbers $a_1, a_2, \dots, a_n$ with at most one $a_i < 1$ and $a_1^2 + a_2^2 + \cdots + a_n^2 = n$.

Proposed by Vasile Cîrtoaje – Romania

JP.600 Calculate the limit of sequence $(a_n)_{n\geq1}$ defined by the following relationship:

$$a_n = \frac{1}{n}\int_0^{\frac{1}{2}} \ln\left(1 + e^{n\cdot \arcsin x}\right)\, dx$$

Proposed by Vasile Mircea Popa – Romania

PROBLEMS FOR SENIORS

SP.586 Solve for real numbers:

$$\begin{cases} \left(\sqrt{x+y} - \sqrt{x}\right)\left(\sqrt{x^2+xy} + 1\right) = xy \\ x + y + z = 3 \\ \left(\sqrt{y+z} - \sqrt{y}\right)\left(\sqrt{y^2+yz} + 1\right) = xy \end{cases}$$

Proposed by Daniel Sitaru – Romania

SP.587 Let a,b,c be sides in ΔABC. If $\tan B=2$; $\tan C=3$ then:

$$a^2+b^2+c^2>\frac{2F}{3}\left(3\sqrt{2}+3\sqrt{5}+2\sqrt{10}-11\right)$$

Proposed by Daniel Sitaru – Romania

SP.588 For given $n\ge 3$, prove that 2 is the least positive value of k such that:

$$\frac{1}{ka_1+1}+\frac{1}{ka_2+1}+\cdots+\frac{1}{ka_n+1}\ge\frac{n}{k+1}$$

for any positive real numbers a_i with at most two $a_i>1$ and $a_1a_2\dots a_n=1$.

Proposed by Vasile Cîrtoaje – Romania

SP.589 Prove that 4 is the largest positive value of k such that the inequality

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge a^k+b^k+c^k$$

holds for any positive real numbers a,b,c with at most one of them less than 1 and

$$a+b=c=a^2+b^2+c^2$$

Proposed by Vasile Cîrtoaje – Romania

SP.590 Solve the following system in integers $(x,y,z)\in\mathbb{N}^*\times\mathbb{N}^*\times\mathbb{Z}$

$$\begin{cases}x^3-y^2+2z=0\\x^2+y^2+z^2=179\end{cases}$$

Proposed by Said Attaoui –Algeria

SP.591 If $a,b,c>0$, $a^8+b^8+c^8\le 768$ then:

$$\sum\frac{1}{\sqrt{4+a^5}}\ge\frac{1}{2}$$

Proposed by Marin Chirciu – Romania

SP.592 If $a,b,c>0$ and $n\in\mathbb{N}, n\ge 2$ then:

$$\sum a\sqrt[n]{b^n+c^n}\ge\sqrt[n]{2}(ab+bc+ca)$$

Proposed by Marin Chirciu – Romania

SP.593 Solve for real numbers:

$$(\sin x + \cos y)^2 = (\sin x + 1)(\cos y - 1)$$

Proposed by Daniel Sitaru – Romania

SP.594 Find $x, y > 0$ such that:

$$\ln^2(xy) = \ln(xe) \cdot \left(\ln \frac{y}{e}\right)$$

Proposed by Daniel Sitaru – Romania

SP.595 Solve for real numbers:

$$\tan 2x + \tan 3x + \tan 5x = \tan 2x \cdot \tan 3x \cdot \tan 5x$$

Proposed by Daniel Sitaru – Romania

SP.596 Solve for real numbers:

$$\begin{cases} \sin^2 x = \frac{1}{2} + \sin^2(y - z) \\ \sin^2 y = \frac{1}{3} + \sin^2(z - x) \\ \sin^2 z = \frac{1}{6} + \sin^2(x - y) \end{cases}$$

Proposed by Daniel Sitaru – Romania

SP.597 Let a, b, c, d be positive real numbers with $\sum a \ge \sum \frac{1}{a}$. Prove that:

$$\sum \frac{a + b + c - d}{a^4 + b^4 + c^4 + abcd} \le \frac{4}{3}\left(\frac{ab + ac + ad + bc + bd + cd}{abc + abd + acd + bcd}\right)$$

Proposed by Huseyin Yigit Emekci –Turkey

SP.598 Let x, y, z be positive real numbers. Prove that:

$$\frac{x^3 + 9xy^2}{z^3 + x^2y} + \frac{y^3 + 9yz^2}{x^3 + y^2z} + \frac{z^3 + 9zx^2}{y^3 + z^2x} \ge 3 + \frac{12xyz(x + y + z)}{x^3y + y^3z + z^3x}$$

Proposed by Huseyin Yigit Emekci– Turkey

SP.599 We consider the function $f: D \to \mathbb{R}$

$$f(x) = x\int_x^{x+\frac{3}{x}} t \arcsin\left(\frac{1}{t}\right) dt$$

where D is the maximal domain of the function.

a. Find the domain D

b. Show that the function $f(x)$ is even

c. Calculate $\lim_{x\to-\infty} f(x)$

Proposed by Vasile Mircea Popa – Romania

SP.600 Let a,b,c,d,e,f,g be real numbers such that $a\geq b\geq c\geq d\geq e\geq f\geq g$ and $a+b+c+d+e+f+g=0$. Prove that:

$$a^2+b^2+c^2+d^2+e^2+f^2+g^2\geq 2(ab+bc+cd+de+ef+fg+ga)$$

Proposed by Vasile Cîrtoaje – Romania

UNDERGRADUATE PROBLEMS

UP.586 If $A\in M_{2,1}(\mathbb{R}); B\in M_{1,2}(\mathbb{R}); A\cdot B=\begin{pmatrix}0 & 0\\ 8 & 1\end{pmatrix}$ then find $B\cdot A$.

