ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$r_a^2r_b^2+r_b^2r_c^2+r_c^2r_a^2+n_a^2n_b^2+n_b^2n_c^2+n_c^2n_a^2\ge 2h_ah_bh_c(h_a+h_b+h_c)$$

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Solution by Jenish Rijal-Nepal

$$h_c=\frac{2r_ar_b}{r_a+r_b}\overbrace{\le}^{HM-GM}\sqrt{r_ar_b}\Leftrightarrow r_ar_b\ge h_c^2\Leftrightarrow$$

$$r_ar_b\cdot n_an_b\ge h_c^2\cdot h_ah_b\ [\because n_a\ge h_a\ and\ n_b\ge h_b]$$

$$\therefore\ r_ar_bn_an_b\overbrace{\ge}^{①}h_ah_bh_c^2$$

$Now, via\ AM-GM\ Inequality$:

$$r_a^2r_b^2+n_a^2n_b^2\ge 2\sqrt{r_a^2r_b^2n_a^2n_b^2}=r_ar_bn_an_b\overbrace{\ge}^{\text{Via }①}2h_ah_bh_c^2\Rightarrow$$

$$\sum_{cyc}\left(r_a^2r_b^2+n_a^2n_b^2\right)\ge 2\sum_{cyc}\left(h_ah_bh_c^2\right)$$

$$\Leftrightarrow\ r_a^2r_b^2+r_b^2r_c^2+r_c^2r_a^2+n_a^2n_b^2+n_b^2n_c^2+n_c^2n_a^2\ge 2h_ah_bh_c(h_a+h_b+h_c).$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$