ROMANIAN MATHEMATICAL MAGAZINE

In ΔABC the following relationship holds:

$$\frac{r_a r_b^2 + r_b r_c^2 + r_c r_a^2}{(r_a + r_b + r_c)(r_a r_b + r_b r_c + r_c r_a)} \geq \frac{16r^2 - 3R^2}{3R^2}$$

Proposed by Zaza Mzhavanadze-Georgia

Solution by Jenish Rijal-Nepal

$$\sum_{cyc} r_a r_b^2 = \sum_{cyc} \frac{(r_a r_b)^2}{r_a} \overbrace{\geq}^{\text{BERGSTROM}} \frac{(r_a r_b + r_b r_c + r_c r_a)^2}{r_a + r_b + r_c}$$

$$\therefore\ LHS = \frac{r_a r_b^2 + r_b r_c^2 + r_c r_a^2}{(r_a + r_b + r_c)(r_a r_b + r_b r_c + r_c r_a)} \geq \frac{r_a r_b + r_b r_c + r_c r_a}{(r_a + r_b + r_c)^2}$$

$$Via\ well-known\ identities \sum_{cyc} r_a = 4R + r\ \ and \sum_{cyc} r_a r_b = s^2\ , we\ have:$$

$$LHS \geq \frac{r_a r_b + r_b r_c + r_c r_a}{(r_a + r_b + r_c)^2} = \frac{s^2}{(4R + r\ )^2} \overbrace{\geq}^{\text{Mitrinovic}} \frac{27r^2}{(4R + r\ )^2}$$

$$Thus, it\ suffices\ to\ prove\ that:\ \frac{27r^2}{(4R + r\ )^2} \geq \frac{16r^2 - 3R^2}{3R^2}$$

$$\Leftrightarrow\ 48R^4 + 24R^3 r - 172R^2 r^2 - 128Rr^3 - 16r^4 \geq 0$$

$$\Leftrightarrow\ 4(R - 2r)(12R^3 + 30R^2 r + 17Rr^2 + 2r^3) \geq 0$$

$$which\ is\ true\ via\ Euler's\ Inequality\ (R \geq 2r)$$

$$Equality\ holds\ if\ and\ only\ if\ the\ triangle\ is\ equilateral.$$