Proposed by Daniel Sitaru -Romania

UP.587 Let a,b,c be positive real numbers such that at most one of them is less that 1 and $ab+bc+ca=3$. Prove that:

$$abc(a+b+c)^3\leq 27$$

Proposed by Vasile Cîrtoaje – Romania

UP.588 If $a\geq 0$ then:

$$15\left(\int_0^a \frac{x}{e^x}dx\cdot\int_0^a\frac{x^2}{e^x}dx\cdot\int_0^a\frac{x^3}{e^x}dx\right)^2\leq a^9\left(\int_0^a\frac{x^2}{e^{2x}}dx\right)^3$$

Proposed by Daniel Sitaru – Romania

UP.589 If $X,Y,Z\in M_4(\mathbb{C})$ are matrices such that:

$$\begin{cases} X=2Y+Z\\ X^2=4Y+4Z\\ X^3=8Y+12Z\end{cases}\quad \text{then: } X^{2024}=2^{2024}\cdot Y+2024\cdot 2^{2023}\cdot Z$$

Proposed by Daniel Sitaru – Romania

UP.590 If $A,B\in M_4(\mathbb{R}); A\cdot B=B\cdot A$ then:

$$\det\left(A^4+B^4+AB(A^2+AB+B^2)\right)\geq 0$$

Proposed by Daniel Sitaru – Romania

UP.591 Find:

$$\Omega=\lim_{n\to\infty}\frac{1}{3^n}\sum_{k=0}^{n-1}\sqrt{\binom{n}{k}\binom{n}{k+1}}$$

Proposed by Daniel Sitaru – Romania

UP.592 Solve for real numbers:

$$\begin{cases}\cos x+\cos y+\cos z=1\\ \cos^2 x+\cos^2 y+\cos^2 z=1\\ \cos^3 x+\cos^3 y+\cos^3 z=1\end{cases}$$

Proposed by Daniel Sitaru – Romania

UP.593 For $b\geq a$, prove that:

$$\int_a^b\frac{(x+1)^3-3x}{e^{x^3}}dx\leq b-a+\ln\left(\frac{b^3+1}{a^3+1}\right)$$

with equality if and only if $a=b$.

Proposed by Huseyin Yigit Emekci –Turkey

UP.594 Solve the system

$$\begin{cases}x-2y+z+2=k^2,\quad with\ 3<k<11\\ x^2+y^2+z^2=109659\\ -x^4+y^2+z^2=80929\\ 3<x<y<z,\quad x,y,z\in\mathbb{N}.\end{cases}$$

Proposed by Said Attaoui –Algeria

UP.595 We consider the function $u:\mathbb{R}\to\mathbb{R}$, periodic with period 2π. For the period $[0,2\pi]$ we have: $u(x)=0$ if $x\in\left[0,\frac{\pi}{2}\right)$; $u(x)=-\cos(x)$ if $x\in\left[\frac{\pi}{2},\frac{3\pi}{2}\right)$; $u(x)=0$ if $x\in\left[\frac{3\pi}{2},2\pi\right)$. Prove the equality:

$$\int_0^\infty\frac{u(x)}{1+x^2}dx=-\frac{\pi}{4e}+\frac{e^2+1}{2e}\arctan\left(\frac{1}{e}\right)$$

Proposed by Vasile Mircea Popa – Romania

UP.596 If $x>0,y>0,z>0$ prove that there exists $u>0$ such as

$$\frac{\sin x\sin y+\sin y\sin z+\sin z\sin x}{xy+yz+zx}=\frac{\sin u}{u}$$

Proposed by Cristian Miu – Romania

UP.597 Find the following limit:

$$L=\lim_{n\to\infty}\left(\frac{1}{2^n}\cdot\lim_{x\to\frac{\pi}{n}}\left(\sum_{k=0}^{n}\binom{n}{k}\sin(k+1)\,x\right)\right)$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

UP.598 Find the following limit:

$$L=\lim_{x\to0}\left(\frac{1}{x}\cdot\lim_{n\to\infty}\sum_{k=1}^{n}3^{k-1}\sin^3\frac{x}{3^k}\right),a\in\mathbb{R}$$

Proposed by Marian Ursărescu, Florică Anastase – Romania

UP.599 Calculate the integral:

$$\int_{-\pi}^{\pi}\frac{\operatorname{arccot}(x)}{\sqrt{3-\cos(x)}}dx$$

In this problem we will consider that definition of the function $\operatorname{arccot}(x)$ which has the image the interval $(0,\pi)$.

Proposed by Vasile Mircea Popa – Romania

UP.600 If $0<a\le b$ then:

$$a^3+3\int_a^b\sinh x\cdot\operatorname{arcsinh}x\,dx\ge b^3$$

Proposed by Daniel Sitaru – Romania

All solutions for proposed problems can be finded on the http//:www.ssmrmh.ro which is the adress of Romanian Mathematical Magazine-Interactive Journal.

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