ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^2\frac{B}{2}\sqrt{\sin\frac{A}{2}}+\sin^2\frac{C}{2}\sqrt{\sin\frac{C}{2}}\right)}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}\geq 6\sqrt{2}$$

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$$Let\ (x,y,z)=\left(sin\frac{A}{2},sin\frac{B}{2},sin\frac{C}{2}\right).\quad\Rightarrow x,y,z\in(0,1)$$

$Lemma\ 1$: $xyz\leq\frac{1}{8}$.

$Proof$: $We\ know\ that$: $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\Rightarrow xyz=\frac{r}{4R}\overset{Euler}{\leq}\frac{1}{8}$.

$Lemma\ 2$: $x+y+z\leq\frac{3}{2}$

$Proof$: $Since\ the\ sine\ function\ is\ concave\ on\ (0,\pi),\ by\ Jensen's\ Inequality,$

$$x+y+z=\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq 3\cdot\sin\left(\frac{\frac{A}{2}+\frac{B}{2}+\frac{C}{2}}{3}\right)=3\cdot\sin\left(\frac{\pi}{6}\right)=\frac{3}{2}$$

$Claim\ 1$: $\prod(x+y)\leq 1$

$$Proof:\ \prod(x+y)\overset{AM-GM}{\leq}\left(\frac{x+y+y+z+z+x}{3}\right)^3=\left(\frac{2(x+y+z)}{3}\right)^3=\overset{Lemma\ 2}{\leq}1$$

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^2\frac{B}{2}\sqrt{\sin\frac{A}{2}}+\sin^2\frac{C}{2}\sqrt{\sin\frac{C}{2}}\right)}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}=\sum_{cyc}\frac{x\left(y^2\sqrt{x}+z^2\sqrt{z}\right)}{y^2z^2(y+z)}=$$

$$=\sum_{cyc}\left(\frac{x\sqrt{x}}{z^2(y+z)}+\frac{x\sqrt{z}}{y^2(y+z)}\right)$$

$By\ AM-GM\ Inequality,$

$$\sum_{cyc}\left(\frac{x\sqrt{x}}{z^2(y+z)}+\frac{x\sqrt{z}}{y^2(y+z)}\right)\geq 6\cdot\sqrt[6]{\prod\left(\frac{x\sqrt{x}}{z^2(y+z)}\cdot\frac{x\sqrt{z}}{y^2(y+z)}\right)}=$$

$$=6\cdot\sqrt[6]{\frac{x^3y^3z^3}{x^4y^4z^4(x+y)^2(y+z)^2(x+z)^2}}=6\cdot\sqrt[6]{\frac{1}{xyz\cdot[\ \prod(x+y)\ ]^2}}$$

$From\ Lemma\ 1\ and\ Claim\ 1\ it\ follows\ that$:

$$\sum_{cyc}\frac{\sin\frac{A}{2}\left(\sin^2\frac{B}{2}\sqrt{\sin\frac{A}{2}}+\sin^2\frac{C}{2}\sqrt{\sin\frac{C}{2}}\right)}{\sin^2\frac{B}{2}\sin^2\frac{C}{2}\left(\sin\frac{B}{2}+\sin\frac{C}{2}\right)}=\sum_{cyc}\left(\frac{x\sqrt{x}}{z^2(y+z)}+\frac{x\sqrt{z}}{y^2(y+z)}\right)\geq$$

$$\geq 6\cdot\sqrt[6]{\frac{1}{\frac{1}{8}\cdot(1)^2}}=6\sqrt{2}$$

Equality holds if and only if the triangle is equilateral.